Concepts for Frequency Sweeps and Efficient Repeated Analysis in the Context of Vibroacoustic Optimization and Uncertainty Quantification

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This paper aims at passive noise control for vibroacoustic problems, which are analyzed by finite and boundary element techniques. The author distinguishes interior and exterior problems mainly because of the quantities used as the objective function to assess the acoustic quality. For interior problems, it is common to use local quantities such as the sound pressure at a field point or, in rare cases, energy density at a field point. The situation is different for exterior problems where the radiated sound power accounts for a suitable and global quantity to assess the emission from a vibrating structure. For most engineering purposes, the assessment requires frequency sweeps in which the problem needs to be solved at many discrete frequencies. In vibroacoustic optimization and in sampling based uncertainty quantification, it is very common that structural parameters are varied, while the acoustic field remains the same throughout the entire process. We will review concepts and recent developments of efficient frequency sweeps and repeated analysis with unmodified fluid domain. For many practical cases, the situation for interior problems is rather simple to survey. Either the authors have applied a modal analysis and used a modal superposition for the frequency sweep and repeated analysis, or the concept of unmodified acoustic transfer vectors is applied. Both concepts are quite successful as long as certain conditions are fulfilled. For exterior problems, a modal superposition is possible but, so far, only for a limited number of cases practically applicable as discussed herein. The concept of using acoustic transfer vectors becomes inefficient since the evaluation of the radiated sound power as an integral over a closed enveloping surface would require an excessively high amount of storage capacity. Therefore, other concepts are being followed. For frequency sweeps, a number of methods are using a frequency interpolation based on a limited number of discrete sample frequencies. Often, these techniques are used together with Krylov–subspace model order reduction techniques. Further recently published approaches investigate low–rank approximations, greedy algorithms, and deflated Krylov subspace techniques. A completely different kind of approach is based on multi–fidelity models and Gaussian processes. The field of efficient repeated analysis shows some interesting developments, which can be easily applied to sampling based uncertainty quantification but do not seem to be easily and generally applied to optimization.

1. INTRODUCTION

The work discussed herein is essentially motivated by a concluding sentence in the author's review paper[1] on structural acoustic optimization, which was published more than two decades ago: *Koopmann and Fahnline[2] emphasized that efficient analysis techniques account for the basis of optimization. Though regarding the entire analysis, fast solutions of the fluid problem or even of the coupled problem involving radiation into open space and considering frequency ranges are strongly desired.* Since the efficiency problem occurs in a very similar way for uncertainty quantification, the latter is addressed herein as well.

The literature on efficient solution techniques for acoustic and vibroacoustic problems is full of excellent books and papers that mainly focus on the solution of coupled and uncoupled boundary value problems formulated at one particular preselected frequency, see for example a selection of books published over the last four decades[3–18] and a large variety of journal papers, see for example.[19–41]

The author is interested in clarifying that this paper does not consider fast solution and efficient solution techniques of the Helmholtz equation for a single frequency. Some of the techniques which are referred herein are developed for single frequencies though. When referring to such techniques, the author will point out how the particular method can be efficiently used in optimization and uncertainty quantification. In contrast to efficient single frequency solutions, the number of papers on efficient methods for frequency sweeps in acoustics is most likely two orders of magnitude smaller whereas even far fewer papers are dealing with efficient reanalysis. In addition to some own contributions, the author is aware of very few such works on efficient reanalysis for fully coupled vibroacoustic systems.[42,43] Very often, however, the major problem of efficient reanalysis is encountered for sound radiation into open spaces and then, the bottleneck for efficient reanalysis is identified as the repeated evaluation of the radiated sound power.

This paper reviews concepts of dealing with acoustic and vibroacoustic problems over frequency ranges and efficient reanalysis. It is structured as follows: We will start with a brief motivation explaining the problem because acoustic and vibroacoustic problems often require solutions of large frequency ranges, whereas in optimization and uncertainty quantification, many solutions for varying parameters are required. Thereafter, we will briefly review single frequency solutions using finite and/or boundary element methods. This is followed by a discussion of objective functions. Solution concepts for frequency sweeps and reanalysis will be presented and discussed thereafter in Section 3. The paper will be completed by a short conclusion section.

2. PROBLEM DESCRIPTION

2.1. Prerequisites and Motivation

The analysis in this paper works in the frequency domain, which means that harmonic problems in acoustics and vibroacoustics are considered. A harmonic time dependence of $\mathrm{e}^{\mathrm{i}\omega t}$ or $\mathrm{e}^{-\mathrm{i}\omega t}$, with t as time and ω as angular frequency, is assumed throughout this and in all related work. The latter assumption is used for the equations of this work. Mathematically, this means that the acoustic problem is governed by the Helmholtz equation whereas the elastic problem, which is usually described by the Navier–Lame equation, is a bit more complex to survey as specific structures, such as beams, plates, shells, etc., may result in different governing equations derived from the Navier–Lame equation under specific assumptions. When using (linear) finite element methods for the structure, it is common to yield a system of equations for the structural vibrations as:

$$\boldsymbol{A}_{\mathrm{s}}\boldsymbol{u}_{\mathrm{s}} = \left(-\omega^2\boldsymbol{M}_{\mathrm{s}} - \mathrm{i}\omega\boldsymbol{D}_{\mathrm{s}} + \boldsymbol{K}_{\mathrm{s}}\right)\boldsymbol{u}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{s}}; \qquad (1)$$

where the column matrix $\boldsymbol{u}_{\mathrm{s}}(\omega)$ represents the time–harmonic structural displacement vectors, $\boldsymbol{f}_{\mathrm{s}}(\omega)$ the column matrix of nodal forces of the finite element mesh, $\boldsymbol{M}_{\mathrm{s}}$ and $\boldsymbol{K}_{\mathrm{s}}$ are known as the static, i.e., frequency independent, mass and stiffness matrices, respectively. $\boldsymbol{D}_{\mathrm{s}}$ is the damping matrix often assumed to be independent of frequency, which is a very strong assumption that is often not fulfilled in applications. However, this assumption is very commonly made since more precise descriptions of $\boldsymbol{D}(\omega)$ are seldom available. The dynamic stiffness matrix of the structure $\boldsymbol{A}_{\mathrm{s}}$ is often assumed as a quadratic polynomial of frequency composed of $\boldsymbol{M}_{\mathrm{s}}$, $\boldsymbol{K}_{\mathrm{s}}$ and $\boldsymbol{D}_{\mathrm{s}}$. The column matrix of the surface particle velocity of the structure $\boldsymbol{v}_{\mathrm{s}}$ which denotes the normal component of the particle velocity vector of a surface node of the finite element mesh is related to $\boldsymbol{u}_{\mathrm{s}}$ as:

$$\boldsymbol{v}_{\mathrm{s}} = -\mathrm{i}\omega\boldsymbol{N}\boldsymbol{u}_{\mathrm{s}} \qquad (2)$$

in which $\boldsymbol{N}$ is a matrix to map the displacement vectors in $\boldsymbol{u}_{\mathrm{s}}$ to yield the normal component of the particle velocity just at the surface nodes. Note that the notation is only correct if $\boldsymbol{u}_{\mathrm{s}}$ contains only degrees of freedom of the wet surface, i.e., degrees of freedom with fluid contact. It is used for simplicity though.

For the analysis of the source-free fluid, we assume either a finite element or a boundary element discretization.[44] The finite element discretization may lead to a system of equations as:

$$\boldsymbol{A}_{\mathrm{f}}\boldsymbol{p} = \left(-\omega^2\boldsymbol{M}_{\mathrm{f}} - \mathrm{i}\omega\boldsymbol{D}_{\mathrm{f}} + \boldsymbol{K}_{\mathrm{f}}\right)\boldsymbol{p} = \boldsymbol{\Theta}_{\mathrm{f}}\boldsymbol{v}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{f}}. \qquad (3)$$

Very similar to Eq. (1), $\boldsymbol{M}_{\mathrm{f}}$ and $\boldsymbol{K}_{\mathrm{f}}$ are the static fluid mass and fluid stiffness matrices, respectively. $\boldsymbol{D}_{\mathrm{f}}$ is a kind of a damping matrix. With $\boldsymbol{p}$ as the column matrix of the sound pressure, $\boldsymbol{\Theta}_{\mathrm{f}}$ the boundary mass matrix of the fluid mesh to map the particle velocity of the structure to the nodal forces $\boldsymbol{f}_{\mathrm{f}}$ and the dynamic stiffness matrix of the fluid $\boldsymbol{A}_{\mathrm{f}}$, we arrive to a system of equations which looks the same as for the structure. Some additional remarks about $\boldsymbol{D}_{\mathrm{f}}$ though: For many practical examples, it results either from an admittance boundary condition[45] or from the radiation condition.[46] The admittance condition can be formulated as:

$$v_{\mathrm{f}}(\vec{\boldsymbol{x}}) - v_{\mathrm{s}}(\vec{\boldsymbol{x}}) = Y(\vec{\boldsymbol{x}})p(\vec{\boldsymbol{x}}) \qquad \text{with} \quad (\vec{\boldsymbol{x}}) \in \Gamma_{\mathrm{f}}; \qquad (4)$$

in which Γ_{f} represents the fluid boundary and the product of the boundary admittance, and the sound pressure equals the difference of the fluid particle velocity and that of the structure.[47,48] In the case of the full fluid-structure interaction, it is reasonable to assume the admittance to be zero. A damping matrix resulting from the radiation condition can be static when using a certain type of infinite elements, so-called conjugated infinite elements.[49,50] Other formulations fulfilling the Sommerfeld radiation condition, e.g., unconjugated infinite elements[51] as well as local and non-local absorbing boundary conditions,[52,53] usually result in an implicit frequency dependence of the damping matrix $\boldsymbol{D}_{\mathrm{f}}$. The same holds for perfectly matched layers.[54] When using a boundary element formulation as described in,[44,55] we either yield the system as:

$$\boldsymbol{A}_{\mathrm{f}}\boldsymbol{p} = \boldsymbol{H}\boldsymbol{p} = \boldsymbol{G}\boldsymbol{v}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{f}}; \qquad (5)$$

or, for a non-vanishing boundary admittance:

$$\boldsymbol{A}_{\mathrm{f}}\boldsymbol{p} = (\boldsymbol{H} - \boldsymbol{G}\boldsymbol{Y})\,\boldsymbol{p} = \boldsymbol{G}\boldsymbol{v}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{f}}; \qquad (6)$$

with $\boldsymbol{G}(\omega)$ and $\boldsymbol{H}(\omega)$ as the implicitly frequency dependent system matrices of the boundary element formulation , whereas $\boldsymbol{Y}$ is a sparse matrix containing the boundary admittance values. As such, this $\boldsymbol{Y}$ depends on the frequency if the boundary admittance depends on the frequency.

For fully coupled vibroacoustic analysis, Eqs. (1) and (3) must be coupled. The coupling conditions are given by the continuity of the particle velocities of structure and fluid at the interface and by balancing the momentum at the interface, of course both formulated locally. The coupling conditions can be formulated even for different meshes of structure and fluid.[55,56] Introducing the coupling matrices $\boldsymbol{C}_{\mathrm{sf}}$ and $\boldsymbol{C}_{\mathrm{fs}}$, the resulting system of equations takes the form as:

$$\begin{bmatrix} \boldsymbol{A}_{\mathrm{s}} & \boldsymbol{C}_{\mathrm{sf}} \\ \boldsymbol{C}_{\mathrm{fs}} & \boldsymbol{A}_{\mathrm{f}} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\mathrm{s}} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{\mathrm{s}} \\ \boldsymbol{0} \end{bmatrix}. \qquad (7)$$

There are several options for the solution of this system of equations. While direct solvers are usually quite time-consuming, there have been a number of approaches suggesting modal superposition by using pure structural modes and pure fluid modes,see for example.[57] Originally, these authors tried to do a modal analysis of the whole system[58], but this approach has hardly ever reached practical applications, although implemented in commercial software packages. The major problem has most likely been that the global system matrix in Eq. (7) is asymmetric and may lead to complex modes and eigenfrequencies even for the undamped system. An iterative solution of Eq. (7) has hardly ever been popular since the system is rather poorly conditioned.

In the context of frequency sweeps, reformulations based on the Schur complement have become quite popular in recent two decades. There are basically two options of formulations, one as the structural equation, i.e., the equation for the unknown structural displacements, as:

$$\left(\boldsymbol{A}_{\mathrm{s}} - \boldsymbol{C}_{\mathrm{sf}}\boldsymbol{A}_{\mathrm{f}}^{-1}\boldsymbol{C}_{\mathrm{fs}}\right)\boldsymbol{u}_{\mathrm{s}} = \boldsymbol{f}_{\mathrm{s}}; \tag{8}$$

and the other one as the fluid equation, i.e., equation for the unknown sound pressure, as:

$$\left(\boldsymbol{A}_{\mathrm{f}} - \boldsymbol{C}_{\mathrm{fs}}\boldsymbol{A}_{\mathrm{s}}^{-1}\boldsymbol{C}_{\mathrm{sf}}\right)\boldsymbol{p} = -\boldsymbol{C}_{\mathrm{fs}}\boldsymbol{A}_{\mathrm{s}}^{-1}\boldsymbol{f}_{\mathrm{s}}. \tag{9}$$

This paper is reviewing the variety of techniques which basically solve either Eqs. (7–9) or a combination of Eq. (1) with either (3) or (6) to determine certain objective functions which will be discussed in the subsequent section. While (7–9) consider a full structure–fluid interaction, Eqs. (1) with either (3) or (6) assume a one-way interaction only, i.e., we assume the structure to vibrate in vacuo (1) to produce the boundary conditions of the fluid which is analyzed using either (3) or (6). In some cases, even (1) is omitted, and the boundary conditions for the fluid are presumed.

It is worth mentioning that the literature of the recent three decades is full of so-called Trefftz methods, see the original work by Trefftz[59] and two review papers.[60,61] As these methods are usually using wave-based basis functions, which implicitly depend on frequency, they are often very efficient for a single-frequency solution but are challenging for frequency sweeps so far. The author is unaware of any special frequency range solutions for Trefftz methods.

2.2. Objective Functions

The discussion about the objective function will be kept short because most of it is only a repetition of former work of the author, see, for example, the reviews.[1,62]

2.2.1. Local quantities

Local quantities are quite popular objective functions for assessing the acoustic quality of structures. In vehicle interior noise problems, the sound pressure at one particular point – very often the driver's ear – is used for this ; for references see.[1,62] It is another option to assess the acoustic performance of a cavity based on energy density[63–66], which is still a local quantity. The total energy density has proven to be a more robust measure than just sound pressure for weakly damped cavities in which the sound fields exhibit clear nodal lines and surfaces. While local quantities may be efficiently computed and can be suited for cavity problems, they are hardly suited for exterior problems.

It is a great advantage of these local quantities that they can be easily and efficiently evaluated if the so-called acoustic transfer vectors or influence coefficients are known. The determination of these influence coefficients, which have been introduced several times[67–71], can be understood as an adjoint operator approach in which the influence coefficients account for the sensitivity of the local quantity with respect to the particle velocity on the surface of the fluid. As such, the sound pressure at a particular point $\vec{\boldsymbol{x}}$ in the fluid domain can be written as:

$$p(\vec{\boldsymbol{x}}) = \boldsymbol{b}^T(\vec{\boldsymbol{x}})\boldsymbol{v}_{\mathrm{s}}; \tag{10}$$

i.e., the scalar product of two column matrices where the influence coefficients in $\boldsymbol{b}$ represent the solution of the (adjoint) fluid boundary value problem, whereas the particle velocity of the structure $\boldsymbol{v}_{\mathrm{s}}$ accounts for the solution of the structural problem. Assuming that a structural model is hardly changing its fluid surface in an optimization process, the fluid problem needs to be solved only once; see more elaborate assumptions in the author's paper.[72]

A similar formulation to Eq. (10) can be achieved for the local energy density.[66] For this, we need the formulation for the components of the particle velocity first, i.e.:

$$v_j(\vec{\boldsymbol{x}}) = \boldsymbol{b}_j^T(\vec{\boldsymbol{x}})\boldsymbol{v}_{\mathrm{s}}; \tag{11}$$

with index j indicating the space direction as x, y or z. The total energy density (as the sum of potential and kinetic energy density) at point $\vec{\boldsymbol{x}}$ is found as:

$$e_t(\vec{\boldsymbol{x}}) = \frac{1}{2\rho c^2}|p(\vec{\boldsymbol{x}})|^2 + \frac{\rho}{4}|\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})|^2; \tag{12}$$

where ρ is the fluid density and c is the speed of sound in the fluid. Substituting for $p(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})$ and omitting $\vec{\boldsymbol{x}}$ dependencies, yields:

$$\begin{aligned} e_t = \frac{1}{2\rho c^2}\left(\boldsymbol{b}^T\boldsymbol{v}_{\mathrm{s}}\right)^*\left(\boldsymbol{b}^T\boldsymbol{v}_{\mathrm{s}}\right) \\ + \frac{\rho}{4}\left[\left(\boldsymbol{b}_x^T\boldsymbol{v}_{\mathrm{s}}\right)^*\left(\boldsymbol{b}_x^T\boldsymbol{v}_{\mathrm{s}}\right)\right. \\ \left. + \left(\boldsymbol{b}_y^T\boldsymbol{v}_{\mathrm{s}}\right)^*\left(\boldsymbol{b}_y^T\boldsymbol{v}_{\mathrm{s}}\right) + \left(\boldsymbol{b}_z^T\boldsymbol{v}_{\mathrm{s}}\right)^*\left(\boldsymbol{b}_z^T\boldsymbol{v}_{\mathrm{s}}\right)\right]; \end{aligned} \tag{13}$$

which can be written as a quadratic form as:

$$\begin{aligned} e_t &= \boldsymbol{v}_{\mathrm{s}}^H\boldsymbol{A}_c\boldsymbol{v}_{\mathrm{s}} \quad \text{with} \\ \boldsymbol{A}_c &= \frac{1}{2\rho c^2}\boldsymbol{b}^*\boldsymbol{b}^T + \frac{\rho}{4}\left(\boldsymbol{b}_x^*\boldsymbol{b}_x^T + \boldsymbol{b}_z^*\boldsymbol{b}_z^T + \boldsymbol{b}_z^*\boldsymbol{b}_z^T\right). \end{aligned} \tag{14}$$

In the above equations, superscripts * and ^{H} denote conjugate complex and Hermitian, i.e., transposed conjugate complex, matrix, respectively. It is interesting to see the total energy density at one single point can be written as the sum of four dyadic products of column matrices of influence coefficient, also known as acoustic transfer vectors, in a more generalized meaning. (Note that only $\boldsymbol{b}$ is usually called an acoustic transfer vector.) It is obvious that the matrix $\boldsymbol{A}_c$ is a rank-four matrix, i.e., it has only four non-zero eigenvalues.[66]

2.2.2. Global quantities

In their book, Koopmann and Fahnline[2] suggest global quantities only. They have been very clear about exterior problems in which they suggest assessing the acoustic quality only based on the radiated sound power, which is evaluated as:

$$P(\omega) = \frac{1}{2}\int_{\Gamma} p(\omega)v_f^*(\omega)\mathrm{d}\Gamma_R; \tag{15}$$

and which requires to evaluate the surface integral over an arbitrary closed surface Γ_R around the radiating structure. For convenience, this closed surface is often chosen to be the surface of the radiating structure such that $\Gamma_R = \Gamma$. Evaluation of this integral requires the knowledge of both boundary data, the fluid particle velocity and the sound pressure over the entire surface.

For interior problems, Koopmann and Fahnline[2] suggest energy quantities for assessment, too. They vote for potential, kinetic, and – as their favorite – total acoustic energy within the cavity. In acoustic optimization and uncertainty quantification, the author is aware of only one optimization paper having considered an objective function solving for the volume integral to determine the acoustic (potential) energy in a cavity.[73] However, a recent paper on non–negative surface contributions has dealt with this issue and has even provided ideas for efficient estimations of the acoustic energy in cavities.[74]

2.2.3. Integration over large frequency range

It is commonly accepted – at least in engineering – that single-frequency solutions are hardly useful at all. Such single-frequency solutions in optimization may just favor a very singular parameter set, which is hardly of any practical use. Nevertheless, there are papers in structural-acoustic optimization which focus on single-frequency or narrow-band optimization, see, for example.[75,76]

For an objective function F suggested in the majority of papers on optimization in structural acoustics, we adopt the formulation of[1]:

$$F = \int_{\omega_{\min}}^{\omega_{\max}} \Phi\left\{Q(\omega)\right\} d\omega \quad . \tag{16}$$

Therein, the quantity Q represents the above-discussed local or global quantity. Hence, in the case of a local quantity, it can be sound pressure, sound pressure level, total energy density, total energy density level, etc. In the case of a global quantity, Q may stand for acoustic energy, acoustic energy level, radiated sound power, radiated sound power level, etc. It is the major point here that the evaluated quantity needs to be summed up over a certain frequency range. This can require many evaluations, in particular if large peaks are to be considered, see also discussions on this in.[62,77–80]

The literature on structural-acoustic optimization and vibroacoustic uncertainty quantification shows many examples with very different numbers of discrete frequency evaluations to determine the integral in Eq. (16). According to the author's experience, the number of discrete frequency evaluations varies in the range of $10^2 \dots 10^3$ with outliers to both sides.

2.3. Repeated Computations For Parameter Variations

In both structural-acoustic or vibroacoustic optimization and uncertainty quantification, it is very common that the same problem is solved very often with just different parameter sets. Depending on the optimization algorithm or the sampling method in uncertainty quantification, these repetitions can be required between 10^1 and 10^6 times. In optimization, methods based on genetic algorithms and simulated annealing require the most repetitions, whereas gradient-based algorithms are usually quite economical, in particular if they apply an efficient evaluation of their gradient information. It is similar for uncertainty quantification if it is based on non-intrusive methods. Advanced stochastic approximation schemes, such as sparse polynomial chaos approximations, have turned out to be very economical, whereas a Monte Carlo approximation usually requires quite many function evaluations, which is due to their low convergence rate.

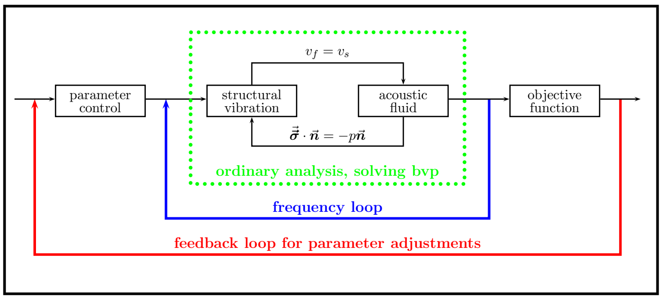

Figure 1. Flowchart of the problem analyzed in this paper: A common vibroacoustic boundary value problem (bvp) encompasses a single solution of a somehow coupled structural-acoustic (vibroacoustic) problem as shown in the green box. In practice, the solution is usually required for a frequency range, which is indicated as the frequency loop in blue. Optimization and uncertainty quantification require yet another feedback loop shown in red. This feedback loop is referred to as reanalysis if the same fluid problem is solved repeatedly just with changing boundary conditions.

As mentioned above, there are many applications in structural acoustics where the parameters of a structure are varied and adjusted in an optimization or uncertainty quantification process, but the fluid domain remains either completely unmodified or experiences just very little modifications that can be neglected. In such cases, the solution of the fluid problem can be understood as a problem with many load cases, i.e., many right-hand sides.

Within the problem description, we can conclude that the efficient single-frequency solution of an acoustic or a vibroacoustic boundary value problem is only one part of the solution when designing quiet structures. We further need to take into account that the solution is required for many frequencies, which accounts for an inner loop in optimization and uncertainty quantification, where the parameter adjustments are required, as shown in Fig. 1. Just two realistic scenario estimations for engineering design:

1. Assume a problem with 100 frequencies and a quick convergence of an optimization after 100 function evaluations: this will require 10^4 solutions of the vibroacoustic problem at a single frequency.
2. Assume a problem with 580 frequencies[81] and between 1000 and 5000 function evaluations: this will require a number of single frequency solutions of order 10^6.

The author has experienced even cases in which the optimizer had required a number of function evaluations of order 10^5 for a problem with approximately 10^2 frequencies. Given that a single frequency solution may take one minute, it is unrealistic to run the simulation $10^6 \dots 10^7$ times as the computation will take some years. Therefore, concepts have been developed to make this process much more efficient. In particular, efficient solutions for the frequency loop have experienced a strong revival over the recent decade. It is the aim of this paper to review concepts for efficient solutions of both the frequency loop and the feedback loop for control of the design variables.

A hot topic for repeated analysis combined with frequency sweeps is known as multi-dimensional (or multi-variate) parametric model order reduction. While first interesting contributions on this are already known,[82–86] as wider exploration on this with application to industrial examples and many parameters is still left and certainly a valuable task for the future.

3. METHODS FOR EFFICIENT FREQUENCY LOOPS AND REPEATED ANALYSIS

3.1. Modal Analysis and Mode Superposition

3.1.1. Mode superposition for interior problems

It is common practice for the simulation of structural vibrations and interior acoustic problems to do a modal analysis first and continuing with a frequency sweep based on modal superposition.[87] This concept can also be applied to reanalysis for vibroacoustic optimization and uncertainty quantification . According to the literature search of the author, Pal and Hagiwara[88] have been the first to apply this concept to vibroacoustic optimization. They are reporting to have calculated acoustic modes first and then used them repeatedly in each optimization step, even for fully coupled structural–acoustic analysis.

While the concept of modal superposition with storage of fluid modes for the entire optimization or uncertainty quantification process seems reasonable at first glance, the author is not clear about the availability of this option in commercial software packages. Furthermore, modal superposition using acoustic modes of the undamped case – usually assuming rigid boundary conditions at all surfaces – may become inefficient if certain boundaries include absorption. The reason for this inefficiency is that in the case of impedance/admittance boundary conditions, the modes of the undamped case are incorrect and do not correctly reconstruct the actual sound field. Even considering impedance/admittance boundary conditions for solving the eigenvalue problem, the convergence of the reconstructed solution by superimposing the complex modes may be slow.[45] Not to speak about the problem that boundary conditions are often frequency dependent, see for example.[89] But even for this, there are options to apply modal superposition and modal reduction if the admittance condition (or other damping terms) is approximated in a polynomial such as suggested in.[47,90]

Even the Schur complement Eqs. (8,9) may offer options for a modal reduction. Matrix $\boldsymbol{A}_{\mathrm{f}}^{-1}$ in Eq. (8) may be reconstructed using eigenvalues and eigenvectors of the matrix polynomial on the left-hand side of Eq. (3). Similarly, matrix $\boldsymbol{A}_{\mathrm{s}}^{-1}$ in Eq. (9) may be reconstructed using eigenvalues and eigenvectors of the matrix polynomial on the left-hand side of Eq. (1). The latter approach has been reported in.[56] Alternatively, a Cholesky decomposition has been used at this point.[91]

3.1.2. Mode superposition for exterior problems

Modal analysis and modal superposition of frequency-independent modes, as known from structural analysis and interior acoustic problems, are rather uncommon for exterior problems of acoustics. The main reason for this consists in the problem that the analytical equations of acoustics result in a continuous spectrum with certain (additional) discrete eigenvalues for unbounded problems.[92] However, the eigenvalues of a discretized model in an unbounded domain are entirely discrete.[45,46] The author has investigated the idea of modal reduction by investigating the number of modes necessary to determine the radiated sound power in 2d by using finite and conjugated Astley-Leis infinite elements.[46] It turned out that for certain problems, the number of modes for reconstructing the radiated sound power is rather small,[93] see also.[94] However, even though it is a priori unknown which modes contribute the most to the radiated sound power, it can be an option to store only the components of the modes on the structure–fluid interface. According to,[93] the radiated sound power is evaluated as:

$$P = \Re\left\{\frac{1}{2}\sum_{j=1}^{N}\frac{\boldsymbol{y}_{j\Gamma}^{T}\,\boldsymbol{f}_{\mathrm{f}}}{\alpha_j + \mathrm{i}k\beta_j}\boldsymbol{x}_{j\Gamma}^{T}\boldsymbol{f}_{\mathrm{f}}^{*}\right\} \qquad \text{with} \quad \boldsymbol{f}_{\mathrm{f}} = \boldsymbol{\Theta}\boldsymbol{v}_{\mathrm{s}}; \tag{17}$$

in which $\boldsymbol{x}_{j\Gamma}^{T}$ and $\boldsymbol{y}_{j\Gamma}^{T}$ are the surface components of the N right and left eigenvectors of the arising state space problem, respectively. The nodal forces $\boldsymbol{f}_{\mathrm{f}}$ result from the distribution of the particle velocity of the radiating structure whereas α_j and β_j represent the eigenpairs resulting as diagonal terms when the modal matrices are applied to diagonalize the state space matrices. It has been interesting to see that for the evaluation of the radiated sound power, only the eigenvector components on this interface are required to evaluate the radiated sound power. As such, it may be an option to store these small parts of the eigenvectors and keep them in memory for both frequency sweep and repeated analysis. In this sense, modal superposition of a complete set of frequency-independent modes for the radiation problem has been applied to structural-acoustic optimization in.[81]

There are papers (of the author) on modes of the fully coupled problem[95,96] in which the formulation of Eq. (9) has been used and a Taylor expansion allowing to formulate a polynomial eigenvalue problem. It has been developed into a matrix polynomial of sixth and eighth order for which the polynomial eigenvalue problem is linearized and solved. For the solution of the eigenvalue problem in comparison with the example in,[95] see also.[97] In,[96] a structure-preserving Arnoldi-based model order reduction technique is used to apply this method to a larger structure and in frequency sweep with modal superposition. It clearly shows how the modal superposition accelerates the frequency sweep in comparison to solve the problem at each single frequency.

Evaluation of frequency-independent modes diagonalizing the system matrices accounts for a hot topic of research.[98,99]

3.1.3. Using radiation modes

Further, in the context of modal analysis, it is necessary to discuss frequency-dependent acoustic radiation modes.[100–105] Acoustic radiation modes suffer from the feature that they depend on frequency. An interesting feature of acoustic radiation modes is that they can be grouped for certain radiator types. This grouping may allow a mapping to certain other radiator geometries provided these are not too different compared to the original geometries.[106,107] As they are usually converging quickly, meaning that only a few of them are required to reconstruct the radiated sound power, they can be considered an alternative to the acoustic transfer vectors. However, the number of required acoustic radiation modes depends very much on the specific problem to be solved. The author is aware of only a few papers utilizing acoustic radiation modes in an optimization process[108–111] and in uncertainty quantification.[112]

The thesis by Kessels[109] who required 180 radiation modes per frequency to reconstruct the radiated sound power of a magnetic resonance image scanner, whereas the paper by Zhang et al.[111] apply to above-mentioned mapped radiation modes for optimization. The advantage of these radiation modes is that – similar to the acoustic transfer vectors – they represent the solution of the fluid domain, which does not change their shape and the eigenvalues during the optimization process. The disadvantage is that they depend on frequency but decay quickly, especially at low frequencies. The frequency dependence requires to store a couple of these modes for each frequency.

3.1.4. Recent developments in solution of non-linear eigenvalue problems and potential application in modal reduction

While the above discussed normal and radiation modes have the advantage that they diagonalize the system matrix, it could be an option to use normal modes from a non-linear eigenvalue problem for modal reduction. The author is unaware of having seen such a method in the past but has successfully tested this for a simple cavity. In such a case, the modes will reduce the system size but do not diagonalize the system matrices.

The numerical solution of the non-linear eigenvalue problem for exterior acoustic problems has been a hot research topic in recent years, even though early approaches reach back more than three decades.[113] The major problem with this is the implicit frequency dependence of the system matrices for the boundary element method and most of the approaches of finite element analysis for unbounded domains. It has been mentioned above that conjugated Astley-Leis infinite elements allow for the formulation and solution of a quadratic eigenvalue problem.[46,105] A series expansion of the implicit frequency dependent part on the left-hand side of Eq. (9) results in a higher order polynomial eigenvalue problem which can be linearized and solved in a traditional way, see Peters et al.[95,96] In both methods, the eigenvectors diagonalize the state-space system matrices. This is not necessarily the case when contour integral methods are applied. Examples for determination of modes by solution of the non-linear eigenvalue problem are found in papers[114–117] for pure acoustic problems and for fully coupled systems in.[91,97,118–122] Chebyshev approximation has been applied to the pure exterior acoustic problem (with a structural vibration as excitation) in.[123] In all of these papers, the fluid is discretized with boundary elements, whereas in the coupled analysis, the structure is meshed by finite elements.

The literature knows solutions of the acoustic eigenvalue problem using finite element approaches for the fluid as well. Hohage and Nannen et al., see for example,[124,125] have developed a different type of infinite element approach to determine eigenvectors and eigenvalues of unbounded acoustic domains. Araujo-Cabarcas et al.[126] have presented an approach for the Dirichlet–to–Neumann map as absorbing conditions. Kim et al.[127,128] have shown an approach for using perfectly matched layers to achieve full absorption at the boundary of the physical domain.

3.2. Acoustic Transfer Vectors

As mentioned in the context of Eq. (10), acoustic transfer vectors may account for a very efficient tool to decouple the acoustic solution from a repeated structural analysis. There are many practical cases in which plate and shell thicknesses, spring stiffnesses, non-structural masses, thin beam parameters, or even small shell geometry modifications show hardly any effect on the fluid domain. Basically, acoustic transfer vectors are computed for each frequency point. This results in one complex value per surface node and per frequency.

Assuming N surface nodes and M frequencies, we end up with storage requirements of $N \times M$ complex values. With the assumption of 10^4 surface nodes and 100 frequencies, the storage requirements of approximately 16 Megabytes are rather moderate. Even the (extreme) case of $N = 10^5$ and $M = 10^3$ resulting in 1.6 Gigabyte storage requirements could be handled easily today. However, there are contributions in literature where authors aimed at storage reduction.[129] That paper applied acoustic transfer vectors to an exterior problem and achieved data reduction using a proper orthogonal decomposition. It is worth mentioning that modal acoustic transfer vectors were proposed for the fully coupled problem of a vehicle body.[130] Therein, the authors claimed to be more efficient than using ordinary acoustic transfer vectors, which is quite likely. However, the authors did not provide a convergence study based on accuracy.

The evaluation and storage of acoustic transfer vectors prior to a vibroacoustic optimization process of vehicle panels has originally been proposed and repeatedly applied by the author,[69,72,131–135] see also.[1,62] In these works, the author called them influence coefficients, whereas the term acoustic transfer vectors in this context has most likely been coined with the patent application.[70]

It has been shown above that these transfer vectors can also be used to determine energy density quantities. For the local total energy density in a cavity, four of this generalized acoustic transfer vectors are required to be stored for each frequency. This means the storage costs compared to storing them just for the sound pressure – or the potential energy density, which is basically the sound pressure squared – is four times as high, but, essentially, it remains in the same order of magnitude and should be easily affordable for an optimization process or an uncertainty quantification.

We can even go further and check the use of acoustic transfer vectors for evaluation of the total acoustic energy in the cavity as suggested by Koopmann and Fahnline in their book.[2] In a recent paper,[74] it has been shown that the total acoustic energy within two different cavities can be well approximated by evaluating the total energy density at 19 and 36 points. It is worth mentioning that this test cannot be considered a full convergence study but is a first indication that the total acoustic energy can be well approximated by the total energy density at a very limited number of field points. Hence, it seems that 10^2 transfer vectors per frequency may do well. Again, assuming 10^2 frequencies and a surface mesh of 10^4 nodes, a storage of 1.6 Gigabytes is required for this data, which represents the full solution of the acoustic field.

It would be interesting whether this concept can be applied to exterior problems in acoustics too. As explained above, the assessment for exterior problems is usually based on the radiated sound power, which is determined as the surface integral over the product of sound pressure and the particle velocity in the normal direction to the chosen surface. Acoustic transfer vectors , as introduced in Eqs. (10) and (11), would require an enveloping surface around the radiating body. In most cases, the author would expect a similarly detailed mesh for the envelope as for the radiator itself. Thus, assuming 10^2 frequencies and two meshes of 10^4 nodes each will result in 10^{10} complex numbers to be stored, i.e., 160 Gigabytes. While even 160 Gigabyte storage may not be prohibitive anymore, the 10^4 nodes on the enveloping surface will require as many acoustic transfer vectors for each frequency, i.e., the (adjoint) acoustic problem needs to be solved 10^6 times prior to the optimization or uncertainty analysis. Although these 10^4 solutions are using the same system matrix and would result in 10^4 right-hand sides, this still sounds unreasonable and is discouraged here. Other concepts will most likely make more sense, although the author is unaware of any convergence analysis for this problem.

3.3. Frequency Approximation Techniques

There seems to be a strong desire for efficient frequency sweeps for numerical methods not easily allowing to describe the frequency dependence of matrices $\boldsymbol{A}_\mathrm{s}$, $\boldsymbol{A}_\mathrm{f}$ and thus the system matrix of the coupled system in Eq. (7) just in terms of a quadratic or even a linear matrix polynomial in the frequency. Well-known examples, as already mentioned above, are (most of) the boundary element methods but also many popular finite element methods for unbounded domains. With respect to BEM, this problem has been discussed in two recent review papers, cf.[136,137] Another nice review paper on frequency sweeps in a finite element context has been published by Hetmaniuk et al.[138] It mainly refers to the model order reduction techniques in combination with Padé approximation and proposes interesting model order reduction techniques. Many of the techniques reviewed in this section are understood as model order reduction techniques in which the solution is determined at certain key frequencies (snapshots) and approximated in between. Some of the methods require the solution of the original system of equations at these key frequencies and reduce the system first and then determine the solution for the reduced order model.

There are a number of different categories of approaches to frequency approximation. The author is trying to set up a couple of specific categories even though the mapping does not always seem to be unique. Overall, it is obvious that there are numerous concepts. The author is unaware that any of them has already been applied in a commercial tool. Reasons for that might be that some of them require too much manual control of parameters. There are a few tailored solutions though.[139]

3.3.1. Matrix polynomial formulation with approximation

In finite element analysis, it is very common to assume the matrices, i.e., stiffness, damping, and mass matrices, respectively, in the formulations (1) and (3) to be independent of frequency. While this is convenient to analyze, it often contradicts reality. The author has come across this for purely structural problems when determining stiffness and damping parameters at different temperatures.[140] Frequency dependence of the boundary admittance is generally accepted, see for example.[141–143] A nice example of the problem with frequency-dependent damping and frequency approximation combined with model order reduction has been presented by Xie et al.[144]

Things are different in the boundary element method. As Green's function contains the frequency in the exponent of the exponential function, it cannot be easily used as an explicit factor for the system matrices as known from the finite element method. There are workarounds, however. In the 1980s and 1990s, a method called the dual reciprocity[145] and another method named the particular integral[146] have proposed techniques allowing for the formulating of frequency-independent mass matrices. (As a side remark, the author believes that both methods are identical, at least with respect to the formulation of mass matrices for elastic and acoustic problems.) These first papers have been focusing on elastic[145,146] and acoustic[147] eigenvalue analysis.

The very interesting paper by Chen et al.[90] has suggested a boundary element method utilizing particular integrals and admittance boundary conditions. As the authors have been aware of the frequency dependence of the boundary admittance, they have suggested the use of a piecewise linear approximation of the frequency dependence of the boundary admittance. Such a behavior would result in a complex mass matrix to host the linear frequency-dependent contribution. There has not been presented an example in[90] for such an admittance behavior. Actually, the author has investigated such an example in his paper[47] because the admittance determined by measurements of the reverberation time in a vehicle cabin have strongly suggested this behavior being linearly dependent on the frequency, see also.[69] The hardly visible conference papers by the author have investigated the accuracy of the particular integral method[148] and mode superposition techniques.[149] While these techniques seem to be an interesting alternative to standard boundary element formulations, the author has encountered serious problems when applying the particular integral method to high frequencies. However, the most serious problem with the particular integral method and the dual reciprocity method is that they are restricted to cavity problems. The author is unaware of any serious application to acoustic radiation or scattering into open domains.

3.3.2. Interpolation of (mostly) boundary element system matrices

Approaches of this category have been the first to be published for efficient frequency sweeps in addition to modal superposition. All of these approaches somehow deal with the $\mathrm{e}^{\pm \mathrm{i}kr}$ term in the Green's function. (In 2d, the oscillating term in the Green's function stems from the Hankel function.) In,[150] the authors introduce a conjugate complex test function for the matrix entries and interpolate linearly between two preselected frequencies, for which they have determined the matrices before. The Green's function has been developed in a Taylor series at the Chebyshev points in a certain frequency interval.[151] Another paper[152] has proposed a separation between the frequency-dependent and the frequency-independent, i.e., singular, part of the Green's function. The formulation looks very much like a regularization formulation which it actually is since the frequency-dependent part does not require to deal with singularities, and the singular part is frequency-independent.

The concept of a Taylor series for the Green's function or its oscillating term has been proposed several times over the recent three decades.[153–159] Usually, these approaches arrive at matrix polynomials of different order to approximate the actual boundary element system matrices. At a side note, the suggestion of[156] to apply a one-point integration rule for all non-singular integrals is clearly discouraged, cf.[160]

It is a common feature of all of these approaches that they may be efficient for the matrix setup but they require to solve the system of equations at each discrete frequency. Therefore, this approach is quite unpopular for large-scale systems. The memory required to store the polynomial coefficient matrices is the bottleneck of using a Taylor expansion or any other interpolation technique for boundary element formulations. That is why it is crucial to combine those techniques with model order reduction techniques. The basis for these techniques however needs to be constructed before the interpolation, such that upon constructing each of these coefficient matrices the reduction may occur. This is in particular relevant for the boundary elements and not so much for finite elements as the latter result in sparse matrices, see, for example, the discussion in.[86]

It is yet another problem of the matrix interpolation schemes that they usually require access to the code, as most of these techniques are intrusive and cannot be easily adopted into existing commercial or open code. Even software companies would need to change their existing code and maybe restructure it to allow for the application of these methods.

The Taylor series approach for the non-linear frequency dependence of the damping parameter arising from viscoelastic material behavior has been combined with a second-order Arnoldi reduction scheme by Xie et al.[144] The paper by Xie et al.[161] proposes a model order reduction technique for large-scale systems. It includes a Taylor series approach in the vicinity of the (high) frequency under consideration. The technique has been extended to fully coupled vibroacoustic problems in[162] and the two-dimensional case in.[163]

The recent work by Chen et al.[164] is utilizing a Taylor expansion for the oscillating part of the system and then applying a second-order Arnoldi scheme to reduce the size of the system matrices, which results as a second-order matrix polynomial. This approach has similarities with.[96]

The Taylor series expansion presented by Yoon[165] for a finite element model, including options for porous material, will be discussed later in Section 3.3.7. It uses preselected frequency intervals as well.

A different approach which is also based on a Taylor series of the matrix entries at a certain discrete key frequency has been presented by Raveendra in.[166] Therein, however, the author transfers the perturbation from the key frequency to the right-hand side so that a certain frequency sweep in the vicinity of the key frequency can be achieved by solving the same system of equations with a number of different right-hand sides. A similar but much more sophisticated approach based on finite elements only has been presented by Hetmaniuk et al.[167] Again, matrix entries are approximated by using a Taylor series of higher order this time. The frequency interpolation between the key frequencies is carried out in a piecewise interpolation. The work presented there is shown for a finite element formulation with frequency-dependent parts such as perfectly matched layers, absorbing boundary conditions, and frequency-dependent admittance boundary conditions.

3.3.3. Padé approximation schemes

The oldest paper on frequency approximation of acoustic problems based on Padé approximation deals with a boundary element formulation by Coyette et al.[168] Actually, it is yet an approximation technique for the boundary element matrices as discussed in the previous subsection. Interestingly, the problem has been formulated on behalf of an implicit differentiation scheme of the system of equations. With that, derivatives of up to the 30th order with respect to frequency have been required and determined by using a symbolic differentiation of the computer code, apparently very similar to what automatic differentiation is doing nowadays. The usage of Padé approximation is usually aiming at approximation of transfer functions on behalf of a rational function, allowing for excellent approximations of resonances.

Overall, the author is aware of only a few more papers on boundary elements and Padé approximation though. A second paper[169] uses a similar approach for the thin body boundary element method, which is often referred as the indirect boundary element method. Note that there are several indirect boundary element approaches, see, for example, the discussion in.[136] The paper[169] adds a model order reduction technique to avoid solving the large system of equations though. A rather recent paper[170] revisits Padé approximation but shows applications for the Burton and Miller method[171] and in optimization. The authors of that paper determine the derivatives with respect to frequency directly at the kernel, i.e., the Green's function. Similarly to the methods discussed in the previous subsection, they solve the whole system of equations for each frequency step.

On the finite element side, the paper by Djellouli et al.[172] has modeled an exterior acoustic problem with finite elements and an absorbing boundary condition. These authors have compared a matrix Taylor approximation, Padé approximation, and a much older method proposed by Wynn.[173] Interestingly, the latter two methods clearly outperform the former, while the older method is performing quite well even for wide frequency ranges. It is just that the newer Padé approximation seems to be more efficient for large-scale models. The follow-up paper[174] develops this approach further into a frequency sweep based on several key frequencies.

In the late 1990s, Malhotra and Pinsky presented an efficient technique in which they combine model order reduction and frequency interpolation.[175] This technique, applied to a finite element model with absorbing boundary conditions, is referred to as Padé–via–Lanczos approximation and has been applied and developed further over the recent two decades, see for example.[176–181] This technique has proven to be very efficient if the solution in a small region has been looked for. Baumgart et al.[182] have applied this technique to the efficient evaluation of the radiated sound power over frequency ranges. According to,[167] the matrix Padé–via–Lanczos scheme[178–180] is only valid for a frequency-independent right-hand side.

A similar technique is using an Arnoldi method for model reduction and is thus called Padé–via–Arnoldi approximation, see in particular the work by Puri and Morrey.[183–185] It is applied to the structural-acoustic problem of a vehicle body with its cavity. Both are discretized using finite elements, and the method is basically applied to Eq. (7). While[183] presents the method,[184] applies it in an optimization. The third paper[185] compares a one-sided Arnoldi reduction scheme with a two-sided and concludes the latter is the most favorable one. A technique looking similar to a Padé–via–Arnoldi scheme has been investigated with respect to an error estimation by Aumann and Müller.[186] The work of Xie et al.[187,188] may also be considered as Padé–via–Arnoldi approximation, although later discussed in the context of proper orthogonal decomposition.

The paper[189] applies Padé approximation to a finite element model surrounded by a perfectly matched layer. The authors apply a multipoint approximation over a frequency range for which they use Chebyshev points as the key frequencies.

A poroelastic problem coupled with acoustics is considered by Rumpler et al.[190] Therein, the system of equations of the elastic part is reduced using the modes which are solution of the purely elastic eigenvalue problem. Thereafter, a Padé approximation is applied to carry out the frequency sweep efficiently. The same authors have published another paper on Padé approximation, this time for a bivariate case, i.e., for a parameter sweep in two directions.[82] Besides the frequency, the flow resistivity is understood as a second free parameter that is scanned over a certain interval. For Padé approximation, it can be useful to know eigenfrequencies in advance as they can account for the key frequencies of the series development.[191]

The same lead author, Rumpler together with Aumann[83] took up a technique from electromagnetics. It is called the Well-Conditioned Asymptotic Waveform Evaluation and can be categorized as a Padé approximation scheme too. The multi-parameter version, briefly called MWCAWE,[83] has been developed to allow for frequency sweeps together with sweeps over other parameters. Similar to the application of the greedy reduced basis algorithms[84], which will be discussed below, applications have been limited to two parameters so far. It is assumed that this method can be extended to efficient sweeps over more than two parameters in the future.

3.3.4. Proper orthogonal decomposition

Proper orthogonal decomposition is known as a powerful tool for model order reduction to approximate time-dependent problems.[192] Substituting the frequency for time allows for application of this method for frequency sweeps. Usually, it requires a number of so-called snapshots and allows thereafter to reconstruct the solution between these snapshots by interpolation. This concept has been applied to boundary element models by snapshots at certain frequencies in.[193] While these authors report a successful application of proper orthogonal decomposition applied to a number snapshots similar to,[192] the author of this manuscript (together with co-authors) has investigated such techniques to substitute for the techniques presented in[95,96] and found it unconvincing. In particular, the accuracy and reliability to generalize this technique had turned out to be insufficient at that time. This may have been due to sharp resonance peaks, which may be challenging to reconstruct using proper orthogonal decomposition.

An apparently more sophisticated technique based on proper orthogonal decomposition has been proposed by Negri et al.[85] They denoted their method as a matrix discrete empirical interpolation method and referred to it as MDEIM. It is a kind of parametric model order reduction scheme, which, in a first application, applied to the frequency as parameter and in a second application, to the frequency together with four geometry parameters in a shape optimization of a two-dimensional horn.

A substantial piece of work on frequency sweeps based on approaches utilizing proper orthogonal decomposition for boundary element techniques is found in papers by Panagiotopoulos et al.[86,194,195] and by Xie et al.[162,163,187,188] The papers[86,187] start from very similar ideas. They are using a combination of proper orthogonal decomposition and Krylov subspaces. They employ user-defined fixed sampling in the frequency domain, but instead of sampling the solution vectors of the systems, they sample the Krylov subspaces of a user-defined dimension that would lead to their iterative solution. Therefore, it is understood as recycling the Krylov vectors of a number of sampled systems to find the solution for the non-sampled systems. Thus, this method is referred to as a combination of proper orthogonal decomposition and Krylov subspace recycling. The advantage of this approach in comparison to traditional proper orthogonal decomposition is the definition of an error estimator and the decreased sampling requirements. Both papers differ with respect to the actual Krylov subspace, which in[86] is based on the actual system of equations and in[187] on the inverted system. Further differences are found for the treatment of irregular frequencies which is, however, of secondary importance at this point. A follow-up paper of[187] deals with coupled vibroacoustic problems[188], whereas the follow-up paper of[86] automates the above procedure with an automatic Krylov recycling algorithm, which makes the sampling and the dimension of the subspaces to construct the reduction basis adaptive.[194] It is somehow equivalent to what will be discussed in the context of the greedy algorithms[84,196], not requiring a pre-selection of the sampling points where the full system needs to be solved. Again, Krylov subspaces are sampled. The advantage of the automated Krylov recycling algorithm is the significantly reduced sampling that is required. So, fewer full-order boundary element systems are required to be assembled and solved to construct the reduced basis. A second follow-up paper[195] extends the two earlier ones[86,194] for the multi-parameter case. It provides examples for source position, shape, and impedance parametrization. It is worth mentioning that the method developed in[86] has been applied in the code OpenBEM[197] by Paltorp et al.[198]

3.3.5. Greedy algorithms

Other than most proper orthogonal decomposition techniques and similar to the work of Panagiotopoulos,[194,195] a greedy algorithm based on Chebyshev approximation as presented in[84,196] does not require predefined frequency positions for its snapshots. After an initially defined snapshot, all other frequency supports are automatically chosen until a certain threshold of the residual is achieved. The examples presented in[196] for just a frequency sweep of simple acoustic problems and in[84] for a fully coupled vibroacoustic problem with two parameter sweeps, i.e., for the frequency and the Young's modulus, still remain simple academic examples. However, the method has the potential to become one of the most popular techniques, in particular for frequency sweeps, as it can be automatically used in (almost) arbitrary cases. The method's main drawback becomes apparent when the setup of the system matrices is quite time-consuming, as it happens for large-scale boundary element techniques. The cases in[84,196] have shown relatively moderate model sizes for the boundary element part of the vibroacoustic examples. For them, the system matrices are ease to be either store or reconstructed in the iterations.

3.3.6. Low-rank scheme

The low-rank approximation proposed in[199] presents an iteration scheme for frequency sweeps such that simultaneously solutions at many frequency points are evaluated. The low-rank scheme benefits from the fact that the information content in a frequency sweep is usually lower than the many solution vectors at all frequencies. The low-rank scheme is based on a polynomial frequency approximation of the boundary element equations and on low-rank factorizations of intermediate matrices. Combining both concepts enables efficient evaluations of matrix vector products and their incorporation into iterative solvers such as the solvers tested therein, i.e., BiCGstab[200] and GMRes,[201] see also.[202] While the results of the paper[199] can be considered as a proof of concept, it is questionable whether this method will prove to be as efficient as other techniques presented in the previous and subsequent sections and subsections.

3.3.7. Other frequency approximation and interpolation methods

There are some papers on this topic which seem to be a bit more challenging to categorize, at least to categorize them within the categories used for this work.

The earlier mentioned paper by Yoon[165] is proposing a Taylor series in terms of the frequency around a certain center frequency. The approximate solution is then yielded by superimposing Ritz vectors weighted by the powers of the frequency difference. The motivation for development of this method has arisen from frequency-dependent models for porous material and the Sommerfeld radiation condition.

The interesting work by Liang et al.[91] uses finite elements for the structure and boundary elements (with a Nyström) discretization (utilizing a fast multipole technique) for the fluid, thus making it a coupled system as given in Eq. (7). They further apply the Schur complement as shown in Eq. (9) and apply the solution of the eigenvalue problem and a frequency sweep to that formulation. The authors suggest Cauchy interpolation to search for complex eigenvalues and Chebyshev interpolation for the frequency sweep with purely real frequency values. The paper by Liang et al.[91] is further interesting since it provides some practically relevant suggestions about sampling points, solvers, and, in particular, relevant from the author's point of view, a useful scaling of the system matrix in Eq. (7).

In,[203] the authors propose a matrix-free approach for model order reduction. Basically, this approach is approximating transfer functions as it is known from the Padé–via–Lanczos and the Padé–via–Arnoldi techniques which had been used in the context of finite element techniques. Here, the authors apply a technique similar to fast boundary element techniques, such as the approach using hierarchical matrices and the fast multipole method.

There are a couple of other techniques for frequency sweeps. Finite element solutions with perfectly matched layers may be accelerated if the frequency-dependent absorption function is frozen to a specific, piecewise constant value.[204] Interestingly, the authors describe that they have tried several polynomial frequency approximations , but none of them seems to perform better than the frozen frequency value in the absorption function. Recent work on new absorbing functions has the community expected more approaches using perfectly matched layers for efficient frequency sweeps.[205,206]

3.3.8. Comparison of Model Order Reduction Techniques Combined with Frequency Sweeps

In particular, the approaches based on the boundary element method for a discrete frequency step show some similarities. These solution methods use the information content of the problem. In several cases, this is approximated by the basis of a recycled Krylov subspace. This is, apparently, similar to how the low-rank scheme works, similar to the greedy algorithm, which is also based on a low-rank approximation, similar to the basis of a proper orthogonal decomposition, and also similar to a reduced modal basis. Essentially, it seems as if all the methods are looking for a reduced basis to reconstruct the spatial and the frequency variation of vibroacoustic – mostly – coupled displacement/pressure fields. It is worth mentioning here that the fast boundary element methods, e.g., the fast multipole method, see for example[207] and the approaches making use of hierarchical matrices,[208] also rely on the information content of the boundary element system matrices and allow for sparse representations.

In their paper,[209] Aumann and Werner have presented a very valuable (and unfortunately often neglected) comparison between different model order reduction schemes combined with frequency sweeps applied to different problems of acoustics, vibrations, and vibroacoustics. The authors have tried to achieve a similarly efficient implementation of these methods to allow for a fair comparison. In addition, they provide a link to their codes and data. However, discussing the results in detail would go beyond the scope of this paper. Furthermore, it has been impossible for these authors to consider a large part of the techniques discussed here. While there are already quite many methods available, this is an emerging field with many new publications every year.

3.4. Krylov Subspace Recycling with Deflation

Krylov subspace recycling techniques have already been discussed in the context of frequency sweeps using proper orthogonal decomposition. Apparently, the work by Panagiotopoulos et al.[86,194,195] has mainly been inspired by a paper by Keuchel et al.[210] It was the idea of these authors that the iterative solver could benefit from a previous solution in a frequency sweep if the frequency point of the previous problem is not too far away from the current one. The results have shown an improvement, which has been moderate, however. It has been much better than what the author has managed some two decades ago when he expected an improved convergence of the initial solution guess for the iterative solver in the context of his paper[202], but this was not the case. However, it seems to be much more sensible to use Krylov subspace recycling instead of relying on a very good initial guess, at least when using either the GMRes algorithm[201] or the generalized conjugate residual with inner orthogonalization and/or Deflated Restart, see[210] and references therein.

It has been the idea of Panagiotopoulos et al.[211,212] that a deflated Krylov subspace can be used efficiently for solutions over a large frequency range. This deflated subspace would be yielded in an offline computation prior to a frequency sweep. It is considering the solution of the entire problem for a couple of master frequencies. Actually, the idea is rather similar to the methods discussed in the previous two subsections. They both focus on the information content and argue that the information content is not that much different between different frequency solutions. Here, the iterative solution for a frequency increment benefits substantially from previous evaluations. As the previous alias offline computations are distributed over the entire frequency range, it is expected (and shown in the papers) that very fast solutions can be achieved even in the gaps in between. These gap solutions are then called online solutions, which refer to the solutions to provide a (virtually) continuous frequency response curve for the entire frequency range. An interesting extension of[211] has been shown in[212] with its combination with conventional preconditioners. The authors show that a conventional preconditioner constructed for the solution at a single frequency can be quite efficient over a certain frequency range.

Other than in the techniques discussed in the previous section, Krylov Subspace Recycling with Deflation is not a frequency approximation technique but solves the (discretized) problem, i.e., the system of equations, with an arbitrary accuracy based on the demanded residual of the iterative solver. It is rather likely that the techniques can also be efficiently applied to the problem of reanalysis. It is just that the author is unaware of any work published on this subject.

3.5. Low- and High-Fidelity Models Utilizing Gauss Processes

A completely different concept for frequency sweeps (and actually for optimization purposes, too) consists in the use of multi-fidelity models. A multi-fidelity model allows the combination of multiple models with differing fidelity levels. Typically, it consists of a low-fidelity and a high-fidelity model. Further fidelity levels may be used, but they are not that common.

Low-fidelity models are attributed to low computational costs and decreased accuracy, whereas high-fidelity models achieve predictions with higher accuracy for the burden of high expenses. Analytical or numerical models at small scale can be regarded as low-fidelity models. Highly resolved numerical models or cumbersome physical experiments can be considered to be high-fidelity models. As such, the advantages of both fidelity levels, namely, fast evaluations and high accuracy, are merged in a multi-fidelity model. The correspondence between the different fidelity models is arranged by using a technique based on the idea that the frequency dependence accounts for a Gaussian process. It allows to substitute a complex model by an efficient surrogate model.[213,214]

Multi-fidelity models have been used to efficiently solve partial differential equations.[215,216] Beyond Gaussian processes, artificial neural networks have been implemented in multi-fidelity schemes for parameter-dependent outputs[217] and, more than two decades ago, in a multi-level scheme for vibroacoustic optimization of a vehicle hat-shelf.[133]

According to the author's knowledge, the first application of Gaussian processes to frequency sweeps in computational acoustics has been published by Gurbuz et al.[214] Therein, a multi-fidelity model has been developed based on boundary element simulations of two different meshes. A coarse and a fine boundary element mesh accounted for the low-fidelity and high-fidelity models, respectively. The Gaussian process model has been trained using frequency responses of the two fidelity models. Although the frequency responses between low-fidelity and high-fidelity models have differed by more than five decibels, only a few computations of the high-fidelity model (in combination with more computations of the low-fidelity model) have been required to precisely predict the frequency response curve of the high-fidelity model in much detail.

The author believes that this approach will lead to a new family of efficient methods for frequency sweeps and maybe even for efficient repeated analysis.

3.6. Efficient Reanalysis

Efficient reanalysis methods account for an important ingredient to uncertainty quantification and optimization in situations where parameter modifications change the vibration of a structure but hardly change the fluid's domain and properties. Only the boundary (or the interface) conditions of the fluid are modified, and thus, it seems unnecessary to solve the same problem on and on.

Previous subsections discussed traditional methods for efficient reanalysis. There are methods to efficiently reconstruct the (frequency-dependent) inverted system matrix based on eigenvectors and eigenvalues, including the well-known radiation modes. Further, local quantities can easily make use of previously evaluated and stored acoustic transfer vectors. However, all these traditional methods come with some shortcomings, which is the reason why additional methods are required in this context.

The literature rarely addresses efficient reanalysis. The author is aware of a few papers addressing the problem of analysis with many right-hand sides in a single computational step. The approach of Meerbergen and Bai[218] is using recycled Ritz vectors and the Lanczos method for efficiently solving the frequency sweep with many right-hand sides. Similarly, the work[219] has presented a combination of a frequency sweep comparable to the one suggested in[138] and many different excitation cases at once.

As mentioned above, Krylov subspace recycling with deflation could account for another technique being suitable for efficient reanalysis techniques as they enable the user to make use of a predetermined Krylov subspace, allowing for a very fast solution of the arising systems of equations.[211,212] Application to parameter dependencies beyond frequency is still to be found in the literature.

The recent paper[122] addresses the problem of many right-hand sides for boundary element matrices with Toeplitz structure. A different approach has been presented in[220], where the authors try to store data of the inverse system matrix for a fast multipole boundary element algorithm. They could show that such a technique is well suited as a preconditioner for solving the frequency-dependent system efficiently but might not be that well suited for an efficient reanalysis. A parametric model order reduction scheme, i.e., a greedy algorithm for two parameters, combined with many right-hand sides, has been presented by Jelich et al.[84]

Overall, the techniques of efficient reanalysis seem to focus on problems with many right-hand sides. Such problems occur, especially in uncertainty quantification. In optimization, efficient techniques for many right-hand sides may be useful if a new parameter set requires many function evaluations at once. Among other situations, such a problem may be encountered if gradient information is purely determined based on global finite difference techniques[1,72] or for methods using quite a number of function evaluations within a single optimization step. One group of such optimization methods is known as genetic algorithms, see, for example, discussions in.[1] Gradient-based optimization techniques with analytic, semi-analytic, and/or adjoint operator-based sensitivity analysis will hardly benefit from methods efficiently solving for many right-hand sides since most function evaluations need to be performed sequentially.

The papers on efficient reanalysis of fully coupled vibroacoustic systems mentioned in the Introduction of this paper[42,43] are essentially based on a modal reduction scheme. This makes sense at first glance. However, the author is not that convinced of the suitability of modal reduction for problems with a high modal density after testing this technique for a simple one-dimensional case of the Helmholtz equation.[45] This holds in particular in the case of[42,43], where extremely many (> 1000) modes are present in the frequency range under consideration.

4. CONCLUSION

Having briefly reviewed some concepts for efficient frequency sweeps and reanalysis for optimization and uncertainty quantification, the author has tried to convince the reader that this field accounts for a field of active research. Most papers in this field deal with either purely acoustic or vibroacoustic content. Even if we assume that the future will allow for efficient solutions to these problems, the author is sure that there will remain many challenging problems related to them. One could be the transfer of these techniques into problems involving flow as discussed here.[221–223] Closely related are efficient time-domain solutions, including radiation or frequency-dependent boundary conditions as discussed here[224] or non-local boundary conditions.[225] Hence, the author expects many more papers on this subject to be published in the future.

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