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Sustainability refers to the coexistence of the biosphere and human civilization, where the use of resources, the way of investments, the development of technologies, and institutional changes are all in harmony, and “meet the needs of the present without compromising the ability of future generations to meet their own needs” (United Nations General Assembly, 1987). Acoustics deals with sound waves in gases, liquids, and solids and its application is present in almost all aspects of modern society, from both a physical and human perspective. It is, therefore, of great significance to consider how research of acoustics and its applications could become an integrated part of sustainability. In this editorial, those are explored from the built environment perspective, in terms of acoustic comfort, urban sound planning, building-envelop designs, sustainability of acoustic materials and products, use of natural means for noise reduction, and the acoustic impacts of sustainable measures.

Acoustic comfort is an important part of the overall physical comfort, a key consideration in a sustainable built environment. If there are acoustic defects in an indoor or outdoor space, remedial treatments are often costly and inefficient. At an urban scale, along with the EU Environmental Noise Directive 2002, mandating each city to identify and protect its quiet areas, there has been a recognized lack of research in urban soundscapes, integrating the conventional physical approach with the consideration of perceived sound environment, as the conventional approach of merely reducing the noise level often does not deliver an improved quality of life (e.g. during the COVID-19 lockdown the sound level decreased by 10–15 dB in London whereas noise complaints doubled). In addition to controlling noise, introducing pleasant sounds is very important. At a building scale, acoustic quality and comfort is not only important in acoustic buildings such as auditoria and recording studios, but also in everyday ‘non-acoustic’ buildings such as hospitals, schools, sport centers, churches, offices, libraries, and shopping centers.

In planning a city, sound propagation needs to be taken into account. There is an increasing urban population and correspondingly higher building density, and the construction industry accounts for 1/3 of our total energy consumption. Although higher building density will normally bring a higher noise level, a comparison between a typical high density city and its application is present in almost all aspects of modern society, from both a physical and human perspective. It is, therefore, of great significance to consider how research of acoustics and its applications could become an integrated part of sustainability. In this editorial, those are explored from the built environment perspective, in terms of acoustic comfort, urban sound planning, building-envelop designs, sustainability of acoustic materials and products, use of natural means for noise reduction, and the acoustic impacts of sustainable measures.

Acoustic comfort is an important part of the overall physical comfort, a key consideration in a sustainable built environment. If there are acoustic defects in an indoor or outdoor space, remedial treatments are often costly and inefficient. At an urban scale, along with the EU Environmental Noise Directive 2002, mandating each city to identify and protect its quiet areas, there has been a recognized lack of research in urban soundscapes, integrating the conventional physical approach with the consideration of perceived sound environment, as the conventional approach of merely reducing the noise level often does not deliver an improved quality of life (e.g. during the COVID-19 lockdown the sound level decreased by 10–15 dB in London whereas noise complaints doubled). In addition to controlling noise, introducing pleasant sounds is very important. At a building scale, acoustic quality and comfort is not only important in acoustic buildings such as auditoria and recording studios, but also in everyday ‘non-acoustic’ buildings such as hospitals, schools, sport centers, churches, offices, libraries, and shopping centers.

In planning a city, sound propagation needs to be taken into account. There is an increasing urban population and correspondingly higher building density, and the construction industry accounts for 1/3 of our total energy consumption. Although higher building density will normally bring a higher noise level, a comparison between a typical high density city in China (with a major road grid of 500 m by 500 m), and a typical low density city in the UK (with many evenly distributed smaller roads), shows that, while the spatial maximum noise levels in the former are mostly higher than those in the latter, the spatial average and minimum noise levels in the latter are generally higher (Appl. Acoust., 2011;72:556–568). This suggests that with a given density, there is a great potential of planning the urban morphology to make a city more noise resistant. Another solution is to develop self-noise-protection buildings by designing appropriate building shapes.

Designing sustainable building-envelops is also related to acoustic issues. For example, a window with two or more layers of glazing could bring benefits in both energy saving and noise reduction. On the other hand, encouraging the use of natural ventilation is an important aspect of the green building movement, but opening windows can often cause noise problems. It is thus important to develop window systems that reduce noise transmission while allowing natural ventilation as well as efficient use of daylight, thus increasing the overall sustainability of the building stock. There have been various attempts to produce suitable systems using passive (Appl. Acoust., 2005;66:669–689) and active control techniques (J. Acoust. Soc. Am., 2011;130:176–188; Sci. Rep., 2020;10:10021), as well as their combinations.

Various acoustic materials and products, including absorbers, insulators, diffusers, silencers, and noise barriers, may have similar acoustic performances but very different characteristics in terms of sustainability. It is therefore important to carry out lifecycle analysis and environmental impact assessment for acoustic materials and products (Build. Environ., 2009;44:2166–2175). A number of acoustic products have also been developed by using recycled materials such as tires and carpets, and by using natural materials (Appl. Acoust., 2020;159:107070), to achieve better sustainability.

Using natural means to reduce noise will also significantly contribute to the overall sustainable development, for example, with vegetated surfaces (applied to building façades and roofs), caged piles of stones (gabions), vegetation belts (tree belts, shrub hedges), earth berms, and various ways of exploiting ground-surface-related effects (Appl. Acoust., 2015;92:86–101; 2020;165:107328). The acoustic effects of vegetation arise through three mechanisms: sound absorption and sound diffusion, which occur when a sound wave impinges on the vegetation and is then reflected back; and sound level reduction, when a sound wave is transmitting through the vegetation. When used on boundaries within a street canyon or square, the effectiveness of absorption and diffusion can be greatly enhanced since there are multiple reflections.

Some sustainable measures, such as wind farms, may be noisy, so the useable land is reduced and the overall sustainability could be negatively affected. It is thus important to examine the noise generation and propagation of such measures, as well as the perception of such noise sources (Renew. Energy, 2017;107:629–638; Landsc. Urban Plan., 2017;165:1–10).

Overall, given the above interrelationships between creating comfortable sound environments and maintaining sustainable developments, acoustics should be an essential consideration in sustainability, where there are a range of research and application potentials, from basic research for new materials to design methods and techniques, and also, it is important to integrate acoustics into various standards and regulations on sustainability.

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Vibration Behaviour Control of a Fabricated One-Passenger Electric Vehicle with Either Mechanical or Air Suspension

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Vehicle vibration transmission associated with the dynamic system depends on the frequency and direction of the input motion and the characteristics of vehicle suspension system and the seat from which the vibration exposure is received. A fabricated one-passenger electric vehicle equipped by a coil spring (mechanical) suspension system is introduced in this study. An air suspension system is used to replace the coil spring suspension system to improve ride comfort performance and intelligent classical adaptive neuro-fuzzy inference system controller is used to control the vehicle seat performance parameters. Accelerometers are mounted on the seat pan and seat base (floor) when measuring vertical acceleration. Data is frequently weighted according to standard BS 6841 in order to model the human response to vibration in terms of location and direction. The Simulink model is developed in Matlab software with the adaptive neuro-fuzzy inference system controller for the vehicle seat weighted vibration acceleration control. The results indicate that the predicted vibration acceleration can track the target vibration acceleration very well. Moreover, the values of the crest factor and kurtosis for the vehicle equipped by air suspension system are lower than those for the vehicle equipped by mechanical suspension system. Furthermore, the seat effective amplitude transmissibility for the fabricated vehicle with air suspension behaves lower value than that for mechanical suspension.

1. INTRODUCTION

A comfortable ride is essential for a vehicle to obtain passenger satisfaction. In this view, vehicle manufacturers are constantly seeking to improve vibration comfort. Many factors influence the transmission of vibration to and through the body. Transmission associated with the dynamic system depends on the frequency and direction of the input motion and the characteristics of the vehicle suspension system and the seat from which the vibration exposure is received. Oscillatory responses are analysed by means of a bus oscillatory model with linear characteristics and three degrees of freedom, with excitation by the Power Spectral Density (PSD) of the roughness of asphalt-concrete pavement in good condition. This analysis is conducted through a simulation, in frequency domain, using statistical dynamics equations. A program created in the software pack MATLAB is used to analyse the transfer functions, spectral density and RMS of oscillatory parameters. The results of the analysis show that the parameters, which ensured good oscillatory comfort for the driver, were conflicting with the parameters which ensured the greatest stability of the bus and the corresponding wheel travel. In terms of the driver’s oscillatory comfort, the bus suspension system should have a spring of small stiffness and a shock absorber with a low damping coefficient. In terms of active safety, it should have a spring of small stiffness and a shock absorber with a high damping coefficient, while minimum wheel motion requests for springs of great stiffness and shock absorbers with a high damping coefficient.

A new pneumatic spring that allows for tuning the stiffness and ride height independently and continuously is pre-
sented. The proposed pneumatic spring comprises a double-acting pneumatic cylinder, two accumulators, and a tuning sub-system. This new pneumatic spring is highly capable of improving vehicle performance and comfort on any road condition according to a driver’s preference or adaptations in real time. The mathematical model is established based on principles of thermo and fluid dynamics. An experimental setup has been designed and fabricated for testing and evaluating the proposed pneumatic spring, spring modeling and validate the tuning capabilities of the new pneumatic spring. This new pneumatic spring provides a more flexible suspension design alternative for meeting various conflicting suspension requirements for ride comfort and performance.

A seat exposed to increasing magnitudes of vibration is initially friction locked. As the vibration magnitude is increased, the seat begins to ‘break away’ producing jolting on the seat surface and causes the SEAT (seat effective amplitude transmissibility) value to increase. As friction becomes less dominant, the SEAT value decreases and the seat acts to reduce the vibration on the seat surface. The SEAT value increases as the end-stop impacts cause severe vibrations at the seat surface. An evaluation of old and new seat suspension systems is given. It starts approximately 20 years ago, when the scissors system as a supporting structure of the suspension system took strong hold. Only the conventional type of seat is taken into consideration (‘full travel’ type of suspension with free travel of 100 mm). This mechanism proved to be very safe and was low in production cost. The same supporting structure was used in all seats under investigation.

Vehicles driven by in-wheel motors have been receiving more and more attention. However, due to the introduction of in wheel motors, the ratio between unsprung and sprung mass is increased. The influence of the change on ride comfort of vehicles driven by in-wheel motors, an 11-degree of freedom of vehicle ride comfort model will be presented and studied with Matlab/Simulink. Then, road tests will be conducted to corroborate the simulation results. It can be obtained that the vehicle ride comfort becomes poor with the increasing unsprung mass. Finally, semi-active air-suspension proportional–integral–derivative control system will be proposed to improve the vehicle ride comfort. Through the simulation results, one can conclude that the proportional-integral-derivative control system for air suspension is feasible and effective to improve the ride comfort of the vehicles driven by in-wheel motors.

In this study, a fabricated one-passenger electric vehicle equipped by a coil spring suspension system was designed. An air spring suspension system was used to replace the coil spring (mechanical) suspension system to improve ride comfort performance parameters from the controlled vehicle seat vertical vibration responses. The relevant system parameters are detailed described. The intelligent, classical, and adaptive neuro-fuzzy inference system (ANFIS) controller was designed according to underlying principles which can be easily applied on this fabricated vehicle. For this special study, a feed-forward compensator is introduced to improve the performance. The frequency range is being set up to 30 Hz and the road surface is paved concrete texture with vehicle forward speed of 10 km/h.

![Figure 1. BS 8641 frequency weighting filter.](image)

### 2. BS 8641 Frequency Weighting Filter

#### 2.1. Road Test Data

The best methodology for comfort evaluation, giving a single number value, was determined. A single number estimate of vibration severity required that the motion be weighted according to the relative importance of different physical variables: magnitude, frequency and duration. Using filters $W_i(f)$ for vertical axis (bounce) as proposed in the British Standard BS 6841 (equivalent to the ISO 2631-1 vertical vibration filter) (Fig. 1) for whole-body vibration evaluation, all frequencies from the vehicle body vibration acceleration data with lower contribution to the discomfort feeling were lowered in value, where frequencies with high contribution were raised in value. The former are mainly the frequencies under 4 Hz and above 8 Hz.

#### 2.2. Vibration Signal

##### 2.2.1. Crest factor

The crest factor is defined as the ratio of the peak value to the RMS of the signal. The crest factor (CF) was calculated to determine the most suitable method of analysis (i.e. RMS or VDV), and was given by:

$$CF = \frac{\max(a_n)}{a_{w_rms}}.$$  

##### 2.2.2. Kurtosis of vibration signals

Kurtosis (Kurt) is defined as the fourth statistical moment signal, known as a global statistical parameter, that is highly sensitive to the impulsiveness of the time-domain data. For discrete data sets was approximated by:

$$ Kurt = \frac{1}{(1/N) \sum_{n=1}^{N} (x(n) - \bar{x})^4}{\left[\frac{1}{(1/N) \sum_{n=1}^{N} (x(n) - \bar{x})^2}\right]^2}; $$

where $N$ was the number of data points taken in the signal, $x_i(n)$ was the amplitude of the signal at the $n^{th}$ point, and $\bar{x}$ was the mean value of all amplitudes.
The kurtosis statistical parameter was measured in different speeds as a road surface characteristic parameter. The variation due to road surface and speed conditions will be evaluated as well. The kurtosis value was approximately 3.0 for a Gaussian distribution. Higher kurtosis indicated the existence of numerous extreme data values, inconsistent with a Gaussian distribution, while lower than 3.0 designated a relatively flat distribution.

### 2.2.3. International road roughness index

The three main sources of vibration transmitted to the vehicle driver are: road roughness, vehicle engine and driver’s behaviour (including choice of speed). Within reasonable variations in these factors, road roughness plays a considerably greater part than the other two. In vehicle research into comfort quality, all the three sources are playing important role. To include the road roughness in this study, the International roughness index (IRI) was used. It was a general pavement condition indicator that summarized the roughness qualities that affected tractor response and was most appropriate when a roughness measure was desired that relates to overall ride quality and overall surface condition. The IRI value was defined approximately from the following equation:

\[
\frac{(a)}{IRI} = 0.16 \left( \frac{\nu}{80} \right)^{1.2};
\]

where \(\nu\) (km/h) was the tractor forward speed and \((a)_{rms}\) was the vehicle floor RMS acceleration vibration (m/s\(^2\)).

### 2.3. Frequency Analysis

Vibration evaluations were performed according to the recommendations in the BS 6841. This involved the application of frequency weightings, multiplications of factors to allow for different sensitivities of the body in different axes, calculation of root-mean-square (RMS), and vibration dose value (VDV).

The frequency weighted data was converted into a comfort value using RMS, or VDV. BS 6841 proposed using RMS when CF is less 6. RMS is the time dependency according to the ISO 2631. This is a very complex procedure and it was not appropriate for signals containing shocks.

\[
RMS = \sqrt{\frac{1}{T} \int_0^T a_w^2(t)dt};
\]

where \(a_w(t)\) was the frequency-weighted acceleration. According to the BS 6841, vibration magnitudes and durations that produce RMS in the region of 0.825 m/s\(^2\) A (8) will usually cause severe discomfort. The exposure period required for the RMS to reach a tentative action level of 0.825 A (8) can be calculated as:

\[
T_{0.825A} = \frac{8 \times 0.825^2}{(a)^2_{rms}};
\]

where \((a)_{rms}\) was the weighted RMS vibration magnitude.

The data acquired was measured for 10 min. However, this was measured in such a way as to represent the vibration levels experienced by the passenger related to the normal 8-hour work period. The required parameters were then computed and extrapolated to cover the entire duration of exposure. Subsequently, the weighted RMS parameter was then computed.

### 2.4. Seat Effective Amplitude Transmissibility

Seat comfort is usually assessed by making vibration measurements on the surface of the tractor seat based on the BS 6841. Seat-isolation performance was indicated by Seat Effective Amplitude Transmissibility (SEAT) values, which can be calculated from frequency-weighted RMS accelerations on the vehicle seat pan surface and vehicle floor, \(RMS_{seat}\) and \(RMS_{floor}\), respectively:

\[
SEAT\% = 100 \times \frac{RMS_{seat}}{RMS_{floor}}. \tag{6}
\]

The SEAT value is a measure of how well the transmissibility of a seat is suited to the spectrum of entering vibration, considering the sensitivity of the seat occupant to different frequencies. SEAT values less than 100% indicate isolation or attenuation of vibration. It allows for the comparison of seat performance on a variety of road surfaces.

### 3. VEHICLE SEAT CONTROL PERFORMANCE PARAMETERS

#### 3.1. Adaptive Neuro Fuzzy Inference Scheme Technique

The final output was the weighted average of each rule’s output. Adaptive Neuro Fuzzy Inference Scheme (ANFIS) is functionally equivalent to a Takagi-Sugeno (T-S) fuzzy inference system. By using stipulated input-output training data pairs, ANFIS tunes the membership functions and other associated parameters by back propagation gradient descent and least square type method, respectively. Such methodologies make the ANFIS modeling more systematic and less reliant on expert knowledge, thus more objective. The corresponding equivalent ANFIS structure and learning process can be found in.

#### 3.2. Training Process for ANFIS Model

The ANFIS model mainly consisted of the NF network placed in series with the vehicle. The training process for obtaining the model of the vehicle is shown in Fig. 2, where the input and output data set was used to reflect the input-output characteristics of the vehicle. The training data set was used based on: FSO, FSN, RSO, RSN, ORS, URS, CORS\(_\phi\), where FSO was the offside front suspension vibration acceleration, FSN was the nearside front suspension vibration acceleration, RSO was the offside rear suspension vibration acceleration, RSN was the nearside rear suspension vibration acceleration, ORS was the vehicle seat pan vibration acceleration, URS was the vehicle under seat (floor) vibration acceleration, and CORS\(_\phi\) was the desired vehicle seat acceleration. The ANFIS network was trained by the least square estimation method to minimize the cost function error \((E_c)\) defined by:
\[ E_1 = \sum_{i=1}^{N} (ORS - CORS_d). \]  

ORS was the input signal and CORS\(_d\) the corresponding actual (desired) output of the component’s ANFIS. The iterative learning tuner was designed to improve the tracking performance of inverse-ANFIS control which repeats the desired task over a finite interval.

### 3.3. Inputs Selection

Prior to training the individual ANFIS model, the number of input membership functions was determined, which is an important factor in the initial condition. In the case of ANFIS which has \(N_i\) inputs and \(K\) membership functions for each input, the number of fuzzy rules \(R\) was:

\[ R = K^{N_i}. \]  

It was seen from Eq. (8) that too many inputs and input membership functions will lead to substantial increase of inference rules and thus increase training time and computer memory space consumption. Therefore, it was necessary to do input selection that found the priority of each candidate inputs and used them accordingly. A quick and straightforward way of input selection for neuro-fuzzy modeling was to use ANFIS. The proposed input selection method assumed that the ANFIS model with the smallest RMSE (root mean squared error) after one epoch of training had a greater potential of achieving a lower RMSE when given more epochs of training. In this study, five models had been tried with different combinations of two inputs and trained them with a single pass of the least square method.\(^{16}\) After several trials, the smallest training error was achieved when the inputs were FSO, FSN, RSO, RSN, ORS, URS at the current time step and CORS\(_d\) from previous time step to produce output CORS\(_d\) at the current time step. Then the selected inputs were trained using the hybrid learning rule to tune the membership functions as well.

### 3.4. Adaptive Neuro Fuzzy Inference Scheme Controller Design

#### 3.4.1. Background

In this study, Adaptive Neuro Fuzzy Inference Scheme (ANFIS) technique was used to design a controller to reduce the vibration of the vehicle seat pan by control the vibration response transmitted from both engine and road surface. The design process is as follows: A neuro-fuzzy model for the vehicle seat pan was built to represent the relation between the variation of the vibration acceleration transmitted through to driver seat pan i.e. (obtain outputs given the inputs).

#### 3.4.2. Training neuro-fuzzy vehicle seat pan controller

Given input/output data sets, ANFIS constructs Fuzzy Inference System (FIS) whose membership function parameters were adjusted using a back-propagation algorithm. The size of input-output data needed to be large enough and cover all ranges to fine tune the membership function. The offside front suspension vibration acceleration (FSO), nearside front suspension vibration acceleration (FSN), offside rear suspension vibration acceleration (RSO), nearside rear suspension vibration acceleration (RSN), vehicle seat pan vibration acceleration (ORS) and vehicle under seat (floor) vibration acceleration (URS) at previous step were used as inputs and controlled (desired) vehicle seat vibration acceleration (CORS\(_d\)) from current step as output as seen in Fig. 3.

The ANFIS controller was developed using MATLAB software based on the experimental data sets. 3048 data samples were collected corresponding to vehicle speed of 10 km/h). These data was divided to 1524 points (odd number, where an odd number is number that cannot be divided evently by 2) for training and 1524 points (even number) for checking.\(^{17-20}\)

#### 3.4.3. Development of Simulink model

The simulink model for the control of the vehicle seat pan vibration acceleration was developed using MATLAB software. This Simulink model with the ANFIS controller was developed using the various toolboxes available in the Simulink library such as the power system, power electronics, control system, signal processing toolboxes and from its basic functions. The entire system modeled in Simulink is a closed loop feedback control system consisting of the NF forward model and the inverse ANFIS controller. The developed Simulink model for the control of various parameters of the NF controller is shown in Fig. 4.
4. EXPERIMENTAL METHODOLOGY

4.1. Classical Passively Vehicle Suspension System

The major purpose of any vehicle suspension system is to isolate the body from road unevenness disturbances and to maintain the contact between road and the wheel. Therefore, the suspension system is responsible for the ride quality and driving stability. The design of a classical passive suspension system is a compromise between this conflict demands. However, the improvement in vertical vehicle dynamics is possible by developing an air spring suspension system. Figure 5 shows the schematic diagram of the electric vehicle with air suspension system.

4.1.1. Mechanical suspensions system

The mechanical type suspension generally consists of spring and damper elements as shown in Figs. 6(a) and 6(b). No computer control is associated with this type of suspension. The suspension characteristics are fixed by the mass, spring, and damper elements, although some passive systems may have limited adjustments for the spring and/or damper elements. Passive suspensions do not automatically change or optimize their spring or damper characteristics based upon a changing environment. Therefore, they are most effective over a narrow range of disturbance inputs. The suspensions are designed based on a nominal mass load and disturbance environment expected to be most encountered over the design life of the suspension. The equivalent spring constant and damping coefficient of the mechanical suspension are 20000 N/m and 1200 Ns/m, where maximum travel $-25 \times 10^{-3} \text{m}$. These data have been originated based on the simulation presented in reference.1

4.1.2. Air suspensions system

In this paper, the separate functions of spring and damper were combined into a system that contained an air spring, valve, and accumulator. The sizing of the air spring and the accumulator provided the spring function, and the valve provided the damping function by controlling the air flow between the spring and the accumulator. A computer algorithm controlled the opening and closing of the valve to optimize the damping provided by the system.

For passive vibration suppression applications, the air spring’s engineering function was the same as a steel spring, where a force was generated as a function of the displacement of the load. Since both air springs and steel springs contain relatively small amounts of damping, they were typically used in conjunction with separate dampers or shock absorbers. Although integrated air spring and damper units were available for some applications, as shown in Fig. 7(a), Fig. 7(b) shows the spring and damper functions were typically separate. In many cases, air or some other gas was the fluid medium for the air spring, and a hydraulic fluid was the medium for the damper.

Traditionally, the air suspension ball control system, the buoyancy of suspending comes from the wind-force from a DC fan. Once the floater reaching the setting height, the error signal is zero. The control signal will be zero soon and the DC fan will stop. If the floater was lower than the setting point, the control signal would restart the DC fan to lift the floater. However, the wind-power had to fill with the chamber first. Then, the additional wind-power rose to the tube to float up the floater. Due to the gravity of the floater and the physical dead-time of the fan, the floater would fall to the bottom of the tube in the time difference.

Therefore, a proper buoyancy to maintain the height of the floater was necessary. The equivalent spring constant and damping coefficient of the air bellow rubber diaphragm. The rubber diaphragm had to be thick enough to resist the force due to the pressure difference and payload. However, the introduced equivalent spring constant and damping coefficient were 300 000 N/m and 20 000 Ns/m respectively, where maximum travel $-25 \times 10^{-3} \text{m}$.1

4.2. Experimental Setup

Figure 8 shows photograph of a sample from the vibration measurement on the mechanical suspension system, while Fig. 9 shows the recording and analysis instrumentation.

The test vehicle was a one-passenger fabricated electric vehicle prototype, where its working conditions were set at constant speed of 10 km/h. The road surface used in this study was
paved concrete texture, where its characteristics had no wear or weathering, small stones, a smooth surface, low overall vibration level and higher frequency greater proportion of noise, and no “roar”. Figure 10 shows photographs for this road surface and the vehicle was driven at 10 km/h, where GPS sensor was used for the measurement of vehicle speed which provides an accurate information of the speed of the vehicle estimated through the Doppler’s effect.

The vibration measurements were made on the road surface mentioned above and the vibration samples were acquired with an integration period of 1.0 second and each individual mea-
measurement with a 10 second duration using a multi-channel analyser. Unweighted accelerations on the vehicle seat base (floor) and on the passenger seat pan were measured with Bruel & Kjaer accelerometers Type 4514B-001. The vibration amplitudes recorded from the seat base (floor) and on the passenger seat pan during travel were investigated for possible artifacts and any unclear signals that were detected were removed. The accelerometer must move with the interface, they must not alter the dynamic properties of either the vehicle seat pan or the passenger body and it must offer little impedance to movement over the frequency range of interest. For this reason, a thin disc of 0.25 m in diameter was used (seat interface for accelerometer indicating passenger body acceleration received). It was a rigid device that compressed the seat in the same manner as a passenger’s buttocks. As only the vertical direction was considered, only vertical accelerations were measured. Under each working condition, the vibration signals at the vehicle floor and the passenger body/seat were analysed by using the Bruel & Kjaer portable, multi-channel PULSE Type 3560-B-X05 analyser, Bruel & Kjaer PULSE labshop, and the measurement software type 7700 and saved in the computer. Figure 11 presents a schematic of the measurement assembly.

5. RESULTS AND DISCUSSION

Figures 12(a) and 12(b) show the experimental raw signals of the vibration acceleration in terms of time history results taken at the connection of unsprung and sprung points on the left hand side (LHS) and right hand side (RHS) of the vehicle while Fig. 12(c) shows the corresponding raw signals the vibration acceleration in terms of time history results taken on the vehicle floor (seat base) and seat pan based on the measurement methodology presented in section 4. Figure 12(d) shows frequency domain by comparatively presenting raw and weighted floor acceleration for the vehicle when equipped by mechanical. Figures 13 (a) to 13 (c) show the results of the same parameters at the same points when the vehicle was equipped with an air suspension system. The road surface is being paved concrete texture and at vehicle speed of 10 km/h. Figure 13(d) shows frequency domain comparatively by presenting raw and weighted floor acceleration for the vehicle when equipped by air suspension system.

The developed ANFIS model structure with two input neurons and one output neuron along with four hidden layers (input membership function, rule base, membership function, and aggregated output), which are shown in Fig. 2. 2048 data points from experimental data are used to validate the accuracy of the forward model. Figures 14(a) and 15(a) show the comparisons between the vehicle’s seat measured and predicted vibration accelerations based on the ANFIS model in terms of time-history for the mechanical and air suspension system respectively. The predicted vibration acceleration can track the target vibration acceleration very well. The experimental data belonged to speeds of 10 km/h. The training of the neural network by using the fuzzy rule base for selection of proper and optimal rule is considered in the next section (designed ANFIS controller).

Figures 14(b) and 15(b) show the comparisons between the vehicle’ seat measured; and predicted vibration accelerations based on the ANFIS controller in terms of time-history for the mechanical and air suspension system respectively. The predicted vibration acceleration can track the target vibration acceleration very well. The experimental data belonged to speeds of 10 km/h. It is observed from the analysis of results in the figures that, by using the ANFIS controller, the vehicle seat vibration acceleration controlled reaches an acceptable error values. It is also observed that, with the designed neuro-fuzzy controller, the vehicle seat vibration acceleration characteristics curves take less time to settle with the system stabilization because of the training process of the ANN used and the proper selection of the rule base.19

The conditions during the tests are illustrated in the previous figures. Perhaps the most powerful analysis technique is the frequency-domain analysis (FFT) for the following reasons:

1. Changes in minor spectral components, which may be the first indication of incipient failure, will not always affect...
Figure 12. Vehicle vibration acceleration — mechanical suspension system.

Figure 13. Vehicle vibration acceleration — air suspension system.
the overall vibration level. These changes can be picked up by spectrum monitoring.

2. A rise in overall level indicates that something has been changed but does not give any information regarding the source. This will often be indicated by the frequency at which the change occurs.

In Figs. 14(c) and 15(c), the FFT of weighted vibration acceleration measurements in vertical direction were truncated to show the frequency range of interest up to a 30 Hz clarity for the vehicle floor, seat, and seat controlled by ANFIS for both mechanical and air suspension system respectively. In both figures, a good seat should have a peak frequency of at least 1.4 times lower than the peak frequency of the vehicle where the seat is mounted (uncontrolled), but for the ANFIS — a seat should have a peak frequency of at least 2.8 times. In the present study, it is noticed that the weighted vibration acceleration at the peak frequency for either a seat or the ANFIS-seat is amplifying rather than attenuating due to poor design and improper maintenance of seat. Therefore, it could be possible that the main source of discomfort.

Table 1 tabulates the floor vibration signal parameters calculated values of crest factor (CF) kurtosis (kur) and the international road roughness (IRI) based on Eqs. (1) to (3) respectively. According on the data presented in the figures and table, it can be observed that the CF values ranged from 3.87 (mechanical) to 3.38 (air). On the other hand, the kurtosis value of approximately 3.0 (3.07 air) is for a Gaussian distribution. Higher kurtosis of (4.45 mechanical) indicates the existence of numerous extreme data values, inconsistent with a Gaussian distribution. Moreover, the international road roughness (IRI) values are proportional to kurtosis values which indicate that
a human perceives more peaks and impulses when driving on road surfaces with greater roughness. Generally, the values of crest factor (CF) and kurtosis (kur) for the vehicle equipped by air suspension system are lower than those for the vehicle equipped by mechanical suspension system.

In Table 2, since the fabricated vehicle produces low amounts of shocks (crest factor less than 6) and as time dependency is incorporated in weighted RMS. Therefore, the RMS will be used to evaluate the SEAT ratio rather than VDV. The calculated weighted RMS value and exposure period $(T_{0.825A(8)})$ for the vehicle seat weighted vibration acceleration based on Eqs. (4) and (5) are presented in Table 2, where their values are lower for air suspension than mechanical suspension. On the other hand, the weighted RMS controlled values by the ANFIS are lower than the uncontrolled values while the exposure period $(T_{0.825A(8)})$ controlled values by the ANFIS are higher than the uncontrolled values.

In terms of seat effective amplitude transmissibility (SEAT), estimations based on Eq. (6) and RMS values are used to make a comparison between the types of suspension system, where the SEAT for the fabricated vehicle with air suspension behaves lower value than that for the fabricated vehicle with mechanical suspension either seat RMS uncontrolled or controlled (See Table 2).

### 6. CONCLUSIONS

Several different analysis methods have been discussed, all of which involve capturing the acceleration at the seat (and/or floor) of the vehicle. Some of the techniques include methods to calculate expected passenger’s comfort from the vibration magnitudes measured. Most techniques give results that indicate that the vibration in vehicles is not severe but could occasionally cause some discomfort.

The ANFIS can be used with systems handling more complex parameters. Another advantage of the ANFIS is that its speed of operation is much faster than the other control strategies; the tedious task of training membership functions is done in ANFIS. Collectively, these results show that the ANFIS controller provides faster settling times, has very good dynamic response and good stabilization. A simulink model was developed in Matlab software with the ANFIS controller for the vehicle seat weighted vibration acceleration control. The control strategy was also developed by writing set of fuzzy rules according to the ANFIS control strategy with the back-propagation algorithm in the back end.

The influence of vibration on comfort is generally known and the consequent origination or even deterioration of health. It must be kept in mind that low frequency vibration has essentially higher energy severity than a vibration of middle and higher frequencies. The long-term exposition of the energy rich low frequency vibration can potentially cause harm to human health, and not only the health but also functionality of other organs such as the central nervous system. Therefore, it is important to improve the criteria of energy rich, low frequency, vibration assessment by means of control by any control strategy so that the influences of energy on human health and comfort are assessed correctly.

The predicted vibration acceleration can track the target vibration acceleration very well. Moreover, the values of crest factor and kurtosis for the vehicle equipped by air suspension system are lower than those for the vehicle equipped by mechanical suspension system. Furthermore, the seat effective amplitude transmissibility for the fabricated vehicle with air suspension behaves lower value than that for mechanical suspension.

In terms of seat effective amplitude transmissibility (SEAT), an estimation based on weighted root mean square (RMS) is used to make a comparison between the vehicle equipped by either mechanical or air suspension system. For the assessment of vibration, with strong low frequency content, in the frequency range from up to 30 Hz, it is more logical to use SEAT estimation based on weighted RMS.

### REFERENCES


As critical components, rolling bearings are widely used in a variety of rotating machinery. It is necessary to develop a suitable fault diagnosis method to prevent malfunctions and breakages of bearings during operation. However, the current methods for the fault diagnosis of rolling bearings are too cumbersome to be applied in practical engineering. In addition, the working condition of rolling bearings is generally tough, complex, and especially variable. These conditions cause fault diagnosis methods to be less effective. This paper aims to provide a simple and effective method for the fault diagnosis of rolling bearings under variable conditions. The main contribution of this paper is as follows: (1) The refined composite multiscale fuzzy entropy (RCMFE) is applied in bearing fault feature extraction because of its simplicity and high efficiency; (2) The improved support vector machine (ISVM), based on the whale optimization algorithm (WOA), is proposed to identify the fault pattern of rolling bearings. The ISVM is proposed in this paper to solve the problem that parameter setting affects the classification effect of SVM. In the ISVM, the WOA is employed to optimize both the regularization and kernel parameters of the SVM. Compared with the traditional optimization methods, the WOA has the advantages of high optimization speed and better optimization ability; (3) Combining the RCMFE and the ISVM to diagnose bearing fault under variable working conditions. The effectiveness of the RCMFE-ISVM has been validated via experimental vibration signal of bearings faults under variable working conditions.

1. INTRODUCTION

Rolling bearings are widely used in various industrial machines such as electric motors, pumps, gearboxes, and turbines. The faults of bearings are prone to occur spalls or cracks on surfaces due to their complex working conditions such as high speed, heavy load, and high temperature. The fault probability of rolling bearings is 30% in all faults of rotating elements. It is necessary to detect where the faults occurred in bearings because the faults will lead to both equipment shutdown and safety accidents.

Various methods have been introduced in bearing fault diagnosis, such as vibration analysis, acoustic emission, debris analysis, and infrared thermal imaging. Among those techniques, vibration analysis has served as an efficient tool for bearing fault detection. Various vibration analysis methods have been applied to bearing fault diagnosis, such as autoregressive model, spectral kurtosis, and kurtogram, wavelet transform, matching pursuit order tracking, and empirical mode decomposition. However, the above methods are complex and require a lot of specific knowledge for their practical application. Moreover, the non-stationary, non-linear characteristics of the bearing fault signal and the complex and diverse working conditions will make fault diagnosis more difficult. Therefore, an intelligent fault diagnosis method for bearings that is based on refined composite multiscale fuzzy entropy (RCMFE) and improved support vector machine (ISVM) is proposed in this paper.

An important and difficult step in rolling bearing fault diagnosis research is fault feature extraction, which directly affects the effect of fault diagnosis. Time domain analysis and frequency analysis are commonly used feature extraction methods, which can extract statistical feature parameters such as energy, root mean square, etc. These methods are widely used in bearing fault diagnosis because of their simplicity and practicability. However, time and frequency domain analysis can only be applied to stationary and linear signals. To extract fault features from non-stationary and non-linear signals, many methods for time-frequency analysis must be introduced into bearing fault diagnosis. For example, wavelet transform (WT) is generally effective to extract the fault feature of the rolling bearing. However, the certain bands of the defect information and the selection of the base function are the short-
comings of WT. Then some self-adaptive time-frequency are applied to the fault diagnosis of the rolling bearing. For instance, empirical mode decomposition (EMD),\textsuperscript{26} local mean decomposition (LMD),\textsuperscript{27} and ensemble empirical mode decomposition (EEMD)\textsuperscript{28} are introduced for feature extraction of the rolling bearing. The problem of mode aliasing has always puzzled the application of EMD. Although LMD and EEMD can alleviate this shortcoming, they have not been completely overcome. To address this shortcoming, Konstantin proposed a signal processing method called variational mode decomposition (VMD).\textsuperscript{29} VMD was quickly introduced into bearing fault feature extraction and produced good results. VMD is introduced into bearing fault feature extraction and there is no obvious mode aliasing,\textsuperscript{30} but VMD needs to set the mode number artificially. Whether the mode number is appropriate or not will affect the effect of fault feature extraction. In addition, the feature extraction process of the above methods is mostly complex, which limits their application in engineering practice. Therefore, it is necessary to explore a simple and effective feature extraction method in engineering practice. Recently, the nonlinear dynamic analysis method has been widely used in bearing fault feature extraction due to its simple and effective characteristics, such as fractal,\textsuperscript{31} sample entropy,\textsuperscript{32} and fuzzy entropy.\textsuperscript{33} However, these methods are all single scale analysis methods, which cannot describe the complex characteristics of vibration signals. Therefore, multi-scale analysis methods are introduced into bearing fault diagnosis, such as multi-scale sample entropy\textsuperscript{34} and multi-scale fuzzy entropy.\textsuperscript{35} Li et al. summarized the application effect of the existing entropy method in the field of rotating machinery fault diagnosis.\textsuperscript{36, 37} Because sample entropy uses step function to calculate similarity measure function, there will be a sudden change in measuring similarity. Sample entropy is greatly affected by the sample length. In addition, the MSE and MFE have shortcomings in coarsening. In order to alleviate these problems, Azami proposed a new irregular index called RCMFE and applied it to biomedical signal analysis.\textsuperscript{38} The RCMFE can extract the complexity characteristic related with fault information from vibration signals of rolling bearing, and the RCMFE is quickly applied to bearing fault diagnosis and achieved good results.\textsuperscript{39, 40} In literature, the RCMFE needs dimension reduction to be used as feature parameters.\textsuperscript{37, 38} The dimension reduction process may be cumbersome and time-consuming, which is not conducive to practical engineering application. In this paper, RCMFE is directly extracted as a feature parameter, and the process of dimension reduction is omitted.

After fault features extraction, it is necessary to classify fault features accurately. The commonly used classification methods mainly include the naive bayesian network (NBN),\textsuperscript{41} the K-nearest neighbor algorithm (KNN),\textsuperscript{42} and the artificial neural network (ANN).\textsuperscript{43} However, each of these methods have their own shortcomings. For instance, the NBN cannot deal with the change results based on feature combination. The KNN calls for a large amount of calculation in classifying test samples. So far, there is no unified and complete theoretical guidance for the selection of the ANN structure, and it can only be selected by experience. Over-selection of network structure, inefficiency in training and over-fitting may result in low network performance and low fault tolerance. If the selection is too small, the network may not converge. Therefore, how to choose the appropriate network structure in application is an important problem.\textsuperscript{44} The support vector machine (SVM) can solve the problem of small sample classification, and it can overcome shortcomings of overfitting and local optimal solution based on structural risk minimization principle.\textsuperscript{45} Therefore, the SVM is applied to early fault detection of bearings.\textsuperscript{46} However, the setting of the parameters will affect the classification effect of SVM. Therefore, several optimization methods are used to optimize the SVM parameters, such as genetic algorithm (GA),\textsuperscript{47} particle swarm optimization (PSO), artificial bee colony algorithm (ABC),\textsuperscript{48} and so on. But the above optimization algorithms all have their shortcomings. For example, the inappropriate choice of operators and parameters used in the evolutionary process makes the GA susceptible to premature convergence. Moreover, the algorithm has other shortcomings, such as time-consuming computation, difficulty in dealing with high-dimensional problems, and so on.\textsuperscript{49} The PSO algorithm is simple in calculation and fast in convergence, but it is easy to fall into local optimum.\textsuperscript{50} The performance of the ABC is better than other methods, but there are still some shortcomings of unbalanced search ability and inadequate resolution accuracy.\textsuperscript{51} The WOA is a novel nature-inspired meta-heuristic optimization algorithm proposed by Mirjalili,\textsuperscript{52} and it has been proved that the performance of the method is more competitive than that of the traditional optimization methods. Therefore, this paper proposed an improved SVM which employed the WOA to optimize the parameters of the SVM.

It can be seen from above discussion, there are some problems in fault diagnosis of bearings with variable conditions. Those include:

1. The bearing fault data under variable working conditions have the characteristics of non-linearity and non-stationarity, and the changing working conditions may cause similar characteristics to the fault, which increases the difficulty of fault diagnosis.

2. The method needs to be simplified to facilitate the practical application of engineering.

To cope with the challenges, a novel method based on RCMFE and ISVM is proposed for fault diagnosis of rolling bearing under variable working conditions. This method can automatically identify fault patterns and complete diagnosis with small sample data. The flowchart of the proposed method is shown in Fig. 1. Firstly, the raw vibration signal is collected from rolling bearing fault test rig by sensors, then the RCMFE is extracted as fault features for fault diagnosis. Subsequently, the ISVM is trained by using training data and obtain the ISVM optimal model. Finally, the result of fault identification is obtained by the testing data.

The rest of paper is organized as follows. In Section 2, the refined composite multiscale fuzzy entropy (RCMFE) is introduced. Section 3 describes the improved support vector machine (ISVM) based on whale optimization algorithm (WOA) proposed in this paper. Section 4 and Section 5 investigate the effectiveness of the proposed method by using experimental data. The conclusions are drawn in Section 6.
2. FAULT FEATURE EXTRACTION USING Refined Composite Multiscale Fuzzy Entropy

2.1. Refined Composite Multiscale Fuzzy Entropy

Firstly, the concept of fuzzy entropy is introduced in this section. For the time series \( y = \{y_1, y_2, \ldots, y_N\} \), \( m \) dimensional vectors were obtained as follows:

\[
U_t^m = \{y_t, y_{t+1}, \ldots, y_{t+m-1}\} - \bar{y}_t; \tag{1}
\]

where \( \bar{y}_t = \frac{1}{m} \sum_{j=0}^{m-1} y_{t+j} \). The distance between \( U_{t_1}^m \) and \( U_{t_2}^m \) was defined as the maximum difference between the corresponding elements, which was calculated as follows:

\[
d_{t_1t_2} = d(U_{t_1}^m, U_{t_2}^m) = \max\{|U_{t_1+k}^m - U_{t_2+k}^m| : 0 \leq k \leq m-1, \ t_1 \neq t_2\}. \tag{2}
\]

The similarity between \( U_{t_1}^m \) and \( U_{t_2}^m \) was defined by using the fuzzy function, which was shown as follows:

\[
\mu(d_{ij}^m, n, r) = \exp\left(-\frac{(d_{ij}^m)^n}{r}\right); \tag{3}
\]

where \( n \) was the power of fuzzy entropy and \( r \) was the tolerance. Then the function \( \phi^m \) was defined as follows:

\[
\phi^m(y, n, r) = \frac{1}{N-m} \sum_{t=1}^{N-m} \frac{1}{N-m-1} \sum_{t_1, t_2 \neq t} \exp\left(-\frac{(d_{t_1t_2})^n}{r}\right). \tag{4}
\]

Similarly, for dimension \( m+1 \), repeat the above computation and obtain the \( \phi^{m+1} \). The fuzzy entropy was defined as follows:

\[
FuzEn(y, m, n, r) = \lim_{N \to \infty} (\ln \phi^m - \ln \phi^{m+1}). \tag{5}
\]

When \( N \) was a finite number, the above expression was expressed as follows:

\[
FuzEn(y, m, n, r) = \ln \phi^m - \ln \phi^{m+1} = -\ln\left(\frac{\phi^{m+1}}{\phi^m}\right). \tag{6}
\]

Then the “coarse-graining” process was applied to the original time series. For the original time series \( \{x_1, x_2, \ldots, x_N\} \) whose length is \( C \), the embedding dimension \( m \) and similarity tolerance \( r \) are given in advance, and a new coarse-grained vector was established based on the original time series. And the standard deviation \( \sigma \) was proposed as measurement in the coarse-graining process

\[
\sigma_{y_1}^{(\tau)} = \frac{1}{\tau} \sum_{b=(i-1)\tau+1}^{i\tau} \left( x_b - \frac{1}{\tau} \sum_{b=(i-1)\tau+1}^{i\tau} x_b \right), \quad 1 \leq i \leq \left\lceil \frac{C}{\tau} \right\rceil = N; \tag{7}
\]

where the \( \tau \) was time scale factor. Therefore, the \( u \)-th coarse-grained time series \( z_u^{(\tau)} = \{y_{u,1}, y_{u,2}, \ldots, y_{u,\tau}\} \) was generated and it is shown as follows:

\[
\sigma_{y_u,ij}^{(\tau)} = \frac{1}{\tau} \sum_{b=(u+j-1)\tau+1}^{u+j-1} \left( x_b - \frac{1}{\tau} \sum_{b=(u+j-1)\tau+1}^{u+j-1} x_b \right). \tag{8}
\]

The different time series \( z_u^{(\tau)} \) were obtained for different scale factor. The \( \phi_{\tau,k}^m \) and \( \phi_{\tau,k}^{m+1} \) are computed separately for a deterministic scale factor \( \tau \) and embedding dimension \( m \). Finally, the refined composite multiscale fuzzy entropy was obtained as follows:

\[
RCMFE(x, \tau, m, n, r) = -\ln\left(\frac{\phi_{\tau}^{m+1}}{\phi_{\tau}^m}\right); \tag{9}
\]

where the \( \phi_{\tau}^{m+1} \) and \( \phi_{\tau}^m \) were the mean values of \( \phi_{\tau,k}^{m+1} \) and \( \phi_{\tau,k}^m \) on \( 1 \leq k \leq \tau \) respectively.
The “coarse-graining” process is a concept introduced by Costa to describe the basis and implementation of multiscale entropy method. Given a one dimensional discrete time series, \{x_1, x_2, \ldots, x_k, \ldots, x_C\}, a coarse-grained vector corresponding to the scale factor \( \tau \) was constructed as follows. The original time series are divided into nonoverlapping windows of length \( \tau \) firstly. Then average the data points inside each window. The “coarse-graining” process is illustrated as Fig. 2.

The coarse-grained vector corresponding to the scale factor \( \tau \) can be calculated as follows:

\[
y_j^{( \tau )} = \frac{1}{(j - 1) \tau + 1} \sum_{i=(j-1) \tau + 1}^{j} x_i, \quad 1 \leq j \leq C/\tau.
\]  

(10)

2.2. Parameters Selection of RCMFE

According to the description of RCMFE in the previous section, RCMFE has some parameters that need to be set, such as time delay, similarity tolerance and embedding dimension. Therefore, to make the RCMFE better reflect bearing fault characteristics, it is necessary to select appropriate parameters of RCMFE. In this paper, Euclidean distance, and standard deviation were used to analyze the influence of parameters on RCMFE. Euclidean distance was used to measure the influence of parameters on RCMFE’s ability to reflect faults, and standard deviation was used to measure the influence of parameters on RCMFE’s stability at different scales.

In this section, an evaluation index based on Euclidean distance was proposed to evaluate RCMFE’s ability to reflect fault characteristics. The evaluation index was calculated as follows:

\[
FCID = \sum_{k=1}^{N} \left[ \sum_{i=1}^{M} \sum_{j=1}^{M} (RCMFE_{ik} - RCMFE_{jk})^2 / 2 \right],
\]

where \( N \) was scale number, and the \( M \) was the number of fault states. The \( FCID \) was standardized to ranges 0 to 1. The larger the \( FCID \), the stronger the ability of reflect faults for RCMFE, and vice versa. The smaller the standard deviation, the better the stability of RCMFE, and vice versa. Therefore, the selection criterion of RCMFE parameters was to make the \( FCID \) as large as possible and the standard deviation as small as possible.

3. THE IMPROVED SUPPORT VECTOR MACHINE BASED ON WHALE OPTIMIZATION ALGORITHM

3.1. Description of Whale Optimization Algorithm

The WOA is a novel meta-heuristic optimization algorithm that is inspired by the bubble-net hunting strategy for humpback whale. The optimization method consists of three stages: encircling prey, bubble-net attacking method and search for prey.

3.1.1. Encircling Prey

Humpback whales can identify their prey and encircled them. It was assumed that the current optimal whale group’s individual position (candidate solution) was the target prey location (target prey or close to the best target prey), and its position updating mathematical expression was shown as follows:

\[
X_{j+1} = X_j - A \times D;
\]  

(12)

\[
D = |C \times X_j^* - X_j|;
\]  

(13)

where \( A \) and \( C \) were coefficient vectors, \( X_j^* \) was the position vector of the best solution obtained so far, \( X \) was the position vector, and the \( j \) was the current iteration.

The coefficient vectors \( A \) and \( C \) were computed as follows:

\[
A = 2a \times r - a;
\]  

(14)

\[
C = 2r;
\]  

(15)

where \( a \) was linearly decreased from 2 to 0 over the course of iterations and \( r \) was a random vector in \([0, 1]\).

3.1.2. Bubble-net Attacking Method

Based on the foraging behaviour of the humpback whale, the mathematical model was established as follows:

1. Shrinking encircling mechanism: this behaviour was achieved by reducing the value of \( a \) in the Eq. (14). It should be noted that the range of variation of \( A \) also shrinks with the decrease of \( a \). The set of random numbers \( A \) is between \([-1, 1]\), and the search position of the new whale group can be defined at any position between...
the current whale group’s individual location and the best
whale group’s individual location.

2. Spiral updating position: The mechanism first calculated
the distance between the whale group and the prey, and
then created a spiral mathematical model between the
whale group and the location of its prey to mimic the spi-
ral movement of the humpback whale as follows:

\[ X_{j+1} = D'e^{bl} \cos(2\pi l) + X_j^*; \]

where \( D' = X_j^* - X_j \) was the distance between \( i \)-th whale
to the prey, \( b \) was a constant for defining the shape of the
logarithmic spiral, \( l \) was a random number in \([-1, 1]\).

The humpback whales swim around the prey within a
shrinking circle and along a spiral-shaped path simultane-
ously. To simulate the synchronization behaviour, it was assumed that a
50% probability was used as the selection threshold in the
process of updating the individual location of the whale
group. That is, either the shrinking encircling mechanism was cho-

3.1.3. Search for Prey

The humpback whales also search for prey randomly. The
method based on the variation of \( A \) vector was used to search for
prey. The whales can randomly search prey according to its
positions. Therefore, \( A \) took a random value greater than
1 or less than \(-1\), and forced whales to deviate from their prey
to search for other more suitable prey to enhance the algo-
rithm’s search ability, enabling WOA to perform global search.

The mathematical model is shown as follows:

\[
X_{j+1} = \begin{cases} 
X_j - A \times D, & p < 0.5; \\
D'e^{bl} \cos(2\pi l) + X_j^*; & p \geq 0.5,
\end{cases}
\]

where \( p \) was a random number in \([0, 1]\).

3.2. ISVM Classification Model

The support vector machine classification model used in this
paper was a soft interval nonlinear support vector machine.
Given a training data set \( D = \{x_i, y_i\}_{i=1}^n \), where \( x_i \in \mathbb{R}^n \)
was the \( i \)-th input feature vectors, \( y_i \in \{+1, -1\} \) was the class
label of \( x_i \), \( n \) was the number of all samples. Using linear soft
interval support vector machines to construct optimal classifi-
cy hyperplanes should be satisfied:

\[
\begin{align*}
\mathbf{w} \cdot x_i + b & \geq 1 - \xi_i, & y_i = +1, \\
\mathbf{w} \cdot x_i + b & \geq -1 + \xi_i, & y_i = -1.
\end{align*}
\]

Two constraint conditions are merged into:

\[ y_i[\mathbf{w} \cdot x_i + b] - 1 \pm \xi_i \geq 0, \quad i = 1, 2, \ldots, n; \]

where the \( w \) was weight vector, \( b \) was a scalar, and the \( \xi_i \geq 0 
was slack variables. And in the case of nonlinearity, it was
necessary to map the input \( x_i \) to a new high-dimensional fea-
ture set \( \varphi(x_i) \) through a nonlinear mapping \( \varphi \). Therefore, the
classification constraint was converted to as follows:

\[ y_i[(\mathbf{w} \cdot \varphi(x_i)) + b] - 1 + \xi_i \geq 0, \quad i = 1, 2, \ldots, n. \]

The objective function was

\[
\min \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_{i=1}^n \xi_i;
\]

where \( C \) was the regularization parameter, which determined
the trade-off between experience risk and complexity. To solve
the problem the following Lagrange function was constructed

\[
L(w, b, \xi, \alpha, \beta) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_{i=1}^n \xi_i - \sum_{i=1}^n \beta_i \xi_i - \\
\sum_{i=1}^n \alpha_i [y_i((\mathbf{w} \cdot \varphi(x_i)) + b) + \xi_i - 1];
\]

where \( \alpha_i \geq 0, \beta_i \geq 0 \) were Lagrange multipliers. The partial
derivatives of \( w, b \) and \( \xi_i \), and they were equal to 0, which can
be obtained as follows:

\[
L(w, b, \xi, \alpha, \beta) = \sum_{i=1}^n \alpha_i - \\
\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \varphi(x_i) \cdot \varphi(x_j).
\]

According to the Karush-Kuhn-Tucker (KKT) condition, the
solution of the optimization problem must also be satisfied as
follows:

\[
\begin{align*}
\alpha_i[y_i((\mathbf{w} \cdot \varphi(x_i)) + b) + \xi_i - 1] &= 0, \\
\beta_i \xi_i &= 0 \Rightarrow (C - \alpha_i) \xi_i = 0, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

According to the Mercer condition, kernel function
\( K(x_i, x_j) \) can replace the inner product in the feature space.
Therefore, the soft interval nonlinear support vector machine
can be transformed into the following dual programming
problem:

\[
\max \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j K(x_i, x_j),
\]

\[
s.t. \left\{ \begin{array}{l}
\sum_{i=1}^n y_i \alpha_i = 0, \\
0 \leq \alpha_i \leq C, \quad i = 1, 2, \ldots, n.
\end{array} \right.
\]

After solving the above problem, the optimal classification
function was obtained as follows:

\[
f(x) = \text{sgn} \left( \sum_{i=1}^n \alpha_i y_i K(x_i, x) + b \right).
\]

In this paper, the kernel function of the support vector ma-
chine was a radial basis function (RBF) which shown as follows:

\[
K(x_i, x) = \exp(\gamma \|x_i - x\|^2), \quad \gamma > 0;
\]

A
where $\gamma$ was the kernel parameter.

Therefore, the regularization parameter $C$ and RBF kernel parameter $\gamma$ are two parameters that can affect the classification efficiency of SVM. In this paper, the WOA is used to optimize these parameters. A fitness function is proposed to evaluate the optimization effect of the algorithm which is shown as follows:

$$F_{\text{fit}}(C_i, \gamma_i) = 1 - \frac{1}{1 + a_{\text{cv}}(C_i, \gamma_i)}; \quad (30)$$

where $a_{\text{cv}}(C_i, \gamma_i) \in [0, 1]$ is the cross validation accuracy of SVM using the parameters $C_i$ and $\gamma_i$. The flow chart for optimization of SVM parameters by WOA is shown as Fig. 3.

4. CASE STUDY 1

In this section, the fault data provided by Society for Machinery Failure Prevention Technology (MFPT) was used to verify the effectiveness of the proposed method. The fault data sets contained the data of outer race fault, inner race fault and normal state. The location of outer race fault and inner race fault are shown in Fig. 4. Figure 5 displays the time domain waveform of raw signals for outer race fault, inner race fault and normal state. The rotation frequency was 25 Hz in the experiment. There were seven working conditions for the outer race fault data, including 25, 50, 100, 150, 200, 250 and 300 lbs of load. There were seven working conditions for the inner race fault data, including 0, 50, 100, 150, 200, 250 and 300 lbs of load. The load of normal state data was 270 lbs. The data of the above each mode was divided into 84 samples, a total of 252 samples, half of which were used as train data, and the rest were used as test data.

The influence of parameters on RCMFE was analyzed before applying the proposed method to fault diagnosis. Firstly, the influence of different time delay $d$ on RCMFE was analyzed. The analysis result of different time delay $d$ on RCMFE...
was displayed in Fig. 6(a). As shown in Fig. 6(a), the abscissa was standard deviation and the ordinate was FCID. After standardization of FCID and standard deviation, the ranges of values both were 0 to 1. According to the description of parameter selection in Section 2, when FCID was 1 and standard deviation was 0, the optimal result was obtained. Therefore, several concentric circles were drawn with (0, 1) as the centre of the circle, and scattered points calculated with different parameters fell into the concentric circle. The closer the scatter point was to the centre of the circle, the better the parameter was. As can be seen from Fig. 6(a), when time delay \( d \) took 1, the scattering point fell at the centre of the circle, which showed that the result was the best. So, the time delay is set to 1. Moreover, it was found that the greater the value of \( d \), the worse the ability to reflect fault characteristics and stability of RCMFE.

The influence of embedding dimension \( m \) on RCMFE was analyzed in this section. The analysis results for different embedding dimension \( m \) on RCMFE are shown in Fig. 6(b). When \( m \) equals 1, the value of FCID reached the maximum, but the standard deviation reached the maximum simultaneously. Therefore, for the sake of stability, \( m \) cannot take 1. The standard deviation was very small and stable when \( m \) was in the range of 2 to 9, and the value of FCID was the largest when \( m = 2 \). Therefore, the embedding dimension \( m \) was set to 2.

The influence of different similarity tolerance \( r \) on RCMFE was analyzed in this section. The analysis results of different similarity tolerance \( r \) on RCMFE are shown in Fig. 6(c). When \( r \) increased gradually from 0.01, the distribution of scatter points first approached the centre of the circle, and then gradually moved away. It can be found from the graph that the scatter point at \( r = 0.03 \) was closest to the centre of the circle. Therefore, the similarity tolerance was set to 0.03 in the case.

After parameters analysis of RCMFE, the proposed method was employed to diagnose the bearing fault data mentioned above. Then the RCMFE of all the samples were extracted as the fault feature of bearings. The feature vector composed of RCMFE was used as the input of the ISVM, and then the bearing fault diagnosis was carried out. The WOA parameter settings in the ISVM are shown in the Table 1. After optimization, the optimized \( C \) and \( \gamma \) were 6.6872 and 0.0073, respectively. The diagnosis result is presented in Fig. 7. As shown in the Fig. 7, the fault diagnosis results were in good agreement with the actual state. After calculated, the recognition accuracy of bearing fault pattern was 100%. It is illustrated that the proposed method was effective for fault diagnosis of rolling bearings.

To verify the advantage of RCMFE in fault detection for rolling bearings, two typical and popular feature parameters of multiscale fuzzy entropy (MFE) and multiscale sample entropy (MSE) were selected as the fault features. The parameters of MFE and MSE were consistent with RCMFE. After calculation, the feature vectors of MFE and MSE were used as input for the ISVM, respectively. The diagnosis results of using MFE and MSE as fault feature are presented in Fig. 8. As shown in Fig. 8(a), two samples are misdiagnosed. One inner race fault sample is misdiagnosed as outer race fault and the other outer race fault sample is misdiagnosed as normal state. After calculation, the recognition accuracy of the bearing fault pattern with using MFE-ISVM was 98.4127%. As shown in Fig. 8(b), three samples are misdiagnosed. Two inner race fault samples are misdiagnosed as outer race fault, and the other outer race fault sample is misdiagnosed as inner race fault. The recognition rate of using MSE-ISVM was 97.6190%. Therefore, it is illustrated that the performance of RCMFE in bearing fault diagnosis was more competitive than the performance of MFE and MSE.

To further verify the advantages of the WOA in the ISVM,
three traditional optimization methods of genetic algorithm (GA), particle swarm optimization (PSO) and artificial bee colony algorithm (ABC) were applied to optimize the parameters of the SVM. The input parameters of the three algorithms were consistent with the WOA. The performance of these methods is displayed in Table 2. The optimization iteration process of these methods is presented in Fig. 9. As shown in Fig. 9, the fitness functions reached maximum and kept unchanged when the four optimization methods iterated to a certain number of times. However, the WOA was faster than the other algorithms in searching for the optimal solution. The optimal solution was found in the second iteration. It shows that the performance of WOA was more competitive than other traditional optimization methods.

To prove the effectiveness of the ISVM, three traditional classification method of naive bayesian network (NBN), artificial neural network (ANN) and K-nearest neighbors (KNN) were employed to classify the bearing faults combined with RCMFE. Figure 10(a) displayed the diagnosis result of using the NBN as the classification method, and there were twelve samples being misdiagnosed by using this method. Three of the normal state samples were misdiagnosed as outer race faults, and nine of the outer race fault samples were misdiagnosed as the normal state. The diagnosis result of using ANN as the classification method is presented in Fig. 10(b). There were six samples of outer race fault being misdiagnosed as inner race fault in the Fig. 10(b). The diagnosis result of using the KNN as a classification method is shown as Fig. 10(c). As shown in Fig. 10(c), eight samples were misdiagnosed. Two samples of normal state were misdiagnosed as outer race fault state, and six samples of outer race fault state were misdiagnosed as normal state. The recognition rates of these methods were presented in Table 3. It can be found from Table 3 that the recognition rates of the ISVM were higher than other traditional classification methods. It is illustrated that the performance of the ISVM was more competitive than other traditional methods.

The influence of different numbers of training samples on diagnostic recognition rate was analyzed. As shown in Fig. 11, with the decrease of training samples, the recognition rate of the ISVM decreased, but all recognition rates remained above 95%. In contrast, with the decrease of training samples, the overall recognition rate of other methods decreased more seriously. It demonstrates that the ISVM proposed in this paper was more robust to training samples than other methods. The ISVM can still maintain a high recognition rate in the case of small samples, which was more suitable for practical application.

In addition, the influence of the different scale factor on diagnostic recognition rate was analyzed in this section. As
shown in Fig. 12, the recognition rate increased with the increase of scale factor. When the scale factor was greater than or equal to 3, the recognition rate reached 100%. It shows that the fault recognition rate reached the maximum value and remained unchanged after the scale factor was greater than 3. If the scale factor was too large, the amount of calculation increased. Therefore, it was more appropriate to set the scale factor at $3 \sim 6$.

5. CASE STUDY 2

In Case 1, three fault patterns of rolling bearings were diagnosed by the proposed method and the results show that the method was effective. However, the data in Case 1 had fewer fault patterns and it was not difficult to diagnose the faults in this case. In this section, the fault data of the Case Western Reserve University were also employed to verify the performance of the proposed method in the bearing fault detection, and the test rig is shown in Fig. 13. In the last case, the proposed method was just used to diagnose the data of different fault pattern. In this case, the data with different fault and different fault degrees was used to verify the effectiveness of the proposed method in the bearing fault detection. The data contained signals of three fault modes of bearings, such as outer race fault, inner race fault and rolling element fault. Each fault pattern has three levels of fault: 0.007 inches, 0.014 inches and 0.021 inches. So, there were nine kinds of fault pattern and one kind of normal state data. The data for each state had four working conditions. The details of the data are shown in Table 4. Figure 14 displays the time domain waveform of ten patterns row signal under working condition 1. The data of each state was divided into 40 samples, a total of 400 samples, half of which were used as train data, and the rest were used as test data.

<table>
<thead>
<tr>
<th>Pattern number</th>
<th>Fault level (inch)</th>
<th>Fault location</th>
<th>Working condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.007</td>
<td>Rolling element</td>
<td>Working condition 1: 0 horsepower load, 1797 rpm speed;</td>
</tr>
<tr>
<td>2</td>
<td>0.007</td>
<td>Inner race</td>
<td>Working condition 2: 1 horsepower load, 1772 rpm speed;</td>
</tr>
<tr>
<td>3</td>
<td>0.007</td>
<td>Outer race</td>
<td>Working condition 3: 2 horsepower load, 1750 rpm speed;</td>
</tr>
<tr>
<td>4</td>
<td>0.014</td>
<td>Rolling element</td>
<td>Working condition 4: 3 horsepower load, 1730 rpm speed.</td>
</tr>
<tr>
<td>5</td>
<td>0.014</td>
<td>Inner race</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.014</td>
<td>Outer race</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.021</td>
<td>Rolling element</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.021</td>
<td>Inner race</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.021</td>
<td>Outer race</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Normal state</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The proposed method was employed to diagnose the bearing fault data mentioned above. The fuzzy power and embedding dimension of refined composite multiscale fuzzy entropy were both set to 2, and the tolerance was 0.03. The scale factor of refined composite multiscale fuzzy entropy was set to 6. Then, the feature vector composed of RCMFE was extracted as the input of the ISVM, and the bearing fault diagnosis was carried out. The WOA parameter settings in the ISVM were consistent with Case 1. After optimization, the optimized $C$ and $\gamma$ were $9.3051$ and $6.4356$, respectively. The diagnosis result was presented in Fig. 15. As shown in Fig. 15, one sample of fault state 4 was misdiagnosed as fault state 5. The recognition rate was 99.5% after calculation, which showed that the proposed method was effective for bearing fault detection.

As in Case 1, MFE and MSE were used as fault features for diagnosis and the diagnosis results are presented in Fig. 16. As shown in Fig. 16(a), there were four samples misdiagnosed. In details, one sample of pattern 1 was misdiagnosed as pattern 7, two samples of pattern 4 were misdiagnosed as pattern 2 and pattern 7 respectively, one sample of pattern 7 was misdiagnosed as pattern 2. As shown in Fig. 16(b), there were six samples misdiagnosed, in which two samples of pattern 4 and two samples of pattern 7 were misdiagnosed as pattern 2, one sample of pattern 4 was misdiagnosed as pattern 1, one sample of pattern 5 was misdiagnosed as pattern 4. After calculation, the recognition accuracy of bearing fault pattern using the MFE-ISVM and MSE-ISVM was 98% and 97% respectively. Therefore, it was illustrated that the performance of the RCMFE in bearing fault diagnosis is more competitive than the performance of the MFE and MSE.

To prove the optimization efficiency of the WOA, the traditional optimization methods such as GA, PSO and ABCA were also applied to optimize the parameters of the SVM. The input parameters of the three optimization methods were the same with those of the WOA. The optimization iteration process of these methods was presented in Fig. 17. As shown in Fig. 17, the iteration times of the WOA, ABC, GA and PSO to reach the optimal fitness function were 2, 2, 31 and 6 respectively. The optimal fitness of the WOA is higher than that of other methods. The optimization results and recognition accuracy of the above methods are shown in Table 5. The recognition rates of the WOA-SVM, GA-SVM and PSO-SVM were 99.5%, while that of the ABC-SVM was only 85.5%. In addition, the run time of the WOA-SVM was shorter than the other methods. It shows that the performance of the WOA was more competitive than other traditional optimization methods.

To prove the effectiveness of the ISVM, the NBN, KNN and ANN were employed as classifiers respectively to detect the bearing. The fault diagnosis results of the above methods are displayed in Fig. 18. As shown in Fig. 18, most of the misdiagnosis samples were concentrated in pattern 4 and pattern 7. It shows that the RCMFE values had some confusion between the samples of pattern 4 and samples of pattern 7. The recognition rates of these methods are presented in Table 6. It can be found from the Table 6 that the recognition rates of the ISVM were higher than other traditional classification methods. It is illustrated that the performance of the ISVM was more competitive than other traditional methods.

Like Case 1, the influence of different numbers of training samples on diagnostic recognition rate was analyzed. As shown in Fig. 19, with the decrease of training samples, the recognition rates of all methods showed a downward trend. But compared with other methods, the ISVM had a gentle down-
ward trend, and the recognition rate was always higher than other methods. It demonstrates that the ISVM proposed in this paper was more robust to training samples than other methods. The ISVM can still maintain a high recognition rate in the case of small samples, which was more suitable for practical application.

Similarly, the influence of different scale factors on diagnostic recognition rates was analyzed in this section. As shown in Fig. 20, when the scale factor was greater than or equal to 3, the fault recognition rate reached 98% and above. It was shown that when the scale factor was greater than 3, the fault recognition rate tended to be stable. If the scale factor was too large, the amount of calculation increased. Therefore, the analysis results were consistent with Case 1, and in this case, it was more appropriate to set the scale factor at 3 ~ 6.

6. CONCLUSIONS

In this article, a novel method based on the RCMFE and ISVM is proposed for rolling bearings fault diagnosis under variable working conditions. To deal with the non-stationary and non-linear characteristics of bearing fault vibration signal, the RCMFE is employed to provide representative features. Further, the ISVM based on the WOA is proposed to identify the fault pattern of rolling bearing. The WOA is used to optimize the regularization parameter and the RBF kernel parameter of the SVM, which can affect the classification efficiency of the SVM. After verification of the proposed method using bearing fault data of the MFPT and Case Western Reserve University, the following conclusions are drawn.

1. The proposed method is a simple and effective tool for the fault diagnosis of rolling bearings in variable working conditions. It has the advantages of a simple operation process, less historical data, and no manual operation. It can detect the fault of rolling bearing variable working conditions automatically. It shows that the proposed method is more versatile. These characteristics show that the proposed method is more suitable for engineering practice.

2. The RCMFE is applied to extract bearing fault features under variable operating conditions. The experimental data prove that the RCMFE can effectively reflect the bearing fault characteristics under variable working conditions. This feature extraction method is simple and does not need other processing such as dimension reduction, which is conducive to application in engineering practice. The performance of the RCMFE in bearing fault diagnosis is more competitive than other nonlinear dynamic analysis methods such as the MFE and MSE. The
influence of parameter setting on the application of the RCMFE in bearing fault diagnosis is analyzed, which has reference significance for the practical application of engineering.

3. The experimental results show that the ISVM proposed in this paper is an effective classifier for bearing fault diagnosis under variable working conditions. The ISVM has proved to be more effective than traditional classification methods and requires less training samples. Moreover, the WOA in the ISVM has a better optimization effect and a faster convergence speed than traditional optimization algorithms such as the GA, PSO and ABC.

REFERENCES


Quality-Factor Enhancing Locations for Substrate Mounted Resonators

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An important but often overlooked factor that affects the performance of a meso/micro electro mechanical vibratory sensor is the structural interaction between the sensor’s resonator and the substrate on which it is mounted. Situating resonators at node points eliminates this interaction and thereby helps to improve a resonator’s quality-factor for a particular mode of vibration. This paper addresses the problem of locating a single degree of freedom spring-mass resonator on a generic cantilever substrate. The loci of natural frequencies obtained when the resonator’s mounting location is varied are developed, and the nodal locations are identified. Thereafter a method to obtain these locations from the characteristic equation without solving the associated eigenvalue problem is described. Lookup tables detailing the nodal locations and the corresponding natural frequencies for various resonator parameters are presented. It is found that at these special nodal locations, the magnitude of the power transmitted through anchors is negligible, which ensures minimal structural interaction between the resonator and the substrate.

1. INTRODUCTION

Mechanical resonators are high performance devices designed to generate sinusoidal signals at precise frequencies. To excite these vibrations, energy is pumped into the device as sinusoidal forcing waves at the resonator’s natural frequency. The resulting large amplitude resonant oscillations are subsequently utilized to generate the required signals.

The performance of micro/meso-scale resonators such as the ones used in Coriolis vibratory gyroscopes depends on the Q-factor of the resonators’ operating mode of vibration.\(^1,2\) The accuracy of measurements made by such gyroscopes improves with an increase of its Q-factor and hence it is desirable to fabricate resonators with Q-factors as high as possible. To realize high Q-factors, it is a common engineering practice to choose quartz based meso-scale resonators designs.\(^3\) Although the Q-factor associated with quartz is extremely high (reported to be over one billion),\(^4\) in reality, several factors reduce the effective Q-factor of a resonator mode. The various factors that affect the Q-factor of a resonator’s vibratory mode are summarized in Table 1. The resultant Q-factor of the resonator’s mode of operation can be expressed as a sum of the contributions from each of these factors in the following form:\(^5\)

\[
\frac{1}{Q} = \frac{1}{Q_{\text{material}}} + \frac{1}{Q_{\text{pressure}}} + \frac{1}{Q_{\text{TED}}} + \frac{1}{Q_{\text{anchor}}} + \frac{1}{Q_{\text{other}}}.
\]

Therefore the resultant Q-factor of a resonator mode is dictated by the factor with the least contributory Q-factor. Among these, the reduction of Q-factor due to improper anchorage of the resonator to its substrate motivates this work.

The type, the number and the location of the anchors attaching the resonator to a substrate significantly affects the Q-factor of the resonator’s mode of vibration.\(^6,7\) An improper anchor location leads to structural interaction between the resonator and the substrate which degrades resonator performance. Several researchers have investigated anchor induced reduction of Q-factor and have developed methods to predict the energy transmitted away from the resonator through the anchors.\(^7-9\) The energy carried by structural waves through the anchor into the substrate on which the resonator is mounted per cycle, is expressed for a cycle of harmonic motion with a period \(T\) as:

\[
\pi L = \frac{1}{T} \int_0^T F \cdot \dot{u} \, dt;
\]

where \(F\) is the vector of forces and moments acting at the anchor location and \(\dot{u}\) denotes the vector of velocities of the anchor point.\(^10\) The energy lost at an anchor depends on the velocity of the attachment point. Hence the obvious design strategy to be adopted to minimize anchor losses (and to maximize \(Q_{\text{anchor}}\)) is to locate the resonator such that the point of attachment is a node point (of the mode shape corresponding to the resonant frequency at which the resonator is designed to operate) of the entire assembly comprising the resonator as well as the substrate to which it is mounted.

Several researchers have investigated the direct problem of evaluating the vibratory characteristics of combined systems comprising masses elastically mounted to continuous media. Free vibrations of beams with elastically mounted masses has been explored in the past.\(^11–14\) Multi-span beams carrying multiple spring mass systems have also been investigated.\(^15,16\) Non-uniform beams carrying an arbitrary number of spring mass systems have also been investigated as well. However, the inverse problem of estimating a design parameter, such as a stiffness or a mass, so that the combined system meets certain requirements has received less attention. A few notable inverse problems investigated include the imposition of nodes at certain locations on a continuous structure by attaching resonators,\(^18–20\) constructing a physically realizable system from a beam’s eigen solutions\(^21\) and generating by design certain natural frequencies in a beam with an added mass.\(^22\)
Table 1. Summary of the various factors that contribute to the final Q-factor of a resonator.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Contributing factor</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{\text{material}}$</td>
<td>Material friction</td>
<td>The energy lost due to the friction inherent to the material constituting the resonator</td>
</tr>
<tr>
<td>$Q_{\text{pressure}}$</td>
<td>Viscosity of the gaseous medium</td>
<td>The energy utilized to overcome the resistance imparted by the gas surrounding the resonator</td>
</tr>
<tr>
<td>$Q_{\text{TED}}$</td>
<td>Thermoelastic effect</td>
<td>The loss which arises due to the localized vibratory heating of a region within the oscillating body</td>
</tr>
<tr>
<td>$Q_{\text{anchor}}$</td>
<td>Anchoring</td>
<td>The energy transmitted from the resonator to the substrate through the anchors</td>
</tr>
<tr>
<td>$Q_{\text{other}}$</td>
<td>Others</td>
<td>The dissipation due to other effects such as surface defects and Akhiezer effect</td>
</tr>
</tbody>
</table>

Table 2. Operating frequencies, Q-factors and damping levels for various high-Q resonators

<table>
<thead>
<tr>
<th>Resonator geometry</th>
<th>Frequency (MHz)</th>
<th>Quality factor, $Q$</th>
<th>Damping factor, $\zeta$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stemless disk</td>
<td>74.3</td>
<td>99000</td>
<td>5.1e-6</td>
<td>23</td>
</tr>
<tr>
<td>Disk</td>
<td>0.247</td>
<td>100000</td>
<td>5.6e-6</td>
<td>23</td>
</tr>
<tr>
<td>Hemispherical</td>
<td>0.0184</td>
<td>143000</td>
<td>3.45e-6</td>
<td>24</td>
</tr>
<tr>
<td>Dual ring</td>
<td>19.7</td>
<td>150000</td>
<td>3.33e-6</td>
<td>23</td>
</tr>
<tr>
<td>Cantilever</td>
<td>0.0083</td>
<td>403000</td>
<td>1.24e-6</td>
<td>24</td>
</tr>
<tr>
<td>Tuning fork</td>
<td>0.0022</td>
<td>800000</td>
<td>6.25e-7</td>
<td>25</td>
</tr>
<tr>
<td>Lame square beam</td>
<td>6.35</td>
<td>1700000</td>
<td>2.94e-7</td>
<td>26</td>
</tr>
</tbody>
</table>

Unlike previous investigations, this work addresses the inverse problem of finding the locations on a substrate where an attached resonator experiences no structural interaction with its substrate for an intended operating resonant mode. A designer who aims to locate a resonator on a substrate with established natural frequencies, modes and nodal locations may perhaps be tempted to locate the resonator at the node points of the (bare) substrate. However, with the addition of the resonator, the location of the node points (or the assembly) deviate from those of the bare substrate. The deviation of the nodal locations upon attachment of the resonator is less significant for resonators with low mass and becomes substantial as the resonator mass increases. The location of the (new) node points of the assembly for a particular mode of vibration depends on the location of the resonator itself. The work explores the possibility of finding locations on a substrate where an attached resonator of a predetermined mass and operating frequency makes the same location a node point of the assembly comprising the substrate and the resonator. The investigation considers the problem of locating a generic single degree of freedom (DoF) spring-mass resonator on the commonly encountered cantilever substrate. The equation of motion governing the dynamics of such a resonator-substrate assembly is presented in Section 2. Thereafter, in Section 3, the natural frequencies and mode shapes of the system are obtained by solving the eigenvalue problem, and the nodal locations are identified from their loci. In Section 4, a method is developed to calculate the nodal mounting locations from the characteristic equation without solving the eigenvalue problem. The nodal locations for various sets of resonator mass and stiffness are identified and tabulated.

Figure 1. Schematic representation of a resonator mounted on a continuous substrate.

2. THE MODEL

The Q-factors and the associated damping factors\(^1\) for various high-Q resonators as reported in literature are presented in Table 2. It is evident that the damping factors associated with such high performance systems are extremely low. The time scale on which the resonator oscillations decay is much larger than the time scale associated with the resonator’s oscillatory frequencies. The decay in the amplitude for each cycle can therefore be assumed to be negligible, and hence the vibratory characteristics of the high performance systems motivating this work is investigated in the following sections using an undamped structural model. The effect of damping is considered later in Section 5.

The system investigated here comprises a resonator mounted on a continuous elastic substrate. The resonator is modeled as a single-DoF spring-mass oscillator with a mass $M_R$ and a spring stiffness $K_R$ as represented in Fig. 1. The elastic substrate to which the resonator is mounted, is modeled as an Euler-Bernoulli beam\(^27\) of length $L$ with a flexural rigidity $EI$ (where $E$ is the modulus of elasticity of the homogeneous material constituting the beam and $I$ the area moment of inertia with respect to the neutral axis of bending) and a mass per unit length $\rho A$, where $\rho$ is the uniform material density and $A$ is the constant cross sectional area through out the length of the beam. The transverse deflection of the beam at a location $X$ along the length of the beam at any time $T$ is denoted by the variable $U(X, T)$. The resonator is assumed to be mounted at a location $X = X_0$. The displacement of mass $M_R$ is denoted by the variable $Z(T)$. In terms of the dimensionless variables, $x = \frac{X}{L}$, $u = \frac{U}{L}$, $\zeta = \frac{Z}{L}$, $x_0 = \frac{X_0}{L}$, $k = \frac{K_R L^2}{EI}$, $m = \frac{M_R}{\rho AL}$ and $t = T\sqrt{\frac{EL}{\rho AL}}$, the equations governing the motion of the system as presented in Reference\(^{12}\) are the following:

\[
\begin{aligned}
\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} &= k (z - y) \delta (x - x_0); \quad x \in [0, 1]; \\
\frac{m}{\partial t^2} + k (z - y) &= f \sin \Omega t.
\end{aligned}
\]

\(^1\)For a single-DoF oscillator, the damping factor $\zeta$ and the Q are related as $Q = 1/2 \zeta$. As $\zeta$ tends to zero, Q-factor tends to infinity.
Here the variable \( y = u( x_0, t) \) denotes the instantaneous displacement of the point of attachment of the resonator to the continuous substrate. The following boundary conditions are employed to model the clamped-free (cantilever) elastic substrate.

\[
\begin{align*}
  u(0, t) &= 0; \\
  \frac{\partial u}{\partial x}(0, t) &= 0; \\
  \frac{\partial^2 u}{\partial x^2}(1, t) &= 0; \\
  \frac{\partial^3 u}{\partial x^3}(1, t) &= 0.
\end{align*}
\]

(5)

The equations presented above governing the dynamics of the system are solved and the solutions are examined in the following section.

3. EIGEN STRUCTURE

The Galerkin’s method is employed to solve the above system of equations. Cantilever beam functions (denoted here by \( \phi_i \) where \( i = 1...n \)), obtained from the solution of the equation governing the transverse vibrations of a classic cantilever Euler-Bernoulli beam, are used as comparison functions to expand the variable \( u \). Note that these functions satisfy the boundary conditions presented in Eq. (5). The expansion:

\[
u(x, t) = \sum_{i=1}^{n} \phi_i(x) \eta_i(t);
\]

(6)

is substituted in Eqs. (3) and (4), and the resulting expressions are multiplied \( n \) times, each time by \( \phi_i \) and integrated with respect to \( x \) from \( x = 0 \) to \( x = 1 \). The obtained system of coupled ordinary differential equations, for \( f = 0 \) can be expressed in the matrix form as:

\[
M \ddot{\eta} + K \eta = 0;
\]

(7)

where the vector \( \eta \) is the transpose of the vector \( \{ \eta_1 \eta_2 \ldots \eta_n \} \). A general solution of the form:

\[
\eta = \omega^{n}\eta M;
\]

(8)

is assumed and on substitution, Eq. (7) yields:

\[
(-\lambda^2 M + K) u = 0;
\]

(9)

which is an algebraic eigenvalue problem, where \( \lambda^2 \) represents the eigenvalues and \( u \) the eigenvectors. The eigenvalue problem is solved; the results obtained are bench marked for certain parameter combinations in Appendix 1. The elements of the mass matrix \( M \) and the stiffness matrix \( K \) depend on the dimensionless system parameters \( m \) and \( k \). The values of the resonator mass \( m \) and stiffness \( k \) depend on the designed resonator and is usually decided early in the product design cycle. For fixed values of \( m \) and \( k \) as the location \( x_0 \) changes, the natural frequencies, which are the square roots of the eigenvalues, and the mode shapes (obtained from the eigenvectors using expansion (6)) of the system vary. At \( x_0 = 0 \) the natural frequencies of a standard clamped-free beam are recovered. In addition, a frequency \( \omega_r = \sqrt{\frac{k}{m}} \) corresponding to the uncoupled resonator is also obtained. The loci of the first four natural frequencies as the attachment location \( x_0 \) is varied from 0 to 1 for mass \( m = 1 \) and stiffness \( k = 150 \). The (red) straight line parallel to the \( x \) axis marks the resonator natural frequency \( \omega_r = \sqrt{\frac{k}{m}} \).

Figure 2. The loci of the lowest four natural frequencies as the attachment location \( x_0 \) is varied from 0 to 1 for mass \( m = 1 \) and stiffness \( k = 150 \). The (red) straight line parallel to the \( x \) axis marks the resonator natural frequency \( \omega_r = \sqrt{\frac{k}{m}} \).

For such a nodal location, \( y(t) = 0 \) and Eq. (4) becomes:

\[
m \frac{d^2 x}{dt^2} + k z = f \sin \Omega t.
\]

(10)

Equation (10) represents a single-DoF spring-mass oscillator with natural frequency \( \sqrt{\frac{k}{m}} \), and which is forced at the frequency \( \Omega \). Therefore, if \( x_0 \) is a nodal location, one natural frequency of the system modeled by Eq. (7) has to be \( \sqrt{\frac{k}{m}} \). Moreover, to excite this particular mode, for which \( x_0 \) is a node point, the forcing speed \( \Omega \) has to be \( \sqrt{\frac{k}{m}} \). The nodal location \( x_0 \) is that value of \( x_0 \) at which any one of the natural frequencies equals \( \sqrt{\frac{k}{m}} \). The frequency \( \omega_r = \sqrt{\frac{k}{m}} \) is marked in Fig. 2 using the (red) solid line-type. The nodal location is obtained from the loci as the value of \( x_0 \) at which any locus intersects this horizontal line. The required nodal location for the pair of parameters \( m = 1 \) and \( k = 150 \) is found to be \( x_0 = 0.5692 \).

The loci of the natural frequencies when location \( x_0 \) is varied for the case when \( m = 1 \) and \( k = 300 \) is presented in Fig. 3. The locus of the second mode intersects the frequency \( \omega_r = \sqrt{\frac{k}{m}} \) at two distinct points, \( x_0 = 0.6697 \) and \( x_0 = 0.9430 \), and therefore there are two resonator mounting locations along the
substrate where the attachment point of the resonator to the substrate is a node point. The mode shapes corresponding to the lowest four natural frequencies of the system when the resonator is attached to these locations are presented in Fig. 4(a) and Fig. 4(b) respectively. The (red) solid line-type corresponds to the mode shape for which the attachment point of the resonator is a node point. The resonator locations are marked on these mode shapes with the (red) circle marker.

For resonator designs with parameters \( m \) and \( k \) that correspond to multiple possible nodal locations \( x_0 \) as presented in Fig. 3, the selection of one mounting location over the other is the designer’s choice. However in most engineering cases, it is required that the oscillations of the resonator are of as high amplitude as possible when compared to the oscillations of the substrate. In other words, the total energy associated with the vibrating substrate (which is the sum of the kinetic and strain energies of the cantilever beam) should be as small as possible when compared to the oscillations of the cantilever beam. A technique that can aid designers to choose one nodal location over the other is detailed in Appendix 2.

Figure 3. The loci of the lowest four natural frequencies as the attachment location \( x_0 \) is varied from 0 to 1 for mass \( m = 1 \) and stiffness \( k = 300 \). The (red) straight line parallel to the \( x \) axis marks the resonator natural frequency \( \omega_r = \sqrt{\frac{k}{m}} \).

Figure 4. The first four mode shapes corresponding to the nodal locations marked on Fig. 3; (a) when \( x_0 = 0.6697 \) and (b) when \( x_0 = 0.9430 \).

Figure 5. The loci of the lowest four natural frequencies as the attachment location \( x_0 \) is varied from 0 to 1 for mass \( m = 1 \) and stiffness \( k = 550 \). The (red) straight line parallel to the \( x \) axis marks the resonator natural frequency \( \omega_r = \sqrt{\frac{k}{m}} \).

4. NODAL ATTACHMENT LOCATIONS

The system under investigation has two points of fastening: the clamped boundary of the elastic substrate (at \( x = 0 \)) and the location \( x_0 \) where the resonator is attached to the substrate. The displacement \( u(0, t) \) is zero due to the imposed boundary conditions. Hence for minimum anchor loss, the condition that needs to be satisfied in locating an already designed resonator, of predefined values \( m \) and \( k \), is to select \( x_0 \) such that it coincides with a node point of the combined assembly of the resonator and the substrate. A technique to obtain such nodal locations without tracing the loci of the natural frequencies is presented next.

The characteristic polynomial, associated with the eigenvalue problem presented in Eq. (9), is obtained by computing the determinant of the matrix \((-\lambda^2 M + K)\) and is of degree \( 2(n + 1) \). The characteristic equation is of the form:

\[ \lambda^{2(n+1)} + P_{n+1} - \lambda^{2(n+1)}\lambda^{-2} + \ldots + P_0 = 0. \]  

For instance, if \( n = 3 \) we obtain \( \lambda^8 + p_3\lambda^6 + p_2\lambda^4 + p_1\lambda^2 + p_0 = 0 \). The \( p_i \) of Eq. (11) are functions of the system parameters \( m \), \( k \) and \( x_0 \). For a particular resonator (with fixed values of \( m \) and \( k \)), the only system parameter that the coefficients of the characteristic equation depend on is the anchor location \( x_0 \). At the nodal location, \( x_0 = x_{0n} \), as explained earlier, one of the natural frequencies of the system must equal the uncoupled resonator frequency \( \omega_r = \sqrt{\frac{k}{m}} \). To find the value of \( x_0 \) at which the attachment location of the resonator to the substrate becomes
In general, as the value of the frequency \( \omega \) increases, a chart depicting the number of possible nodal locations depends on the values of these resonator parameters. The variation of the nodal locations as stiffness \( k \) changes for mass \( m = 1 \). These parameters correspond to the (red) dashed line of Fig. 6. Regimes P and S have one nodal location, Q has two and R has none.

As was seen in Section 3, it is possible that there exists multiple solutions for \( x_0 \) (i.e. multiple nodal locations) for a resonator of mass \( m \) and stiffness \( k \). The number of such nodal locations depends on the values of these resonator parameters. In general, as the value of the frequency \( \omega \) increases, the number of possible solutions for \( x_0 \) within the regime \( 0 < x < 1 \) also increases.

A chart depicting the number of possible values of \( x_0 \) for various combinations of parameters \( m \) and \( k \) is presented in Fig. 6. The hatched regions represent parameter combinations with at least one possible value of \( x_0 \). There is an interesting unhatched region (marked R) banked by hatched regions on both sides. For combinations of values of \( m \) and \( k \) within this region, there are no locations \( x_0 \) along the substrate where the resonator can be mounted so that the attachment point is a node point for any mode shape of the system. The (red) dashed line corresponds to parameter \( m = 1 \). For this value of \( m \), there are various regimes of stiffnesses within \( 0 < k < 1000 \) for which the nodal location \( x_0 \) can have either no solution, one solution or two solutions. These nodal locations as a function of stiffness \( k \) are presented in Fig. 7. The stiffness values corresponding to regions marked P and S have one possible nodal location \( x_0 \). Q has two and R has none. The parameters chosen to develop Figs. 2, 3 and 5 fall in the regions marked P, Q and R of Fig. 7, respectively.

The nodal attachment locations \( x_0 \) for various values of \( k \) when \( m = 0.05 \) is presented in Table 3. For \( k = 1, 10, 50 \) and 300, the system has one nodal point each. However, for \( k = 150 \), there are four possible attachment locations at which the mounting point is a node point for the assembly comprising the resonator and the substrate. The table also presents the lowest four natural frequencies of the system for each of these combinations of mass \( m \) and stiffness \( k \).

The nodal locations and natural frequencies for the cases when \( m = 0.1, 0.25, 0.5, 1 \) and 2 are presented in Tables 4, 5, 6, 7, and 8 respectively. As evident in Figs. 5, 6 and 7, a nodal location \( x_0 \) does not exist for all combinations of resonator parameters \( m \) and \( k \). For those values of \( m \) and \( k \), which do not yield a nodal location \( x_0 \), the corresponding table entries for \( x_0 \) are marked ‘x’ and the natural frequencies are left blank.

Throughout this investigation, the number of basis functions \( n \) was taken to be 6. The resulting system had 7 natural frequencies (6 of which originated from the previously mentioned basis functions \( \phi_i \) and 1 from the resonator DoF) out of which the lowest four changed by less than 0.0005 percent when \( n \) was increased from 5 to 6. For instance, when \( m = 1 \) and \( k = 150 \), as \( n \) was incremented from 5 to 6, the value of \( x_0 \) changed negligibly and the fourth natural frequency, \( \omega_4 \) decreased by just 0.0002 percent from 62.3425 to 62.3423.

### Table 3. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass \( m = 0.05 \). The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

<table>
<thead>
<tr>
<th>Stiffness (k)</th>
<th>Nodal locations</th>
<th>Natural frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1460, 3.5149</td>
<td>( \omega_1 = 4.4721 )</td>
</tr>
<tr>
<td>10</td>
<td>0.6118, 3.4341</td>
<td>( \omega_2 = 14.1421 )</td>
</tr>
<tr>
<td>50</td>
<td>0.2024, 3.5148</td>
<td>( \omega_3 = 31.6238 )</td>
</tr>
<tr>
<td>150</td>
<td>0.4301, 3.4931</td>
<td>( \omega_4 = 54.7723 )</td>
</tr>
</tbody>
</table>

### 5. Power Transmitted Through the Anchor

The energy transmitted per second through the anchors is a measure of the structural interaction between a resonator and...
Table 4. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass \( m = 0.1 \). The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

<table>
<thead>
<tr>
<th>Stiffness ( k )</th>
<th>Nodal locations</th>
<th>Natural frequencies</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
<td></td>
<td>3.4246</td>
<td>22.6038</td>
<td>61.6970</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( x_0 )</td>
<td>( 10.0000 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>( x_0 )</td>
<td>( 20.5303 )</td>
<td>38.7298</td>
<td>65.9878</td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>( x_0 )</td>
<td>( 19.8956 )</td>
<td>54.7723</td>
<td>67.8976</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>( x_0 )</td>
<td>( 20.3772 )</td>
<td>57.7723</td>
<td>67.9836</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>( 21.9757 )</td>
<td>57.7723</td>
<td>66.4487</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 20.9090 )</td>
<td>57.7723</td>
<td>77.3137</td>
<td></td>
<td></td>
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</table>

Table 5. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass \( m = 0.25 \). The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

<table>
<thead>
<tr>
<th>Stiffness ( k )</th>
<th>Nodal locations</th>
<th>Natural frequencies</th>
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<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
<td></td>
<td>3.4547</td>
<td>19.8956</td>
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</tr>
<tr>
<td>10</td>
<td>( x_0 )</td>
<td>( 3.3885 )</td>
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<td></td>
</tr>
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<td>( x_0 )</td>
<td>( 3.1829 )</td>
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<td></td>
</tr>
<tr>
<td>300</td>
<td>( x_0 )</td>
<td>( 3.0214 )</td>
<td></td>
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</table>

Table 6. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass \( m = 0.5 \). The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

<table>
<thead>
<tr>
<th>Stiffness ( k )</th>
<th>Nodal locations</th>
<th>Natural frequencies</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
<td></td>
<td>3.4641</td>
<td>19.8956</td>
<td>54.7723</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( x_0 )</td>
<td>( 3.4547 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>( x_0 )</td>
<td>( 3.1569 )</td>
<td></td>
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<td></td>
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<tr>
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<td>( x_0 )</td>
<td>( 3.1569 )</td>
<td></td>
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<td></td>
</tr>
<tr>
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<td>( x_0 )</td>
<td>( 3.0505 )</td>
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Table 7. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass \( m = 1 \). The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

<table>
<thead>
<tr>
<th>Stiffness ( k )</th>
<th>Nodal locations</th>
<th>Natural frequencies</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
<td></td>
<td>3.4641</td>
<td>19.8956</td>
<td>54.7723</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( x_0 )</td>
<td>( 3.5045 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>( x_0 )</td>
<td>( 3.5045 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>( x_0 )</td>
<td>( 3.5045 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>( x_0 )</td>
<td>( 3.5045 )</td>
<td></td>
<td></td>
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<td></td>
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Table 8. Nodal locations and the lowest four natural frequencies for various resonator stiffnesses when resonator mass \( m = 2 \). The frequency corresponding to the mode for which the resonator attachment location is a node point, is underscored.

<table>
<thead>
<tr>
<th>Stiffness ( k )</th>
<th>Nodal locations</th>
<th>Natural frequencies</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>( x_0 )</td>
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<td>3.4641</td>
<td>19.8956</td>
<td>54.7723</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>( x_0 )</td>
<td>( 3.4641 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>( x_0 )</td>
<td>( 3.4641 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>( x_0 )</td>
<td>( 3.4641 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>( x_0 )</td>
<td>( 3.4641 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 8. (a) The frequency of forcing, and the magnitudes of (b) the force transmitted through the anchor, of (c) the velocity of the anchorage location and of (d) the power transmitted by the force, as the attachment location of the resonator on the substrate is varied from \( x_0 = 0 \) to \( x_0 = 1 \) for three different sets of damping coefficients \( \alpha = 0.0122, \beta = 0 \) (marked by the dashed line type and corresponds to a Q-factor of 1000 for the mode under forcing), \( \alpha = 0.0122, \beta = 8.165 \alpha = 5 \) (marked by the solid line type and corresponds to a Q-factor of 500), and \( \alpha = 0.0122, \beta = 1.635 \alpha = 4 \) (marked by the dot line type and corresponds to a Q-factor of 333).

the substrate on which it is mounted. It is important for the performance of vibratory sensors, such as the ones motivating this investigation, that this interaction is minimal. The ideal anchoring location is the value of \( x_0 \) for which the amplitude of the power transmitted through the anchor is minimum. The effect of the attachment location \( x_0 \) on the force and the power transmitted through the point of anchorage (of the resonator to the substrate) is investigated here.

Vibratory sensors are operated at the resonance corresponding a mode of interest. Here we select the forcing speed \( \gamma \) to be equal to the second natural frequency of the system. Note that \( \gamma \) changes with the attachment location \( x_0 \) as presented in Fig. 8(a). The parameters \( m = 1 \) and \( k = 150 \) are selected (the loci of natural frequencies for the same set parameters was presented in Fig. 2). Note that the undamped, unforced variant of the same discretized system was investigated previously in Sections 3 and 4. A lightly dissipative version of the system presented in Section 2 is explored here. Damping is introduced directly in the discretized governing equations (Eq. (7)) and is assumed to be of the proportional type.\(^{30}\) The equations governing the forced system are obtained as:

\[
\frac{M}{\ddot{y}} + C \dot{y} + K y = Q \sin \gamma t;
\]

where \( Q \) (which is of size \( (n + 1, 1) \)) is a matrix of zeros except the \( n + 1 \)th element which is one, and \( C = \alpha M + \beta K \) where \( \alpha \) and \( \beta \) are scalar constants. The viscous damping
location \( x_0 = 0.5692 \) which is a node point for the assembly of the resonator and the substrate. It is interesting to note the occurrence of a sharp peak (labeled ‘M’ on Fig. 8(d)). If the resonator is attached at the location corresponding to the peak, the frequency of forcing which corresponds to the second mode of vibration in this case is maximum and the power transmitted has greatest amplitude.

It is observed that attaching a resonator at a nodal location minimizes the power transferred and the structural interaction between the resonator and the substrate. Therefore, to achieve the expected performance metrics of high Q-factor vibratory sensors, it is recommended to locate resonators at the node points of the assembly.

6. SUMMARY

Nodal positioning helps to minimize anchor losses and therefore enhances the overall Q-factor of high performance resonators. This work discusses the problem of positioning resonators at the nodal locations of a substrate. The technique developed is demonstrated in the problem of positioning a single-DoF spring-mass resonator on a generic cantilever substrate. The equations governing the dynamics of the system are presented and the associated eigenvalue problem is solved. The loci of natural frequencies as a function of the resonator location are developed and the nodal locations are identified. The nodal locations are calculated numerically from the characteristic equation as well. A chart depicting the number of possible nodal locations for various values of resonator parameters is presented. For certain resonator parameters, multiple nodal locations exists along the substrate. It was found that there also are additional resonator parameter combinations without a possible nodal location anywhere along the span of the substrate. The nodal locations for various combinations of resonator mass and resonator stiffness are presented in the form of lookup tables. The ideas presented here can be extended to more complicated structures. The method employed is expected to assist designers who aim to position a high performance resonator on a substrate without compromising its Q-factor.

REFERENCES


A. Anilkumar, et al.: QUALITY-FACTOR ENHANCING LOCATIONS FOR SUBSTRATE MOUNTED RESONATORS


Table 9. Comparison of frequencies obtained with existing literature for parameter $z_0 = 1.$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Author(s)</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 0.2, k = 0.1$</td>
<td>Rossit and Laura$^{24}$</td>
<td>0.6952</td>
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</tr>
</tbody>
</table>


APPENDIX 1: BENCHMARKING

Several researchers have investigated the problem of evaluating the natural frequencies of a cantilever beam carrying a mass mounted elastically at its free end. To validate the solution procedure employed in this work, the natural frequencies of this system were obtained using the procedure presented in Section 3 and are compared with those of existing literature in Table 9. The results are in excellent agreement. Note that in Reference,13 the author presents the values of parameter $\beta_i$ which need to be squared to obtain the natural frequencies being compared.

APPENDIX 2: CRITERION TO LOCATE RESONATORS WHEN MULTIPLE NODAL LOCATIONS EXISTS

The mode shapes and natural frequencies of the assembly are obtained from the eigenvalue problem presented in Eq. (9). When multiple nodal locations exists (as for the case presented in Fig. 3), to pick one over the other, it is required to compare the eigenvectors (corresponding to the eigenvalues of magnitude $\sqrt{\frac{T}{m}}$) of these cases. Let the eigenvector corresponding to a nodal location $x_0^m$ be $u_q = \{u_{q,1} u_{q,2} u_{q,3} \cdots u_{q,n} u_{q,n+1}\}^T$ where $T$ denotes the transpose operation. The first $n$ elements of this eigenvector correspond to the mode shape of the substrate and the $(n+1)$th element to the displacement $z$ of resonator mass. To compare distinct eigenvectors for the same resonator amplitude, these eigenvectors are normalized with respect to the $(n+1)$th element. Therefore the $z$-normalized mode shapes are obtained as:

$$\tilde{\psi}_q(x) = \frac{1}{u_{q,n+1}} \sum_{i=1}^{n} u_{q,i} \cdot \phi_i(x). \quad (A.1)$$

The $z$-normalized mode shapes are those configurations of the substrate which correspond to the same $z$ displacement of magnitude unity. The most energy efficient mounting location out of all possible $x_0^m$ is that for which the total energy associated with the oscillatory motion of the substrate at the corresponding $z$-normalized mode shape is minimum. The motion of the substrate at such a mode shape with the frequency $\Omega = \sqrt{\frac{k}{m}}$ can be expressed as:

$$u(x,t) = \psi(x) e^{i\Omega t}. \quad (A.2)$$

The total energy associated with the cantilever beam is the sum of its kinetic ($T$) and potential ($V$) energies and can be expressed as:

$$E_{\text{Substrate}} = T_{\text{Substrate}} + V_{\text{Substrate}} = \int_0^1 \frac{1}{2} \left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^1 \frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx. \quad (A.3)$$

Therefore the energy associated with a normalized mode shape (of Eq. (A.2)) of the substrate is:

$$E_{\text{Substrate}} = \frac{1}{2} \int_0^1 \left(-\frac{k}{m} \psi^2 + \left(\frac{\partial^2 \psi}{\partial x^2}\right)^2\right) dx. \quad (A.4)$$

Out of all the possible nodal points $x_0^m$, it is recommended to select the location for which the associated $z$-normalized mode shape returns a minimum value for the integral presented in Eq. (A.4).
Diagnosis of Ball Bearing Faults Using Double Decomposition Technique

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The rolling element bearing is one of the most significant components of any rotating machinery. However, the foremost cause of malfunction in any rotating machine is due to defects like cracks, dents, spall, pits, etc. in ball bearings. Early diagnosis of these bearing faults is highly essential to avoid an accidental shutdown of rotating machinery. In the present work, a novel technique of bearing fault diagnosis is proposed following double decomposition of the vibration activity. The experimentally recorded vibration signals are processed through two stages of decomposition viz. Empirical Mode Decomposition and Tunable Q-factor Wavelet Transform based Time-Frequency decomposition. Subsequently, sub-bands of decomposed time-frequency activity are acquired and discriminable features are computed. Fractal Dimension (FD) based features are extracted from each decomposed sub-band as complexity measures of time-frequency sub-bands. In order to classify bearing faults, a Support Vector Machine classifier is trained with acquired features and classification performance is evaluated. The results of classification reveal that the proposed double decomposition technique is a potential candidate in extracting viable vibration signatures for fault identification. The study is conducted on Case Western Reserve University bearing datasets.

1. INTRODUCTION

The rolling element bearing is a component of great significance used in manufacturing industry. These bearings undergo unexpected failures by the impacts of corrosion, wear, fatigue and overload conditions. Unexpected failure in a bearing results in sudden shut down of machinery and in turn causes heavy loss to the manufacturing industry. This is the reason why autonomous bearing fault diagnosis has gained major attention of researchers across the globe. Efficient bearing fault diagnosis using vibration signal analysis can prevent sudden failure of the bearings and cut down the cost of repair and shut down of machinery. However, vibration signal based bearing fault detection is a challenging task, since vibration signals are non-stationary and non-linear in nature and are often severely affected by the external noise from other parts of the machines. This makes it obligatory to exploit an efficient fault diagnosis methodology in order to generate highly discriminative features. Feature extraction and classification of given vibration activity are the two vital steps in the bearing fault diagnosis task.1

In the past, various signal processing techniques were proposed for diagnosing faults in the bearings. The earliest signal processing techniques of fault diagnosis were simple and were mainly dependent on calculation of statistical parameters viz. mean, kurtosis, root mean square features from the time series data. For instance, the use of normalized kurtosis and normalized skewness values as features of vibration signals was suggested by Honarvar and Martin for early fault diagnosis.2 Honarvar and Martin also used statistical moments in rolling element bearing health monitoring.3 In another work, Mechefske and Mathew proposed the use of an autoregressive model for bearing fault diagnosis purposes.4 Li and Qu developed a signal demodulation technique using the concept of cyclic spectrum and cyclic autocorrelation for recovering fault information present in the modulators of bearing vibration signals.5 Considering the multi-dimensional and multiscale nature of vibration signals, Zvokelj developed an approach by combining the Ensemble Empirical Mode Decomposition (EEMD) method with Principal Component Analysis (PCA) to analyze the non-stationary behavior of bearing signals.6 However, these methods were not capable of minimizing the effect of interference and noise generated from other parts of the machines. In addition, these methods failed to provide any additional information about the frequency content of the signals.

In order to capture information content from both time and frequency domains, researchers suggested use of time-frequency transformation techniques viz. Wigner-Ville distribution, Short Time Fourier Transform (STFT), Hilbert-Huang Transform (HHT) and Wavelet Transform (WT) for bearing fault diagnosis. For instance, Mori et al. introduced a Discrete Wavelet Transform (DWT) based feature extraction technique to predict spalling of ball bearings.7 In another work, Seker and Ayaz suggested WT based Multi-Resolution Analysis (MRA) for diagnosing faults of electric motors.8 Lou and Loparo used WT based feature vectors drawn from normalized vibration signals to train an adaptive neural-fuzzy inference system for classification purposes.9 In similar work, Yang et al. applied an HHT to decompose vibration activity of faulty bearings and extracted substantial features.10 In another work, He et al. proposed a hybrid method of Morlet wavelet filtering and sparse code shrinkage for the detection of impulses generated from bearing faults.11 Researchers also applied several other supplementary methods viz. histograms, complexity measures and entropy based schemes to retrieve fault information from
vibration activity spawned by bearings. For instance, Wyk et al. used Difference Histograms for extracting faulty features of rolling element bearing vibration activity. In another work, Hong and Liang employed normalized Lempel-Ziv complexity values to assess the fault severity of a rolling element bearing. In recent years, various modern feature extraction techniques have been proposed by the researchers. For instance, Zhao et al. employed an Empirical Mode Decomposition (EMD) technique with Approximate Entropy (AE) features for this purpose. In similar work, Dbyala and Zimroz Rolling developed a bearing diagnosing methodology based on EMD of machine vibration signals. Liu et al. introduced a novel methodology of feature extraction, stated as a Local Mean Decomposition (LMD) technique, in which vibration activity was decomposed into a series of product functions defined as the product of an amplitude envelope signal and a subsequent frequency modulated signal. The method of Hilbert Transformation (HT) supplemented by the Duffing oscillator for recognizing localized defects in ball bearings was suggested by Patel et al. A new dimension in this area was added by Klein et al. by applying image processing techniques viz. ridge tracking on the Time-Frequency Representation (TFR) of the bearing vibration signals. In another work, Ming et al. introduced a new technique of Spectral Auto-Correlation Analysis (SACA) by performing autocorrelation on the Fast Fourier Transform (FFT) transformed signals for fault diagnosis of rolling element bearings. Lei et al. suggested an improved kurto-spectrogram methodology in combination with Daubechies–Wavelet based Wavelet Packet Transform (WPT) filtering for this purpose. In another work, Li et al. suggested the Continuous Wavelet Transform (CWT) based reassigned Wavelet Scalogram (WS) technique of bearing fault diagnosis.

With the advancement of nonlinear dynamic modelling, studies advocating nonlinear parameter estimation from bearing vibration signals have been suggested in the recent past and been proven very effective in bearing fault diagnosis. Such studies extract non-linear information hidden in the vibration activity by estimating a range of nonlinear dynamic parameters viz. correlation dimension, Approximate Entropy (ApEn), Permutation Entropy (PE), and Multiscale Entropy (MSE). The advancement in signal processing techniques impelled bearing fault diagnosis research widely. Various significant and effective feature estimation methods have been developed in recent years. For instance, Liang and Faghidi proposed an intelligent fault identification method using a calculus enhanced energy operator. Caesarendra et al. developed a largest Lyapunov exponent algorithm to estimate the degree of non-linearity for low speed slow bearing condition monitoring. In order to anticipate the Remaining Useful Life (RUL) of bearings, Boskoski et al. proposed bearing fault prognostics method using Renyi’s entropy and Gaussian process model. Considering non-linearity, non-stationarity and multifractal properties of vibration signals, a method based on Local Characteristic-scale Decomposition–Teager Energy Operator (LCD–TEO) and Multifractal Detrended Fluctuation Analysis (MF–DFA) was first proposed by Liu et al. In another work, Han et al. suggested a feature extraction method using improved Fast-ICA (Independent Component Analysis) algorithm and the wavelet packet energy spectrum. In order to identify multi-faults in rotating machinery a method combining Spectral Kurtosis (SK) and Minimum Entropy Deconvolution (MED) techniques was proposed by He et al. A kurto-sis based weighted sparse model with a convex optimization technique and Multipoint optimal MED based bearing fault diagnosis techniques are proposed in recent work. Other entropy-based bearing fault diagnosis methods include composite multiscale fuzzy entropy and fusion information entropy-based feature extraction. In similar work, an improved maximum correlated kurtosis deconvolution method was suggested by Miao et al.

Due to its excellent band pass filtering characteristics, the WT has been extensively used in many fault diagnosis applications in the past. However, WT suffers from the problem of constant Q-factor i.e. it has a constant central frequency to bandwidth ratio. In order to overcome shortcomings of classical WT, Selesnick proposed the concept of Tunable Q-factor Wavelet Transform (TQWT). In another work, Cai et al. suggested the concept of Sparsity-enabled signal decomposition using TQWT for fault feature extraction of a gearbox. A TQWT has also been used in combination with a Hilbert transform for investigating the vibration features of motor bearing faults. Weak bursts of vibration signals of an angular contact bearing were extracted by Kumar et al. using TQWT along with envelope demodulation. In the present work, the capabilities of TQWT are further extended using EMD in addition to TQWT. The proposed double decomposition technique ensures frequency bound TQWT coefficient estimation and helps in detailed analysis of decomposed high frequency vibration activity. An EMD technique decomposes given vibration activity into set of signals of specific frequency known as Intrinsic Mode Functions (IMFs). Thereafter, TQWT is applied onto specific set of IMFs with high Fractal Dimension (FD) value and TQWT coefficient sub-bands are attained. Further, Higuchi’s Fractal Dimension (HFD) estimation is carried out on decomposed sub-bands for extraction of useful vibration signatures and preparation of the feature vector. The obtained feature vector is fed as input to a soft computing technique for training and validation purposes. In the present work, a Support Vector Machine (SVM) classifier is employed for fault classification purposes. The results of the classification revealed that proposed double decomposition-based fault classification methodology gives better results than the conventional methods used for bearing fault diagnosis in the past. The organization of the present work is as follows: Section 2 provides a detailed description of the material and methods being used in carrying out the present study. The description of proposed double decomposition-based feature extraction methodology is presented in section 3. The outcomes of the study are discussed in section 4 following by conclusion in section 5.

2. METHODS AND MATERIAL

2.1. Experimental Dataset

The bearing vibration data used and analyzed in the present work is obtained from the Case Western Reserve University bearing data center. Bearing faults ranging between 0.007 inches to 0.040 inches (1 inch = 25.4mm) in diameter were introduced in the rolling element, inner raceway and the outer raceway using electro-discharge. All the data files being used are in MATLAB format. Each file contains drive end and fan end vibration data along with the motor rotational speed. Four classes of data are considered in this study namely, Healthy Bearing (HB), Ball Defect (BD), Inner Race defect (IRD) and Outer Race Defect (ORD). However, the data of HB
is considered as the baseline data. Bearing data was collected at sampling rates of 48 kHz and 12 kHz from drive end and fan end. The data is registered at four different speeds of ball bearing i.e. 1730, 1750, 1772 and 1797 rpm. The faults considered in present study includes IRD, ORD at 6 o’clock position and BD with defect sizes of 7 mils, 14 mils and 21 mils (1 mil = 0.001 inch). Figure 1 shows the experimental test rig used in acquiring bearing vibration data. A sample plot of four classes of bearing vibration signals are represented in Fig. 2.

2.2. Empirical Mode Decomposition (EMD)

EMD was proposed by Huang et al. in 1998.41 EMD was developed on the assumption that any time series consists of different simple intrinsic modes of oscillations. It is a self adaptive signal decomposition technique which decomposes any time series into different oscillation modes and the original signal \( s(t) \) can be recovered by a linear superposition of empirical modes as:

\[
s(t) = \sum_{i=0}^{n} c_i(t) + r_n(t);
\]

where \( c_i(t) \) is the \( i^{th} \) empirical mode and \( r_n(t) \) is the final residue after estimation of \( n \) empirical modes. The EMD technique is very useful, particularly in the analysis of non-linear and non-stationary vibration activity, since it decomposes the original vibration signal into simple oscillatory functions called IMFs \( c_i(t) \), while following set of conditions.41

1. The number of extrema and the number of zero crossings must either be equal or differ by at most one.
2. The mean value of the envelopes defined by local maxima and minima at any point should be zero.

Given two conditions ensure that for any function to be an IMF all its local maxima should be positive and local minima should be negative.42 The decomposition of any input signal \( s(t) \) into IMFs is attained by a shifting process, which can be summarized as:

1. Identify all extrema (i.e. maxima and minima) of the given input signal \( s(t) \).
2. Connect extrema (i.e. maxima and minima) separately with cubic spline interpolation and generate upper \( (e_{\text{max}}(t)) \) and lower envelopes \( (e_{\text{min}}(t)) \) to cover complete data between envelopes.
3. Estimate the running mean \( m_n(t) \) between upper and lower envelopes. The difference between \( m_1(t) \) and the signal \( s(t) \) is the component \( I_1(t) \). If \( I_1(t) \) satisfies condition of IMF, then it is the first component of \( s(t) \).
4. If \( I_1(t) \) doesn’t satisfies the conditions, consider \( I_1(t) \) as the original signal \( s(t) \) and repeat steps 1–3 until first IMF is obtained. After repeated sifting for \( j \) times, first IMF \( I_j(t) \) (represented as \( c_1(t) \)) is obtained as:

\[
I_{1(j)} = I_{1(j-1)} - m_{1j} = c_1(t).
\]

The obtained IMF \( c_1(t) \) represents smallest temporal component of the signal \( s(t) \). Further, in order to attain other IMFs, residue \( r_1(t) \) is generated by subtracting \( c_1(t) \) from signal \( s(t) \). Here, \( r_1(t) \) is treated as the original data and the process is repeated to attain second IMF component \( c_2(t) \). The decomposition process continues until the final residue is a constant, a monotonic function or a function from which no other IMFs can be derived. All derived IMFs (i.e., \( c_1(t) \), \( c_2(t) \), ..., \( c_n(t) \)) represent specific frequency band ranging from high to low and are stationary in nature. Once decomposition process completes, original signal \( s(t) \) can be obtained as:

\[
s(t) = \sum_{n=1}^{n} c_n(t) + r_n(t);
\]

where \( n \) signifies number of IMFs, \( c_n(t) \) is the \( n^{th} \) IMF, and \( r_n(t) \) represents the residue of the decomposition process. Figure 3 and Fig. 4 represent the IMFs obtained after decomposition of healthy and ball defect vibration activity consecutively. It is evident from Fig. 3 and Fig. 4 that higher order IMFs effectively epitomize lower frequencies and lower order IMFs are suitable for higher frequencies.

Despite the fact that vibration activity of faulty bearing comprises high information content as compared to healthy bearing, a lower number of IMFs (i.e. 16 IMFs) are required in the decomposition process. Theoretically, a larger number of IMFs should be generated after decomposition of faulty bearing vibration activity, however, this not the case always, as notified by Dybala and Zimroz.43 Vibration activity suffers environmental and instrumental noise which largely reflect in some IMFs and the number of decomposed IMFs significantly depend on the amount of noise present in any vibration activity. Since, early stage fault signatures are well submerged into the system noise, it makes early stage fault diagnosis a challenging task. In recent years, researchers proposed various EMD based hybrid fault diagnosis techniques and reported substantial diagnostic results.44 Hybrid EMD techniques in addition to artificial intelligence have shown promising results in the previous studies.45 In the present work, the EMD algorithm proposed by Rato et al. is implemented to estimate IMFs of vibration signals.46 Rato et al. suggested that reducing the resolution factor reduces the number of obtained IMFs. This property is useful for sidewise analysis of different signals decomposed into the same number of IMFs. In order to optimize resolution factor, it is varied in the band of 40–50 dB with step size of 5 dB and experimental results are evaluated for attaining the highest classification performance. The proposed automatic bearing fault diagnosis methodology using EMD based double-decomposition reduces noise influences on the decision system’s performance by treating decomposed IMFs with TQWT.
Figure 2. Sample plot of four classes of bearing vibration data.

Figure 3. IMFs obtained after applying EMD on ball defect vibration signal.
The DWT is a commonly used TFR technique for bearing fault diagnosis, since, it is capable of apprehending disparities in the morphologies of bearing vibrational activity. The TQWT is the expansion of the DWT in terms of parametric values adjustment ability to attain the desired Time-Frequency response. The parameters of TQWT includes Quality-factor (Q-factor/Q), redundancy parameter (r) and decomposition levels (p). The parameter Q controls the number of oscillations, r limits unwanted excessive ringing and ensures localization of the wavelet in the time domain with preserved shape characteristics. The filters of TQWT have a non-rational transfer function and maintain a direct relationship with the frequency-domain. This property of TQWT helps in gaining perfect reconstruction ability and proficient implementation using an FFT algorithm. The decomposition of vibration activity into specific number of levels (i.e. p levels) is achieved by iteratively smearing two channel filter banks onto given vibration activity. As shown in Fig. 5, the input signal $s(n)$ (having sampling frequency $f_s$) is decomposed into low-pass ($l^0(n)$) and high-pass ($h^1(n)$) sub-band signals with altered sampling frequencies of $\alpha f_s$ and $\beta f_s$, respectively. The low-pass signal $l^0(n)$ is generated by passing $s(n)$ through low pass filter $H_0(\omega)$ followed by low-pass scaling factor, LPS $\alpha$. Similarly, high-pass signal $h^1(n)$ is generated by passing $s(n)$ through a the high-pass filter $H_1(\omega)$ followed by scaling factor, HPS $\beta$. It is established in the literature that, the value of scaling parameters should follow the condition given as: $0 < \beta \leq 1$; $0 < \alpha < 1; \alpha + \beta = 1$; to ensure controlled redundancy and perfect reconstruction of signals.

In TQWT, for a given value of Q-factor, with an increase in the number of decomposition levels $j$, the cut-off frequency and bandwidth reduce. However, with the increment in Q-factor value, the frequency response compresses and it requires more decomposition levels to cover the entire frequency range as depicted in Fig. 6. In order to attain optimum TFR, the value of Q-factor should be maintained low for little or no oscillatory vibrational activity. On the other side, a high Q-factor
value is required to analyze high oscillatory vibrational activities. However, other than CWT, WTs have less capability to adjust its Q-factor value.\textsuperscript{37} The ability of parameter adjustment offered by TQWT makes it an expedient tool for non-stationary, oscillatory vibration signal analysis.

The identical system to yield coefficients of low-pass and high-pass sub-bands (i.e. $l_p(n)$ and $h_p(n)$) with $p^{th}$ number of decomposition levels is presented in Fig. 7. The identical frequency responses of low-pass and high-pass sub-band filters (i.e. $H_p^0(\omega)$ and $H_p^1(\omega)$) with $p^{th}$ level of decomposition is presented in Fig. 7(a), and is expressed as:

$$\begin{align*}
H_p^0(\omega) &= \prod_{m=0}^{p-1} H_0\left(\frac{\omega}{\alpha^m}\right) \\
H_p^1(\omega) &= \prod_{m=0}^{p-1} H_0\left(\frac{\omega}{\alpha^m}\right) (1 - \beta) \\
&\quad \text{for other } \omega \in [-\pi, \pi] .
\end{align*}$$

Figure 7(b) shows synthesis filters engaged in the reconstruction of decomposed vibration activity. The relationship between scaling parameters (i.e. $\alpha$ and $\beta$) and TQWT input parameters is expressed by Eq. (6) and Eq. (7) as:

$$r = \frac{\beta}{1 - \alpha} ;$$

$$Q = \frac{2 - \beta}{\beta} .$$

In TQWT based techniques, identification of suitable number of decomposition levels is a crucial part. However, in present study, criterion of dominant frequency is used to attain suitable number of decomposition level. Number of decomposition levels are selected in a way that the decomposed sub-band frequencies correlated well with the frequency ranges of interest (i.e. ball spin frequency/or ball pass frequency). It is perceived in this study that higher number of decomposition levels help in extracting highly differentiable fractal features from IMFs with variable frequency ranges. Decomposing IMFs into higher number of levels (i.e. $j = 9$) gives better insight into the frequency ranges of interest. In the present work, the redundancy parameter $r$ is selected as $r = 3$. Keeping the redundancy parameter $r \geq 3$ helps in correct localization of the Wavelet’s response in time-domain and broadening the transition bands of low pass and high pass sub-band filters. Hongchao et al. notified that classical Wavelet Transform (WT) with constant Q-factor is inefficient in handling vibrational activity having significantly high/low Q-factor values.\textsuperscript{34} Therefore, in order to obtain appropriate value of Q-factor, multiple experiments are carried out on vibration data with varying Q-factor in the range of 1–50 in this work.
\[ L(m, k) = \left\{ \left( \frac{\sum_{i=1}^{\text{int}(N-m/k)} |x[m = ik] - x[m + (i - 1) \times k]|}{(N - 1) / \text{int}(\frac{N-m}{k}) \times k}} \right) \right\} \]
3. PROPOSED METHODOLOGY
OF FRACTAL FEATURE ESTIMATION

In the present work, each vibration signal is treated individually and corresponding features are estimated. On arrival of each vibration signal to the input of the algorithm, its HFD is calculated as the feature of the signal after performing EMD-TQWT based double decomposition. Once, desired HFD features are estimated from all vibration signals available in the bearing dataset, the feature vectors with varying sizes are prepared and classification is performed. The proposed methodology of double decomposition is carried out in three methodological steps. In the first step, segments of raw vibration signals are decomposed into IMFs of specific frequency using an EMD technique. Thereafter, TQWT is applied on the selected IMFs to attain time-frequency coefficients of the decomposed IMFs. Subsequently, fractal features are estimated from the decomposed time-frequency coefficients of IMFs. Estimated fractal features are arranged to represent different classes of bearing vibration activity. In order to estimate fractal information embedded in the time-frequency coefficients, Higuchi’s algorithm of FDestimation is employed in present work.

HFD is an efficient algorithm for measurement of fractal content of discrete time sequences and was proposed by Higuchi in 1988.\textsuperscript{50} For a given time series \( S = s[1], s[2], \ldots, s[n] \), various steps involved in computation of HFD can be expressed as:

Form \( k \) new time series \( S_k^{m} \) from given time series as:

\[
S_k^{m} = \{s[m], s[m+k], s[m+2k], \ldots, s[m+\text{int}\left(\frac{N-m}{k}\right)\times k]\}; \quad (9)
\]

where \( k \) and \( m \) are integers, \( k \) indicates the discrete time interval between the points, and \( m = 1, 2, \ldots, k \) represents initial time value. The length of each constructed time series is computed as \( (8) \) (see top of the previous page).\textsuperscript{51}

Where \( N \) is the length of the original time series \( S \) and \( \left\lfloor \left(\frac{N-1}{m}\right)\times k \right\rfloor \) is the normalization factor. The length of the curve for the time interval is defined as the average of \( k \) values \( L(m,k) \) for \( m = 1, 2, \ldots, k \) and is given as:

\[
L(k) = \frac{1}{k} \times \sum_{m=1}^{k} L(m,k); \quad (10)
\]

when \( L(k) \) is plotted against \( 1/k \) on double logarithmic scale, with \( k = 1, 2, \ldots, k_{\text{max}} \), the data should fall on a straight line with a slope equal to the fractal dimension of HFD when the criteria suggested by Jindal et al.\textsuperscript{50} and Upadhyay et al.\textsuperscript{51} is taken as \( r = 3 \) and number of decomposition levels are set to 9 which yields 10 sub-bands of decomposed vibration activity. The value of Q-factor is varied in the range of 1–50 and is optimized to attain the highest classification accuracy during fault diagnosis task. The obtained feature vectors are fed as an input to the soft computing technique for fault classification. Fault classification is performed using a SVM classifier on the WEKA toolkit. Different feature vectors are generated with different values of Q-factor and classification results are reported to attain highest classification efficiency. The proposed methodology of fault diagnosis is illustrated by schematic diagram presented in Fig 8.

4. RESULTS AND DISCUSSION

Feature vectors are prepared with varying parametric values and given as an input to a supervised machine learning technique. SVM, for training and validation purposes. Classification of vibrational activity is performed following 10-fold
Figure 10. SVM classification performance with varying Q-factor and feature vector size (considering EMD-40 parameters)
cross fold validation approach to avoid statistical biasing. In 10-
fold cross validation approach, featured data is divided into ten
folds and classification is performed in ten iterations. How-
ever, the final result is the average of all ten iterations. In
the present work, parameters of EMD and TQWT are varied
and classification results are analyzed against these paramet-
ric variations to attain the highest classification performance.
To prepare feature vectors from recorded vibration activity, a
total of 40 instances of raw vibration activity are considered,
out of which 4 were of healthy bearing and 12 each were of
IRD, ORD and BD. Following the criterion of highest HFD
value, four feature vectors with varying vector sizes are pre-
pared after calculating HFD features from four IMFs (40 fea-
tures), three IMFs (30 features), two IMFs (20 features) and
one IMF (10 features) respectively and ranking of the IMFs
was carried out using HFD values. Training and validation
of classifier algorithms is performed corresponding to all four
feature vectors and classification performance is reported. Ta-
ble 1 shows a sample input feature vector obtained after pro-
cessing of a single IMF with the highest HFD value.

Figure 9(a–d) shows the classification performance of SVM
classifier (i.e., classification accuracy) for different sets of
feature vectors (feature vectors with 40 features, 30 features,
20 features and 10 features) with varying parametric values.
Here, EMD,40, EMD,45 and EMD,50 imply that values of
parameter \( q_{Resol} \) and \( q_{Resid} \) are 40, 45, and 50, respectively.

<table>
<thead>
<tr>
<th>HB</th>
<th>BD</th>
<th>IRD</th>
<th>ORD</th>
<th>CLASSIFIED AS</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>HB</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
<td>0</td>
<td>0</td>
<td>BD</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>IRD</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>ORD</td>
</tr>
</tbody>
</table>

Table 2. Confusion matrix of 10-fold cross validation using SVM classifier (\( Q = 20 \), number of features=9).

While preparing classification results presented in the
Fig. 9(a–d), the Q-factor values were varied in the range of
1–50 and step size was kept as 5. It is perceived from Fig. 9(a–
d) that classification accuracy first increases with the increase
in Q-factor value and further decreases as Q-factor value
approaches to 50. Classification accuracy seems constant over
middle ranges of Q-factor values (i.e., 5 to 25) for feature
vectors with smaller number of features, i.e., 20 features and
10 features (ref. Fig. 9(c) and Fig. 9(d)) and decreases after-
wards. Also, it is evident from Fig. 9(a–d) that the classifi-
cation performance improves with the reduction in number of
HFD features and the highest classification accuracy is attained
corresponding to least number of HFD features, i.e., 10. The
EMD parameter variations have a smaller impact on the classi-
fier’s performance for a smaller feature set. However, keeping
EMD parameter values as 40 helped in achieving a high clas-
sification rate in multiple experiments (ref. Fig. 9(d)).

The results of the classification established the fact that IMF
with the highest value of HFD is the most significant and in-
formative IMF for fault classification task. Considering ef-
cicacy of a single IMF with the highest value of HFD in a
bearing fault diagnosis event, a second stage of feature re-
cursion is carried out with different Q-factor values (ranging
from 1–50) and the results are shown in Fig. 10(a–k). It is
observed from Fig. 10(a–k) that reduction in the number of
features significantly improved the classification performance,
but further reduction yielded deteriorating impact on the clas-
sicifier’s performance. The best classification performance
was recorded while considering features extracted from nine sub-
bands of the decomposed IMF (ref. Fig. 10(c–k)). It is also
observed that the classification accuracy is minimum for Q-
factor \( Q = 1 \) and it increases with increasing value of Q-factor
until Q-factor is greater than 25. However, the best classifica-
tion performance is witnessed when Q-factor value lies in the
range of 10–25. Therefore, classification results suggest that
the Q-factor value should be maintained in the range of 10–25
to ensure the highest classification performance in the bearing
fault diagnosis task using the double decomposition technique.

Figure 11(a–d) represents the box plots of extracted HFD
features obtained for four classes of vibration activity. It is ob-
served from Fig. 11(a–d) that the distribution of feature values
is much lower for a healthy bearing as compared to the bear-
ings with faults. However, the maximum distribution of feature
values is observed for bearing with ORD (ref. Fig. 11(d)). Fig-
ure 11(a–d) describes the class discrimination ability of the ex-
tracted EMD-TQWT based HFD features in bearing fault diag-
osis. It is evident from Fig. 11(a–d) that the extracted HFD features have capability to
differentiate among healthy and faulty vibration activity. Ta-
ble 2 presents a confusion matrix for SVM classification; keep-
ing Q-factor value as 20 and considering 9 features for training
and validation purposes. It is reflected from Table 2 that all
the instances are correctly classified using the SVM classifier.
\( q_{Resol} \) and \( q_{Resid} \) are 40, 45, and 50, respectively.
Further, in order to validate the efficacy of proposed feature extraction methodology, three vibration datasets with varying sampling frequency and sensor positioning (with sampling frequency = 12 kHz at drive end & fan end sensor positions; with sampling frequency = 48 kHz at fan end sensor position) are exploit and a comparison of classification performance is carried out. Experiments are performed at four different Q-factor values (i.e., 10, 15, 20 and 25) and the number of features corresponding to each class were considered as 9. Since, EMD parameter variations have a smaller impact on classifier’s performance for smaller feature set (as illustrated from Fig. 9(d)), EMD parameter values are kept 40 consistently across all experiments. Classification performance achieved for drive end & fan end vibration data with a sampling frequency of 48 kHz is presented in Table 3. Classification performance achieved for drive end & fan end vibration data with a sampling frequency of 12 kHz is presented in Table 4. The classification accuracy obtained at varying Q-factor values (i.e. Q-factor=10/15/20/25) for four different vibration datasets is presented in Fig. 12. It is observed from Table 3 and Table 4 that the proposed methodology has the ability to identify faulty and healthy bearings using vibration activity recorded at varying sampling frequencies and sensor placement. The classification performance obtained for drive end vibration data is significantly better as compared to fan end vibration data. It is evident from Fig. 12 that best classification accuracy is obtained at Q-factor=10 for both 48 kHz and 12 kHz sampled vibration activity. A comparative study between the proposed work and previous work published in the literature is presented in Table 5. It is perceived from Table 5 that the methodology proposed in the present work performed better than previously proposed work in terms of classification performance. Double decomposition based HFD feature estimation method helped in achieving 100% classification accuracy with a significantly lower number of features.

5. CONCLUSION

In present work, an effort has been made to develop EMD-TQWT based double decomposition technique for the fault diagnosis of ball bearing. The proposed technique helps in identifying ball defect, inner race defect and outer race defect present in ball bearing. The methodology relies on EMD-TQWT based time-frequency representation of bearing vibration activity for calculation of HFD features as discriminable signatures of faults. It is observed in the study that HFD box-plots have significant distribution over values corresponding to vibration activity of faulty bearings, however, such is not the case with healthy bearings. The box-plot of HFD features for healthy bearing shows positive/zero skewness pattern, however, mixed skewness pattern is witnessed for faulty bearings. The four classes of vibration activity is classified using SVM classifier with varying size of feature vector and parametric values. The results of classification illustrated the significance of IMF with the highest value of HFD in fault diagnosis task. Also, very substantial variation in classification accuracy is discerned with varying value of Q-factor in this work. It is recommended that Q-factor value should be maintained in the range of 10–25 to ensure high classification performance. The highest classification accuracy of 100% is achieved corresponding to Q-factor values of 10, 15, 20, and 25 with 9 HFD features in the present work (for 48 kHz, drive end data). It is observed that increment and decrement in the number of features had negative impact on the classification performance. The results of classification illustrates the efficacy of the proposed double decomposition technique in bearing fault diagnosis. The effectiveness of the proposed methodology is further evaluated on three different vibration datasets and classification performance is analyzed. Considering all other parameters as constant, the highest classification accuracy of 100% is achieved for 12 kHz, drive end vibration data. Comparison results with previous studies revealed that proposed feature extraction methodology has capabilities to identify bearing faults with a significantly lower number of estimated features.

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<table>
<thead>
<tr>
<th>Authors</th>
<th>Number of features used</th>
<th>Classifiers used</th>
<th>Peak efficiency</th>
<th>Remark</th>
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<tr>
<td>Wu et al.(^h)</td>
<td>Eighty (statistical)</td>
<td>SVM</td>
<td>98.5%</td>
<td>Training efficiency</td>
</tr>
<tr>
<td>Zhang and Li(^i)</td>
<td>Twenty</td>
<td>Neighborhood preserving embedding and SOM</td>
<td>99%</td>
<td>Training efficiency</td>
</tr>
<tr>
<td>Tiwari et al.(^k)</td>
<td>sixteen</td>
<td>ANFC</td>
<td>92.50</td>
<td>Ten-fold cross-validation efficiency</td>
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<tr>
<td>Vakharia et al.(^l)</td>
<td>Eleven</td>
<td>ANN, SVM</td>
<td>ANN - 97.5 SVM - 97.5</td>
<td>Ten-fold cross-validation efficiency</td>
</tr>
<tr>
<td>Wang et al.(^m)</td>
<td>Hundred</td>
<td>CART, kNN and SVM</td>
<td>97.26%</td>
<td>–</td>
</tr>
<tr>
<td>Li et al.(^n)</td>
<td>One hundred eighty</td>
<td>SVM-BT</td>
<td>100%</td>
<td>Training efficiency</td>
</tr>
<tr>
<td>Vakharia et al.(^o)</td>
<td>Thirty-five (statistical)</td>
<td>SVM and RF</td>
<td>98.38% (using RF)</td>
<td>Ten-fold cross-validation efficiency</td>
</tr>
<tr>
<td>Present work</td>
<td>Nine</td>
<td>SVM</td>
<td>100% (Q-factor=10)</td>
<td>Ten-fold cross-validation efficiency</td>
</tr>
</tbody>
</table>

SVM: Support Vector Machine, CART: Classification and Regression Trees, kNN: k-Nearest Neighbour, ANN: Artificial Neural Network, RF: Random Forest, ANFC: Adaptive Neuro Fuzzy Classifier


Figure 12. Classification accuracy obtained at Q=10/15/20/25 for four different vibration dataset.


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Slab Track Behaviour under Train Passage and Hammer Impact — Measurements at Different Sites and Calculated Track Interaction with Continuous Soils

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Federal Institute of Material Research and Testing, Berlin, Germany.

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This contribution intends to give an overview on the vibration behaviour of slab tracks in comparison of measurements and calculations and also by comparison of different track types at more than ten different measuring sites. In theory, tracks on continuous soil are calculated by the frequency-wavenumber domain method. In experiment, geophone measurements are transformed to displacement results. Two aspects of track behaviour are considered, the frequency-dependent compliance of the track, measured by hammer impact, and the deflection under a passing axle load. In theory, the response to a single axle can be calculated, whereas in experiment, only the passage of the whole train can be measured. For comparison of theory and experiment, the calculated deflection under a single axle is superposed to get the response of the whole train. As a result, the slab track characteristics are completely different from the ballast track characteristics where each axle can be seen in the time histories. The slab track has a more global behaviour where only a whole bogie can be found in the track response and moreover, the two neighbouring bogies are not completely separated. The measurement of the different track elements (rail, sleeper, track plate, base layer) and the frequency-dependent compliances with possible resonances yield further information about the properties of the track elements. The calculations show that the soil has the dominant influence on the amplitudes and the width of the track-plate displacements. In the measurement results, the following parameters are analysed: slab track vs. ballast track, different types of slab tracks, damaged slab tracks, different trains, switches at different measuring points, voided sleepers, an elastic layer, the mortar layer, and different soils at different places. Finally, a good agreement between measured and calculated results is found for the normal and some special (damaged, floating) slab tracks.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>a</td>
<td>width of the track</td>
</tr>
<tr>
<td>D</td>
<td>damping (of the rail pad)</td>
</tr>
<tr>
<td>EI</td>
<td>bending stiffness</td>
</tr>
<tr>
<td>EI'</td>
<td>bending stiffness</td>
</tr>
<tr>
<td>f</td>
<td>frequency</td>
</tr>
<tr>
<td>F, F'</td>
<td>force, force per track length</td>
</tr>
<tr>
<td>F^T</td>
<td>force vector such as F^T xx</td>
</tr>
<tr>
<td>G</td>
<td>shear modulus of the soil</td>
</tr>
<tr>
<td>H_S</td>
<td>compliance of the soil for a plane wave</td>
</tr>
<tr>
<td>H_S'</td>
<td>compliance of the track-soil interface for wave along the track</td>
</tr>
<tr>
<td>k_b</td>
<td>stiffness of the rail pad</td>
</tr>
<tr>
<td>K'</td>
<td>stiffness matrix of the elastic track elements</td>
</tr>
<tr>
<td>K_S</td>
<td>dynamic stiffness matrix of the soil</td>
</tr>
<tr>
<td>K_T</td>
<td>dynamic stiffness matrix of the track</td>
</tr>
<tr>
<td>K_T S</td>
<td>dynamic stiffness matrix of the track-soil system</td>
</tr>
<tr>
<td>l_k</td>
<td>axle distance in a bogie</td>
</tr>
<tr>
<td>l_g</td>
<td>distance of two neighbouring bogies</td>
</tr>
<tr>
<td>m'</td>
<td>mass per length</td>
</tr>
<tr>
<td>m</td>
<td>mass per length matrix</td>
</tr>
<tr>
<td>p</td>
<td>wavenumber transform of the uniform force distribution across the track</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>t_p</td>
<td>thickness of the track plate</td>
</tr>
<tr>
<td>u</td>
<td>displacement</td>
</tr>
<tr>
<td>u_T</td>
<td>displacement vector xx u_T</td>
</tr>
<tr>
<td>v_s</td>
<td>shear wave velocity of the soil</td>
</tr>
<tr>
<td>v_T</td>
<td>train speed</td>
</tr>
<tr>
<td>x</td>
<td>coordinate across the track</td>
</tr>
<tr>
<td>y</td>
<td>coordinate along the track</td>
</tr>
<tr>
<td>omega</td>
<td>circular frequency</td>
</tr>
<tr>
<td>xi</td>
<td>wavenumber in across the track</td>
</tr>
<tr>
<td>xi'</td>
<td>wavenumber along the track</td>
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</table>

1. INTRODUCTION

Slab track is a track form which becomes increasingly important in railway traffic. Slab track is an alternative track form to the ballast track. The ballast is replaced by a stiff layer which usually is a concrete plate (Fig. 1). By that, the settlement of the ballast is eliminated and the maintenance effort should be reduced. Many lines have been built with slab tracks worldwide but there are still attempts to improve this track form.

The dynamic behaviour of slab tracks is useful for many aspects of railway engineering. Modelling and theoretical results are necessary for the design. The load condition of any track is an important information for understanding the deterioration and for maintenance decisions. Track monitoring can be used to prevent damage. Dynamic measurements can be used to identify track damage or general track characteristics. The soil has a strong influence on the slab track behaviour so that the analysis of the substructure has a great value, particularly at transition zones.
The Federal Institute of Material Research and Testing (BAM) has performed many measurements of dynamic track behaviour in relation to prototype testing (design) and damage detection. The purpose of the present contribution is to evaluate some of these measurements to illustrate the track behaviour of different tracks at different sites, on different soils. The focus is on normal intact slab tracks and their deflections under train passages and their response under hammer impact. Their behaviour is compared with other track forms and other slab tracks. The observed specific phenomena can be used to calibrate theoretical track-soil models.

Hammer impacts are often used, particularly to verify calculation methods and to evaluate input parameters. Alternative methods are wheelset drop tests or measurements from a special rail vehicle. The measurement of train passages is done for monitoring and damage detection and in combination with ground vibration measurements. A special problem in track monitoring is to transform measured accelerations or velocities to track displacements. All measurements should be confronted with calculations. The present article uses wavenumber integrals to describe the deformation of the infinitely long track. It is necessary to include the infinite continuous soil by wavenumber integrals or by Green’s functions. Other methods that account for the continuous soil are the finite element method, and combinations of finite and integral methods, called 2.5D methods. The calculations provide the ranges of the displacements for intact and damaged slab tracks and the influence of the different track and soil parameters.

The novelty of this contribution is to present at the same time:

- measurements of slab tracks and ballast tracks (not only ballast tracks),
- measurements of train passages and hammer impacts (not only one measurement type),
- measurements of all track components (not only the rail),
- measurements at several sites (not only one site),
- deflection shapes under train passage (not only axle-sequence spectra),
- amplitude and phase of the receptances (not only amplitudes),
- calculation of tracks on continuous soil (not only tracks on Winkler soil),
- parameter studies for hammer impact and train passage (not only one example for verification).

where typical shortcomings of the state of the art are given in brackets with exemplary references.

The material, which is based on the conference contribution and which has been considerably extended in theory and experiments, is presented as follows. The measurement methods are presented in Section 2, and the calculation methods in Section 3. The theoretical track response to train passages is analysed in Section 4 for different vehicles, track plates and soils. Section 5 shows the measured results for twelve sites and for normal and special slab tracks. At some of these sites, hammer impacts have been measured (Section 6). The rail, track plate, base plate and base layer response is analysed. The original hammer measurements must be modified to get comparable results for a wheelset excitation. In Section 7, frequency-dependent transfer functions of the tracks and track elements have been calculated for different rail pads, track plates and soils. The measured and calculated results of train passages and hammer impacts are compared in Section 8 leading to the final conclusions (Section 9).
2. METHODS OF TRAIN PASSAGE AND HAMMER IMPACT MEASUREMENTS

The track vibration during the passage of trains has been measured with velocity transducers (geophones) or accelerometers. The sensors are glued to the track elements or fixed with spikes to the soil. The geophone measurement data (Fig. 2a) have been corrected for the frequency-dependent complex transfer function of the geophones (eigenfrequency 4.5 Hz, Fig. 2b) and then time-integrated to obtain the displacements of the track elements (Fig. 2c). To get stable results for these two steps, the frequency content below 1 Hz is filtered out by a 5th order Butterworth filter. The time integration results in reduced high frequencies so that no special filter is necessary to filter out the high-frequency content. In addition to the previous steps, a base-line correction is necessary (Fig. 2d), see a similar procedure in. The results of this procedure have been compared with the results of displacement and laser sensors at several places confirming the good performance, (examples are shown later in Fig. 8).

Using an instrumented hammer, the transfer functions of the track (compliances) and of the soil (admittances) can be evaluated. Usually, the responses of ten hammer impacts on the unloaded track are averaged, and the standard sampling frequency of 2 kHz is chosen for hammer as well as for train measurements. The impact on the right rail, the left rail or both can be measured giving a one-sided transfer function of the track. Averaging the transfer functions from the left-rail and right-rail impact it is also possible to average the left- and right-side response to a single side impact (Fig. 3). Several tests show that taking the half of the one-sided transfer function for the rail and about 0.7 of the one-sided transfer function for the sleeper and track plate yields an acceptable approximation of the two-sided transfer function. Hammer impacts and a measurement line on the soil are also used to determine the wave velocities and the damping of the different soils.

3. CALCULATION OF TRACK AND SOIL VIBRATION BY WAVENUMBER INTEGRALS

The slab tracks can also be modelled as multiple-beam systems (Fig. 4). The first beam represents the two rails, the second beam represents the track plate, (and a third beam could be used as the base plate of a floating slab track). Each beam is described by its bending stiffness $EI_j$ and its mass per length $m_j$, which are assembled in a diagonal stiffness matrix $E$ and a diagonal mass matrix $m$. The beams are connected by elastic elements (rail pads, slab mats) characterised by complex stiffnesses per track length which are assembled in the global stiffness matrix $K'$. The multi-beam system fulfills the set of differential equations for the track displacements $u$ (the rail – base vector $e_1$, the track plate – base vector $e_2$ and the base plate – base vector $e_3$) under the track load $F'(per length)$

$$EI_j \frac{\partial^4 u}{\partial y^4} + m_j \frac{\partial^2 u}{\partial t^2} + K'_j u = F'(per length)$$

This equation is transformed to the frequency-wavenumber domain giving the dynamic stiffness matrix $K'((\xi,f))$ of the multi-beam track model

$$K'(\xi,f) = \xi^4 EI - \omega^2 m + K'$$

where $\omega = 2\pi f$ is the circular frequency and $\xi$ is the wavenumber along the track axis. This dynamic stiffness matrix must be coupled with the dynamic stiffness matrix of the soil which is calculated as follows.

The soil compliance $u/F'(per length)$ is calculated using different matrix methods. This compliance $H_S$ for plane waves along the soil surface can be integrated across the track to get the compliance $u/F'(per length)$ of the track-soil interface for waves along the track

$$\frac{1}{K'(\xi,f)} = H_S(\xi,f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_S(\xi,\xi,f)p_1^2(\xi) d\xi$$

In equation (3), the wavenumber transform

$$p_1(\xi) = \frac{\sin \xi a/2}{\xi a/2}$$

Figure 3. Hammer impact on the right rail of the slab track at site 7, transfer function for the right side of the sleeper, left side of the sleeper, and the average of both (the two-sided transfer function for a wheelset excitation).

Figure 4. Multi-beam-on-soil model for a slab track with rail pads, sleeper pads or a slab mat (floating slab track).
of the uniform load distribution across the track width \( a \) has been used and the average of the response across the track has been taken.\(^{32}\) The inverse

\[
K'_{TS}(\xi_y,f) = \frac{1}{H'_{S}(\xi_y,f)}; \quad (5)
\]

the soil stiffness \( K'_{S} \) per track length, is used for coupling the soil with the track as

\[
K'_{TS}(\xi_y,f) = K'_{T}(\xi_y,f) + \varepsilon_3 K'_{S}(\xi_y,f) \varepsilon_3^T; \quad (6)
\]

and the dynamic stiffness matrix \( K'_{TS}(\xi,\omega) \) of the track-soil system is established.

The displacements in the frequency-wavenumber domain are calculated by the inversion of this matrix

\[
u_T(\xi_y,f) = K'_{TS}(\xi_y,f)^{-1} F_T = K'_{TS}(\xi_y,f)^{-1} F_T(\xi_y,f) \varepsilon_3; \quad (7)
\]

and the wavenumber transform \( F_T'(\xi,\omega) \) of the excitation force on top of the track. Finally, the displacement distribution along the track can be calculated by the Fourier integral of the wavenumber domain solution

\[
u_T(y,f) = \frac{F_T(f)}{2\pi} \int_{-\infty}^{\infty} K'_{TS}(\xi_y,f)^{-1} \varepsilon_1 \exp(i\xi_y y) d\xi_y; \quad (8)
\]

where the special constant wavenumber transform of a point load

\[
F'_T(\xi_y,f) = F'_T(f); \quad (9)
\]

has been introduced.

4. CALCULATED SLAB DISPLACEMENTS UNDER TRAIN PASSAGES

The displacements along the track (\( y \)-axis) can be transformed to time histories (time \( t \)) of a train passage by superposing several axles and using the train speed \( v_T \) for \( t = y/v_T \). Figure 5 shows results for a Talgo train with single axles, a Thalys train with single bogies and an ICE3 train with two bogies per carriage. The displacements of a single axle strongly depend on the stiffness of the soil (Fig. 5a) and less strongly on the bending stiffness of the track (Fig. 5b). The deflection of the track is wider for a soft soil, but narrower for a soft track. The response of a single bogie (Fig. 5c) is stronger than the response of a single axle (Fig. 5a, circle marker) as one axle yields also displacements for the second axle of the bogie (interaction between the two axles). There is only one deflection of the track plate for the two axles of a bogie. The bogie displacement can be almost twice of the axle displacement if the axle distance is short for a freight car (1.8 m, Fig. 5c, triangle marker, 0.2 instead of 0.12 mm). The long bogie of a locomotive (3.0 m) has a wider deflection and a lower maximum of 0.15 mm. The interaction of two neighbouring bogies can be seen in Fig. 5d. The shape of the track response changes with the separation between the two bogies. Namely the displacement in the middle of the two bogies is quite different. For a narrow bogie separation (6 m), the inter-bogie displacement (0.19 mm) is not much smaller than the axle displacement. For a long bogie separation (9 m), the inter-bogie displacement (0.11 mm) is about half the axle displacement, and for the ICE3 (7.5 m), it is about two thirds (0.14 mm). Finally, the influence of the soil and track stiffness on the response of an ICE3 train
Figure 6. Measured passages of 1 and 2 half passenger cars over normal slab tracks a-d) at sites 1-4, e,f) at site 5 with concrete and asphalt slab track, g,h) floating slab track at site 6 with normal and stiff soil; ICE or IC (c) trains. Measurement points • rail, sleeper (□ right, △ mid, ▽ left), track plate (○ right, + mid, × left), base plate (trough) († right, † left), base layer (○ right, ⋆ left), ⋆ soil.

(1 and 2 half passenger cars) is analysed in Figs. 5e,f. The influence of the soil on the maximum displacement is very strong (Fig. 5e), \( u_3/u_1 = 0.33/0.08 \approx 4 = G_3/G_1 \), whereas the influence of the track bending stiffness is very weak (Fig. 5f), \( u_3/u_1 = 0.36/0.33 \approx 1.2 < \approx 27 = EI_3/EI_1 \), weaker than for a single axle (Fig. 5b). The soft track with a thin plate yields less interaction between the bogies and a more local behaviour with narrower deflections and with lower inter-bogie displacements, and in addition, a maximum for each axle occurs (Fig. 5f triangle marker). The soft track with its more local behaviour is an exception between the normal slab tracks which all have a typical global deflection behaviour.
5. MEASURED SLAB TRACK DISPLACEMENTS UNDER TRAIN PASSAGES

The response of different slab tracks to passing trains measured at different sites are shown in Figs. 6 and 7. For each situation, the responses of different measurement points are given in one sub-figure. In the first four Figs. 6a to 6d, normal slab tracks on “normal” soils (wave velocities between 150 and 225 m/s) are presented. In general agreement, the maximum displacements of the sleeper or track plate under an ICE train are in the range of $u_{\text{max}} = 0.3$–$0.5$ mm. Figures 6c, 7c for the passage of an IC train have lower amplitudes due to lower axle loads.

As in Fig. 5, one and two half carriages are shown in each sub-figure, and four deflections for four bogies are present. The measurement results have more irregularities, and namely small vibrations due to the sleeper-distance excitation can be
Two special slab tracks are presented in Figs. 6f-h, both having an additional elastic layer within the track. Figure 6f shows a slab track where an asphalt layer is used instead of the track plate. Corresponding to the soft asphalt, the displacements of the base layer under the asphalt are only 30% of the maximum of the sleepers which is 0.8 - 1.4 mm, the highest of all measured slab tracks. Moreover, each axle can be seen as a small maximum. The slab track of Figs. 6g,h includes a rubber granulate layer between track plate and base plate. Therefore, the amplitude of the base plate is much lower, only 25% of found. The inter-bogie displacements vary between 50% and 80% of the maximum displacement. For normal slab tracks, all measurement points of the track layers (sleepers, track plate, base plate, base layer) have very similar amplitudes from top to bottom of the track. The amplitudes of the soil are much smaller, and the rail has much higher amplitudes in case of a slab track (Fig. 7c). The side- and mid-points of the same track element show sometimes different amplitudes (for example Figs. 6f, 7d) so that some deformation across the track or a rotation of the track can be concluded.

Figure 8. Measured passages of 1 and 2 half passenger cars over switches with voided sleepers at a) site 10, b) site 11. c-d) site 12. □ sleeper displacements from displacement sensor, ○ sleeper displacements from accelerometer, d) displacements of sleeper □-6, ○ -4, △ -2, + +4, ×+6 indicating a void under all these sleepers.

Figure 9. Slab track displacements (track plate) measured under different trains, a) ICE1, b) ICE2, site 1, c) freight train with equally loaded carriages, d) freight train with different and variably loaded carriages, site 4.
the track plate amplitude of 0.45 mm. This special slab track structure has been measured at two sites with very different soils. The soil at the second site (Fig. 6h) was much stiffer including some rock material so that the amplitudes of the same track and the same train were much smaller, 0.2 instead of 0.45 mm.

Figures 7a,b show an intact (repaired) and a damaged slab track where the track plate was loosened from the base layer. This can clearly be seen by the high amplitudes (0.9 mm) of the lose track plate for each axle passage. The amplitudes of the elements below the gap are somewhat smaller than those of the intact track (Fig. 7a, 0.3 mm). The damage and the repair could be checked at some sites by this kind of measurements.

The slab track has been compared to a ballast track in Fig. 7c,d where the different tracks have been measured for the same IC-train passage at two consecutive sections. The low amplitudes of the track plate (0.17 mm) are highly exceeded by the amplitudes of the sleeper on the ballast (0.3–0.5 mm, the difference between side- and mid-sleeper point indicate a considerable sleeper bending). The ballast track reacts with separate bogie responses and with two axle maxima for each bogie. This local behaviour is also found for the rails of both track systems. The rail amplitudes are higher for the slab track, 0.8 mm compared to 0.6 mm for the rail of the ballast track.

Ballasted tracks with switches are shown in Figures 7e–h. At site 8, a normal switch (Fig. 7e) has been compared with a switch on rather stiff under-sleeper pads (Fig. 7f). High amplitudes of 1.5 mm have been measured at the end of the switch (the nose) whereas normal amplitudes of 0.6 mm have been measured at the beginning and the mid of the switch. The switch with under-sleeper pads shows two times higher amplitudes at these points so that all switch points have the same level. A similar effect of slightly increased amplitudes can also be seen at site 9 for the switch with stiff under sleeper pads (Fig. 7g) compared to the normal switch (Fig. 7h).

Figure 8 shows switches over a void between sleeper and ballast. This damage causes high sleeper amplitudes of 2.5 to 3 mm. The passage of the axles, however, are quite different at these three sites. Figure 8a shows high amplitudes but quite normal time histories as for intact ballast tracks. Each axle can clearly be seen and the amplitudes return to zero between the bogies. Figure 8b shows a case with very short axle impulses indicating probably a very local defect, maybe a void under only one sleeper. At the last site (Fig. 8c,d), the axle impulses are not so clear, the deformation is wider. This could be understood as a void under many sleepers which has been confirmed by the measurement of higher amplitudes also at the -6th to +6th sleeper (Fig. 8d).

The analysis of different tracks and different sites is complemented with the analysis of different trains (Fig. 9). Two different ICE trains yield almost identical responses of the slab track. The ICE1 consists of a locomotive + 12 carriages + locomotive whereas the ICE2 consists of two coupled half trains with locomotive + 6 carriages + 1 end car where the axle loads of the locomotive are higher and the axle distance is longer (3 m). The different behaviour under the locomotive can clearly be found in the time histories (around 4.7 s), and the longer inter-bogie distance (at 4.5 s) between end car and locomotive is indicated by the lower inter-bogie displacement. Figures 9c and 8d show freight trains with equally or variably loaded carriages. The maximum displacement is slightly higher than for the ICE trains at this site (Fig. 9d).

6. COMPLIANCE FUNCTIONS MEASURED UNDER HAMMER IMPACT

The hammer impact measurements have been evaluated for the complex transfer functions (amplitude and phase of the compliance) of all measurement points (Fig. 10a,b) and for all measurement sites (Fig. 11a,b). The compliance of the track is higher for the asymmetric one-sided hammer impact compared to the symmetric wheelset excitation of the preceding sections. At site 7 (Fig. 10a), all points of one side yield similar transfer functions, where the excited side has about twice the amplitudes. All track points at this site show a minimum at about 65 Hz and a maximum at 140 Hz which is the resonance of the rail on the rail pads.

The same detailed measurements and evaluations have been done at site 3 (Fig. 10b). Here, the results indicate a slab track damage. Sleeper, track plate and base layer show different and somewhat higher amplitudes. Namely the sleeper amplitudes are considerably higher than what normally has been measured.

The rail points of six different sites are presented in Fig. 11a. Four of these sites have nearly the same resonance at 140 Hz indicated also by a crossing of four phase curves at -90°. Site 3 has its resonance at 215 Hz (not present within the given frequency range), and site 5b has a lower resonance at 90 Hz due to very soft rail pads. The static compliances of the rail are between 1 and 2·10^8 m/N, which would be 0.5 to 1·10^8 m/N for the two-sided wheelset excitation in agreement with the train-passage result in Fig. 6c. The compliances of the track plate (Fig. 11b) are much lower, starting at 1 to 2·10^9 m/N. The rail-on-rail-pad resonance can be found also at the track plate, but not so clear and with an anti-phase, see for example site 5b with the lowest resonance frequency of 90 Hz and a strong phase drop. The phases are linearly decreasing at low frequencies, reach an asymptotic value of -130° before they drop down due to the rail-on-rail-pad resonance at high frequencies.

7. CALCULATED FREQUENCY-DEPENDENT COMPLIANCE FUNCTIONS OF SLAB TRACKS

The frequency-dependent compliance functions of slab tracks have been calculated for some parameter variations. Figure 12 top and bottom shows for each variation the amplitude and phase function of the rail (left) and the track plate (right). If the height of the plate is varied between 0.15 and 0.6 m, there is an influence on the amplitudes of the track plate (relatively higher at high frequencies, Fig. 12a), but only a weak effect on the rail (small differences for the small phases, Fig 12b). The influence of the soil is stronger (Fig. 12c,d), particularly the low-frequency amplitudes and phases are higher for a soft soil. The influence is strongest for the track plate amplitudes which vary between $\nu = 0.7$ and $4.2 \cdot 10^{-9}$ m/N and that is $u \approx v_y \cdot 10^{-6} \approx G \cdot 0.8$ almost indirect proportional to the stiffness $G$ of the soil. The phases of the track plate have a different decrease, the strongest for the softest soil, reach a common asymptote of -90°, and fall further down at the highest frequencies. The latter is due to the rail pad of $k_G = 60$ kN/mm. The influence of the rail pad is analysed in Fig. 12e,f. Soft rail pads result in a clear resonance of the rail at 85 Hz for $k_G = 10$ kN/mm, 120 Hz for 20 kN/mm and 145 Hz for 30 kN (Fig. 12e). At the same frequency, the phase drops down
Figure 10. Compliance transfer functions (amplitude top, phase bottom) from unloaded hammer impact measurements. a) Different measurement points at site 7, and b) at site 3, sleeper (□ right, △ mid, ▽ left), track plate (○ right, × left), base layer (○ right, + left).

Figure 11. Compliance transfer functions (amplitude top, phase bottom) from unloaded hammer impact measurements. a) Rail points and b) sleeper or track plate points at different sites, □ slab track, site 1, ○ slab track, site 2, △ slab track, site 3, + slab track with stiff rail pads, site 5, × slab track with soft rail pads, site 5b, ○ slab track, site 7.
strongly to -130°. The resonances can also be seen at the track plate with smaller amplitudes and a strong phase drop from -100° to less than -250° (Fig. 12f). Finally, the damping of the rail pads has been varied in Fig. 12g,h. The resonance amplitudes of the rail are small for $D = 0.2$ but still visible (Fig. 12g). The resonance can be completely hidden for the track-plate amplitudes (no amplification, Fig. 12h) and the track plate phases (continuous decrease, Fig. 12h bottom) for a rail pad damping of $D = 0.2$ or higher.

The information from this parametric study has been used to find appropriate track models which represent the measurements well. Figures 13a,b show the results for six models which are close to the measured results of six sites (Fig. 11a,b). The agreement between theory and measurement is quite good for both, the rail and the track-plate displacements, and both, the amplitudes and the phases. Note, that the measured one-sided hammer results must be reduced by a factor of 0.5 (rail) or 0.7 (track plate) to be compared with the theoretical results for a two-sided wheelset excitation.
8. COMPARISON OF THEORY AND MEASUREMENT OF TRAIN AND HAMMER EXCITATION

At first, the compliances measured by hammer impact (Fig. 11) are compared with the theoretical compliances for a wheelset excitation (Fig. 12). The measured track-plate compliances of 1 to 2.5 10^{-9} m/N are well within the theoretical range of slab tracks on medium to stiff soils, if the one-sided amplitudes are reduced to the two-sided wheelset excitation (0.5 to 2 10^{-9} m/N). The same theoretical amplitudes hold also for the passage of a single axle (Fig. 5a,b). The amplitudes for the passage of the whole train, when two bogies contribute to the axle displacement, are considerably higher at 0.2 to 0.3 mm per a 100 kN axle load (Fig. 5c-f). These displacements correlate well with the measured track plate displacements under an ICE axle load of 160 kN, if a medium to medium stiff soil (150 to 200 m/s wave velocity) has been assumed. In total all results agree quite well. The train passages match with a little less compliant slab track than the hammer impacts. This could be due to the stiffening effect of the static train load. Whereas the stiffening effect of the static load should be considered, the
moving load effect can be neglected as the train speed is significantly lower than the Rayleigh wave velocity of the soil.

9. CONCLUSION

The calculation and measurement of several slab tracks have revealed the following characteristics. The displacements of the track plate highly depend on the stiffness of the soil. For typical wave velocities between 150 and 200 m/s the displacements are between 0.2 and 0.3 mm per 100 kN axle load. This correlates well with the measured displacements of 0.3 to 0.5 mm for an ICE train with 160 kN axle load. The displacements of the rail are mainly ruled by the stiffness of the elastic rail pads. The measured value of 0.8 mm for the passage of a 100 kN axle load correlates well with the static compliance of 1 mm/100 kN from the hammer excitation of many slab tracks. Some more specific items could be found from the measurements: Resonances, displacement amplitudes, amplitude changes (for damages), amplitude differences of different track elements, deflection shapes with different inter-axle and inter-bogie displacements. These characteristics from train passage and hammer impact measurements can be used to identify track properties such as very soft or rather stiff rail pads, elastic layers in floating slab tracks, an asphalt track or a ballast track, or to identify track damage and the repair effects. The measured track compliances can also be useful for the analysis of the vehicle-track-soil interaction and the determination of dynamic excitation loads.

ANNEX 1. DETAILS OF THE MEASUREMENT SITES

This article reports measurement results of 18 different track-soil systems at 12 different sites. Some details are given in Table 1. The slab tracks consist at least of a track plate and a base layer (site 5). Most slab tracks have an additional element which is a sleeper at sites 3, 4, and 7, or a base plate (a trough) at sites 1 and 2. At sites 1 and 2, there are sleepers integrated in the track plate. At site 6, a floating slab track has been measured where the 6 m long track plates lie on an elastic rubber layer over a base plate and base layer. The typical dimensions of the track layers are 0.3 m for track plate and base layer, and if there is a base plate, then track plate and base plate have 0.15 m each.

Passages of ICE, IC regional and freight trains have been measured. The train type evaluated in this article have been underlined. For each site, the figures with the corresponding results are indicated in the last column.

REFERENCES

5. Bowness, D., Lock, A., Powrie, W., Priest, J. and Richards, D. Monitoring the dynamic displacements of railway track,
### Table 1. Measurement sites, tracks, and excitations (trains).

<table>
<thead>
<tr>
<th>site</th>
<th>track type</th>
<th>structure</th>
<th>excitation</th>
<th>specifics</th>
<th>figure number</th>
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<tr>
<td>1</td>
<td>slab track</td>
<td>integrated sleeper, track plate, base plate, base layer, soil</td>
<td>ICE, IC, freight trains, hammer</td>
<td>intact track</td>
<td>2, 6a, 9, 11</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>damaged</td>
<td>7b</td>
</tr>
<tr>
<td>2</td>
<td>slab track</td>
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<td>ICE, IC, freight trains, hammer</td>
<td>altered repair</td>
<td>7a</td>
</tr>
<tr>
<td>3</td>
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<td>ICE, regional train, hammer</td>
<td></td>
<td>6b, 11</td>
</tr>
<tr>
<td>4</td>
<td>slab track</td>
<td>sleeper, track plate, base layer, soil</td>
<td>ICE, IC, regional and freight trains</td>
<td></td>
<td>6d</td>
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<tr>
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<td>ICE, IC, regional and freight trains, hammer</td>
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<td></td>
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<td>concrete track plate</td>
<td></td>
<td>6f</td>
</tr>
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<td>normal soil</td>
<td>6g</td>
</tr>
<tr>
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<td></td>
<td>short ICE</td>
<td>stiff soil</td>
<td>6h</td>
</tr>
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<td>ICE, regional trains, hammer</td>
<td></td>
<td>3, 7c, 10a, 11</td>
</tr>
<tr>
<td></td>
<td>ballast track</td>
<td>sleeper, ballast, soil</td>
<td>ICE, IC, regional and freight trains</td>
<td>without under-sleeper pads</td>
<td>/e</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>with under-sleeper pads</td>
<td>7f</td>
</tr>
<tr>
<td>9</td>
<td>ballast track</td>
<td>sleeper, ballast, soil, (switch)</td>
<td>ICE, IC, regional and freight trains</td>
<td>without-under-sleeper pads</td>
<td>/g</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>under-sleeper pads</td>
<td>7f</td>
</tr>
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<td>ICE, regional and freight trains</td>
<td>Switch</td>
<td>8a</td>
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<tr>
<td>11</td>
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<td>sleeper, ballast, soil, (switch)</td>
<td>ICE, regional and freight trains</td>
<td>Switch</td>
<td>8b</td>
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<tr>
<td>12</td>
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<td>ICE, regional and freight trains</td>
<td>Switch</td>
<td>8c, 8d</td>
</tr>
</tbody>
</table>

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Zhai, W., Wei, K., Song, X. and Shao, M. Experimental investigation into ground vibrations induced by very high speed trains on a non-ballasted track, Soil Dynamics and Earthquake Engineering, 72, 24-36, (2015). https://dx.doi.org/10.1016/j.soildyn.2015.02.002


Tuned Gyro-Pendulum Stabilizer for Control of Vibrations in Structures

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A special combined gyro-pendulum stabilizer (a gyroscope with coupling to a pendulum) mounted on a vibrating mass is considered for investigation of the vibration responses. This paper mainly focuses on the derivation of the frequency equations and on finding the required angular momentum for vibration control of the system. Besides, there is also an ANSYS simulation model of gyro-pendulum, which was built to verify the mathematical model. The dynamic responses of both that obtained from ANSYS simulation and that obtained from numerical solving of a Lagrangian mathematical model are analyzed comparatively. The angular momentum (\(\Omega I_p\)), in relation to the natural frequency (\(\omega_n\)) of the primary mass, shows that this vibration control device is more adaptable than other conventional ones by producing unidirectional thrust along the forcing excitation axis whilst the gyroscope is spinning.

1. INTRODUCTION

Vibrations are present in many different mechanical systems and may cause undesirable motion and sometimes damage to the mechanical and structural systems. So the effective attenuating of mechanical vibrations has always been a big challenge for engineers. Nowadays, there are different ways of mitigating vibrations by applying one or more passive dynamic vibration absorbers. Passive vibration absorbers have the advantage of not requiring external force or energy.\(^1\)\(^-\)\(^3\) The passive control method is activated by the structural motion without requiring external force or energy to reduce structural responses and utilizes the motion of the structure at the location of the device.\(^4\) The most widely used passive vibration absorbers in engineering are tuned mass absorbers, beam absorbers, pendulum-type absorbers, and even liquid absorbers. Tuned mass dampers (TMDs) are widely used for passive vibration mitigation of mechanical structures due to their simple structure.\(^5\)-\(^10\) One of the first attempts, known as the tuned mass damper (TMD), consists of a mass-spring system with no damping, which was introduced by Hermann Frahm in 1911.\(^5\) As it is well known, classic TMD is tuned to the structural fundamental frequency such that the natural frequency of the absorber is equal to the target frequency to be effective in reducing the primary structure’s response. So, the TMD is a single-frequency vibration absorber that only works within a narrow frequency band and is susceptible to the frequency ratio. However, adding a damper to TMDs has been presented in the work of Ormondroyd and Den Hartog (1928) to have a better application for a wide frequency bandwidth.\(^11\) The authors propose that the addition of the viscous damper to Frahm’s system design can expand the effective frequency band around the principal resonance.

Pendulum vibration absorbers (PVA) can be utilized as TMD to attenuate unfavorable motions of subjected systems and commonly used to control horizontal, vertical and rotational vibrations of main structure because PVAs can oscillate regardless of the direction of base structure motion.\(^2\)\(^-\)\(^3\) Moreover, PVAs have low frequency and use gravitational force instead of elastic stiffness force, so they are widely used in practice for reducing the level of vibrations of buildings or bridges.\(^12\) PVAs can be quite effective and desirable due to their simplicity and lower cost.

In recent years, several research techniques have been used to conduct studies investigating the complicated motions which appear in a gyrostabilizer.\(^13\)-\(^15\) The gyroscopic moment induced by a rotating object offers an attractive means to protect structures against natural hazards in various ways.\(^14\) Compared to conventional active mass dampers for seismic vibration suppression, gyrostabilizers represent a weight and volume saving. The use of gyrostabilizers has emerged, as they have an ability to control vibration at low frequency, and, furthermore, the stored kinetic energy can provide emergency power.\(^13\),\(^15\) The mechanism does not require any other external source of energy, in which the rotor speed is produced by an electric motor in a rotating gimbal. Therefore, this can be classified as a passive control device in a variety of passive vibration control systems.

In this paper, the performance of a gyroscope with a spherical PVA is investigated when attached to a structure that is modeled as a one-degree-of-freedom system under horizontal, harmonic excitation. Since the spherical pendulum can oscillate regardless of the direction of base structure motion, the gimbal of pendulum-like gyrostabilizer (gyro-pendulum) can precess by the rotating disk to convert the angular momentum to linear due to the gravitational force existing on the mass of the pendulum. Hence, the gyro-pendulum in the proposed configuration is supposed to be an effective device that can work in a wide range of excitation frequencies, while much easier changing the disk speed of gyroscope to tune the mode frequencies on the main resonant frequency of the primary system. The classical PVAs have low frequency, and their effectiveness is strongly limited by the energy transfer from the primary system to the PVAs. However, the gyro-pendulum vibration absorbers are much effective in a larger frequency range from pendulum resonance to main system resonance due to the angular momentum of the gyroscope.

Table 1. Physical properties of the system.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Numerical values</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>9.81 m/s²</td>
<td>Gravitational acceleration</td>
</tr>
<tr>
<td>L</td>
<td>0.49766 m</td>
<td>Length of the centroid of the inner gimbal mass</td>
</tr>
<tr>
<td>m_s</td>
<td>99.997 kg</td>
<td>Mass of the inner gimbal</td>
</tr>
<tr>
<td>m_p</td>
<td>20 kg</td>
<td>Mass of the outer gimbal</td>
</tr>
<tr>
<td>m_d</td>
<td>10.001 kg</td>
<td>Disk mass of the gyroscope</td>
</tr>
<tr>
<td>m_c</td>
<td>8000.1 kg</td>
<td>Main mass</td>
</tr>
<tr>
<td>l_c</td>
<td>4 m</td>
<td>Length of the column</td>
</tr>
<tr>
<td>h</td>
<td>0.2 m</td>
<td>Column thickness</td>
</tr>
<tr>
<td>b</td>
<td>0.6 m</td>
<td>Column width</td>
</tr>
<tr>
<td>E</td>
<td>24 × 10^9 N/m²</td>
<td>Young’s modulus of the column</td>
</tr>
<tr>
<td>I_c</td>
<td>4 × 10^-4 m³</td>
<td>Geometrical moment of inertia of the column</td>
</tr>
<tr>
<td>k_c</td>
<td>4.5 × 10^5 N/m</td>
<td>Stiffness of each column</td>
</tr>
<tr>
<td>I_p</td>
<td>18 × 10^5 N/m</td>
<td>Equivalent stiffness of columns</td>
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<tr>
<td>I_o</td>
<td>0.20014 kg m²</td>
<td>The rotary inertia of the disk</td>
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<tr>
<td>I_f_s</td>
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<td>Mass moment of inertia of the disk</td>
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<tr>
<td>I_f_y</td>
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<td>Principal moments of inertia of the inner gimbal (Taken at the center of mass of the inner gimbal)</td>
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<td>I_f_z</td>
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<td>I_p_y</td>
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<tr>
<td>I_p_y</td>
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<td>Principal moments of inertia of the outer gimbal (Taken at the center of mass of the outer gimbal)</td>
</tr>
<tr>
<td>Ω</td>
<td>0 – 50000 rpm</td>
<td>Rotating speed of the disk</td>
</tr>
<tr>
<td>z_o</td>
<td>0.001 m</td>
<td>Harmonic disturbing amplitude</td>
</tr>
</tbody>
</table>

Figure 1. Physical model of the system.

Figure 2. The reduced model of the physical system.

Figure 3. Gyro-pendulum via a pendulum arm of stabilizing mass, m_s, with length L from the centroid (G).
2. SYSTEM SET-UP

The physical model of the system is shown in Fig. 1 and the physical properties of the system are given in Table 1. Sinusoidal excitation is introduced horizontally to the base of the primary mass as shown in Figs. 1 and 2. The base of the spring-mass model harmonically excites with different forcing frequencies ($\omega$) and forcing amplitudes ($z_0$) with the horizontal base excitation of $Z = z_0 \cos(\omega t)$. The horizontal elastic displacement at the free end is $y$. The gyro (the disk) is mounted in a gimbal of a pendulum with coupling to an outer gimbal as shown in Fig. 3. The gyro-pendulum system consists of a disk mass $m_d$, which can spin freely about its geometric axis (by an inner gimbal) mounted to the outer gimbal of the main mass. The disk of the gyro is assumed to have a rotational speed $\omega$ and a precession $\varphi$.

3. EQUATIONS OF MOTION OF GYRO-PENDULUM

There is one spinning disk connected with a pendulum supported by the pivot axis of an outer gimbal which is positioned with respect to the main mass as shown in Fig. 2. As shown in Fig. 3, one gyroscopic disk is spinning at a constant speed to have precession $\varphi$ to compensate for the gravitational force existing on the inner gimbal due to the motion of the pendulum. The gyro-pendulum absorber can be assumed to have three degrees of freedom when the disk is spinning with constant angular velocity $\Omega$. Therefore, this paper is based on the equations of motion, written in the form of three differential equations (Eqs. (4), (5), and (6)) with three unknowns ($y$, $\varphi$, and $\theta$).

Here, the kinetic energies of the gyroscopic (disk), inner gimbal, outer gimbal, and the main mass can be written as follows, respectively:

$$ T_{\text{disk}} = \frac{1}{2} m_d \dot{y}^2 + \frac{1}{2} I_\theta \dot{\theta}^2 + \frac{1}{2} I_p \Omega^2 \sin \theta^2; \quad (1a) $$

$$ T_{\text{inner gimbal}} = \frac{1}{2} m_s \left[ L \dot{\theta} \cos \theta + \dot{y} \right]^2 + \frac{1}{2} I_\varphi \dot{\varphi}^2 \cos \theta; \quad (1b) $$

$$ T_{\text{outer gimbal}} = \frac{1}{2} m_p \dot{y}^2 + \frac{1}{2} I_{fy} \dot{\varphi}^2; \quad (1c) $$

$$ T_{\text{main mass}} = \frac{1}{2} m_c \dot{y}^2. \quad (1d) $$

Therefore, the total kinetic and potential energies of the system can be expressed as follows, respectively:

$$ T_{\text{total}} = T_{\text{disk}} + T_{\text{inner gimbal}} + T_{\text{outer gimbal}} + T_{\text{main mass}}; \quad (2) $$

$$ V_{\text{total}} = -m_s g L \cos \theta \cos \varphi + \frac{1}{2} k (y - Z)^2. \quad (3) $$

Then the equations of motion describing the gyro-pendulum can be obtained from the Lagrange equations. For this system, the Lagrange equations become:

$$ \frac{d}{dt} \left( \frac{\partial T_{\text{total}}}{\partial \dot{y}} \right) - \frac{\partial T_{\text{total}}}{\partial y} + \frac{\partial V_{\text{total}}}{\partial y} = 0; \quad (4) $$

$$ \frac{d}{dt} \left( \frac{\partial T_{\text{total}}}{\partial \dot{\theta}} \right) - \frac{\partial T_{\text{total}}}{\partial \theta} + \frac{\partial V_{\text{total}}}{\partial \theta} = 0; \quad (5) $$

$$ \frac{d}{dt} \left( \frac{\partial T_{\text{total}}}{\partial \dot{\varphi}} \right) - \frac{\partial T_{\text{total}}}{\partial \varphi} + \frac{\partial V_{\text{total}}}{\partial \varphi} = 0. \quad (6) $$

Hence, by applying Lagrange’s equations, the fundamental equations of the gyro-pendulum motion are obtained as:

$$ M_t \ddot{y} + \ddot{\theta} m_s L \cos \theta - \ddot{\varphi} m_s L \sin \theta + k (y - Z) = 0; \quad (7) $$

$$ \left( I_o + I_{fy} + m_s L^2 \right) \ddot{\varphi} + \left( \dot{\varphi} m_s L - I_p \Omega^2 \phi \right) \cos \theta - \left( I_p - I_o + I_{fx} - I_{fy} - m_s L^2 \right) \ddot{\varphi} \sin \theta \cos \theta + m_s g L \sin \theta \cos \varphi = 0; \quad (8) $$

in which

$$ M_t = m_c + m_d + m_s + m_p. \quad (10) $$

3.1. Equations of Motion about the Equilibrium Position for Small Vibration

The kinetic energy is zero when the primary mass ($M_t$) is at the equilibrium position ($\dot{y} = 0$). So, the problem can be simplified by substituting $\dot{\theta} \approx 0$ and $\ddot{\varphi} \approx 0$ in Eqs. (7), (8), and (9) near to the equilibrium position. Now that the motion can be reduced as:

$$ M_t \ddot{y} + \ddot{\theta} m_s L \cos \theta + k (y - Z) = 0; \quad (11) $$

$$ \left( I_o + I_{fx} + m_s L^2 \right) \ddot{\theta} + \left( \dot{y} m_s L - I_p \Omega^2 \varphi \right) \cos \theta + m_s g L \sin \theta \cos \varphi = 0; \quad (12) $$

Neglecting the terms of a higher power from the Eqs. (11)–(13) for small vibrations and substituting $\sin \theta = \theta$, $\cos \varphi = \varphi$, and $\cos \varphi = 1$, the equations can be reduced to:

$$ M_t \ddot{y} + \ddot{\theta} m_s L + k (y - Z) = 0; \quad (14) $$

$$ \left( I_o + I_{fx} + m_s L^2 \right) \ddot{\theta} + \dot{y} m_s L - I_p \Omega^2 \varphi + m_s g L \theta = 0; \quad (15) $$

Therefore, the natural frequencies of the primary mass ($M_t$), the inner gimbal and the outer gimbal can be determined as
\[
\begin{pmatrix}
-\omega^2 M_t + k & -\omega^2 m_s L \\
-\omega^2 m_s L & -\omega^2 (I_o + I_{fz} + m_s L^2) + m_s g L \\
0 & j \omega I_p \Omega
\end{pmatrix}
\begin{pmatrix}
y_0 \\
\theta_0 \\
\varphi_0
\end{pmatrix}
= \begin{pmatrix}
k z_0 \\
0 \\
0
\end{pmatrix}.
\]
(19)

\[
\Omega = \pm \sqrt{\left[-\omega^2 (I_o + I_{fz} + m_s L^2) + m_s g L \right]} \left[-\omega^2 (m_s L^2 + I_{py} + I_o + I_{fy}) + m_s g L \right] \Omega \omega
\]
(28)

\[
(\Omega I_p)_{\text{opt}} = \pm \frac{\sqrt{(\omega_n^2 - \omega_{ig}^2)} (\omega_n^2 - \omega_{log}^2) (I_o + I_{fz} + m_s L^2) (m_s L^2 + I_{py} + I_o + I_{fy})}{\omega_n}
\]
(29)

3.2. Frequency Equations and Optimal Tuning

When the harmonic disturbing base excitation is \( Z = z_0 e^{j \omega t} \), the responses may be written as an angular frequency \( \omega \):

\[
y = y_0 e^{j \omega t}, \quad \theta = \theta_0 e^{j \omega t}, \quad \text{and} \quad \varphi = \varphi_0 e^{j \omega t}.
\]
(18)

Substitution of Eq. (18) into Eqs. (14)–(16) gives in the matrix form of Eq. (19).

By equating the determinant of the coefficients of \( y_0, \theta_0, \) and \( \varphi_0 \) to zero, we obtain the characteristic equation as:

\[
a_0 (\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3) = 0;
\]
(20)
in which:

\[
\lambda = \omega^2;
\]

\[
a_0 = (m_s L^2) (m_s L^2 + I_{py} + I_o + I_{fy}) - M_t (I_o + I_{fz} + m_s L^2) (m_s L^2 + I_{py} + I_o + I_{fy});
\]

\[
a_1 = M_t m_s g L (2 m_s L^2 + 2 I_o + I_{py} + I_{fy} + I_{fz}) + k (I_o + I_{fz} + m_s L^2) (m_s L^2 + I_{py} + I_o + I_{fy}) + M_t (I_p \Omega)^2 - (m_s L^2) m_s g L \right] / a_0;
\]

\[
a_2 = -[M_t (m_s g L)^2 - k m_s g L (2 m_s L^2 + 2 I_o + I_{py} + I_{fy} + I_{fz})] / a_0;
\]

\[
a_3 = k (m_s g L)^2 / a_0.
\]
(21)

The roots of this equation are:16

\[
\lambda_1 = -\frac{1}{2} (S - T) - \frac{1}{3} a_1 + \frac{1}{2} \sqrt{3} (S - T);
\]

\[
\lambda_2 = -\frac{1}{2} (S + T) - \frac{1}{3} a_1 - \frac{1}{2} \sqrt{3} (S - T);
\]

\[
\lambda_3 = S - T - \frac{1}{3} a_1;
\]
(22)
in which:

\[
Q = \frac{3 a_2 - a_1^2}{9};
\]

\[
R = \frac{9 a_1 a_2 - 27 a_3 - 2 a_1^3}{54};
\]

\[
S = \sqrt{R + \sqrt{Q^3 + R^2}};
\]

\[
T = \sqrt{R - \sqrt{Q^3 + R^2}}.
\]
(23)

Besides, Eq. (19) can be solved for \( y_0, \theta_0, \) and \( \varphi_0 \) as:

\[
\frac{y_0}{z_0} = k \left\{ -\omega^2 (I_o + I_{fz} + m_s L^2) + m_s g L \left[ -\omega^2 (m_s L^2 + I_{py} + I_o + I_{fy}) + m_s g L \right] \right\} / \left\{ a_0 (\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3) \right\};
\]
(24)

\[
\theta_0 = \omega^3 z_0 m_s L k \left[ -\omega^2 (m_s L^2 + I_{py} + I_o + I_{fy}) + m_s g L \right] / a_0 (\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3);
\]
(25)

\[
|\varphi_0| = \frac{\omega^3 z_0 k m_s L \Omega}{a_0 (\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3)}.
\]
(26)
In order to make the displacement of the primary mass zero, the numerator of Eq. (24) should be set as:

\[ k \left\{ -\omega^2 \left( I_o + I_{fy} + m_s L^2 \right) + m_s g L \right\} \left\{ -\omega^2 \left( m_s L^2 + I_{py} + I_o + I_{fy} \right) + m_s g L \right\} - (I_p \Omega \omega)^2 = 0. \] (27)

Thus, the disk speed is yielded by Eq. (28).

It can be seen from Eq. (28) and Fig. 4 that the optimum disk speed is equal to zero when the natural frequency of the inner pendulum is equal to the resonant frequency. However, this equilibrium is not needed anymore with an adjustable disk speed of the gyroscope to obtain zero amplitude for the target frequency. From Eq. (28), the calculated optimum disk speed \( \Omega \) for \( \omega = \omega_n \) is 23373 rpm. However, optimum disk speed is about 22500 rpm from the numerical solution of Eqs. (7)–(9). The reason for this 3.88% error is due to the fact that the linearized equations reduced the accuracy of the approximation.

The responses of the system as a function of the frequency ratio \( \omega / \omega_n \) are shown in Figs. 5, 6, and 7. When the optimum disk speed is used, the gyro-pendulum absorber includes two resonant frequencies \( (\omega_2 \text{ and } \omega_3) \) at which the response is characteristic of two-degree-of-freedom systems excited by a harmonic excitation. While eliminating vibration at the known impressed frequency \( \omega \), one of the system’s two natural frequencies is less than the tuned frequency while the other is greater. So, the excitation frequency must be kept away from the resonant frequencies \( (\omega_2 \text{ and } \omega_3) \).

The response of the primary mass as functions of the disk speed ratio \( \Omega / \Omega_{opt} \) is shown in Fig. 8 for the value of the primary mass frequency \( \omega_n \). It can be seen that the amplitude of the primary mass is zero when the disk speed \( (\Omega) \) is tuned to the optimum speed \( (\Omega_{opt}) \). If the gyro operates at other speeds, the amplitude of vibration of the primary mass becomes larger.

4. RESULTS

In the following calculations, Lagrange’s equations of motion may be solved by using a Matlab software tool that involves the fourth-order Runge-Kutta method. The physical parameters of the numerical examples are given in Table 1. In order to identify the dynamical behaviour, time response was simulated with the time step size of 0.005 s, and zero initial conditions.
Table 2. Comparison of resonant frequencies with different disk speeds $\Omega$.

<table>
<thead>
<tr>
<th>$\Omega$ (rpm)</th>
<th>Theoretical $\omega_1$ (Hz)</th>
<th>ANSYS $\omega_1$ (Hz)</th>
<th>Error for $\omega_1$ %</th>
<th>Theoretical $\omega_2$ (Hz)</th>
<th>ANSYS $\omega_2$ (Hz)</th>
<th>Error for $\omega_2$ %</th>
<th>Theoretical $\omega_3$ (Hz)</th>
<th>ANSYS $\omega_3$ (Hz)</th>
<th>Error for $\omega_3$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5898</td>
<td>0.6087</td>
<td>3.1050</td>
<td>0.5965</td>
<td>0.6171</td>
<td>3.3382</td>
<td>2.3793</td>
<td>2.3799</td>
<td>0.0000</td>
</tr>
<tr>
<td>5000</td>
<td>0.4015</td>
<td>0.4200</td>
<td>4.4048</td>
<td>0.4176</td>
<td>0.4340</td>
<td>3.5332</td>
<td>1.2387</td>
<td>1.2589</td>
<td>0.0542</td>
</tr>
<tr>
<td>10000</td>
<td>0.2850</td>
<td>0.3007</td>
<td>5.2839</td>
<td>0.2982</td>
<td>0.3150</td>
<td>5.2384</td>
<td>0.8767</td>
<td>0.8940</td>
<td>0.0833</td>
</tr>
<tr>
<td>15000</td>
<td>0.2147</td>
<td>0.2276</td>
<td>6.4212</td>
<td>0.2277</td>
<td>0.2424</td>
<td>6.4072</td>
<td>1.3387</td>
<td>1.3571</td>
<td>0.0084</td>
</tr>
<tr>
<td>20000</td>
<td>0.1701</td>
<td>0.1808</td>
<td>7.0831</td>
<td>0.1832</td>
<td>0.1981</td>
<td>7.0590</td>
<td>2.0139</td>
<td>2.0377</td>
<td>0.0138</td>
</tr>
<tr>
<td>25000</td>
<td>0.1400</td>
<td>0.1490</td>
<td>7.5504</td>
<td>0.1500</td>
<td>0.1641</td>
<td>7.5165</td>
<td>2.7870</td>
<td>2.8127</td>
<td>0.0257</td>
</tr>
<tr>
<td>30000</td>
<td>0.1186</td>
<td>0.1264</td>
<td>7.9270</td>
<td>0.1265</td>
<td>0.1408</td>
<td>7.8834</td>
<td>3.5422</td>
<td>3.5680</td>
<td>0.0268</td>
</tr>
<tr>
<td>35000</td>
<td>0.1027</td>
<td>0.1095</td>
<td>8.2032</td>
<td>0.1093</td>
<td>0.1236</td>
<td>8.1592</td>
<td>4.2969</td>
<td>4.3227</td>
<td>0.0268</td>
</tr>
<tr>
<td>40000</td>
<td>0.0905</td>
<td>0.0965</td>
<td>8.4786</td>
<td>0.0963</td>
<td>0.1104</td>
<td>8.4344</td>
<td>5.0515</td>
<td>5.0873</td>
<td>0.0358</td>
</tr>
<tr>
<td>45000</td>
<td>0.0808</td>
<td>0.0862</td>
<td>8.7532</td>
<td>0.0861</td>
<td>0.0996</td>
<td>8.7088</td>
<td>5.8062</td>
<td>5.8420</td>
<td>0.0358</td>
</tr>
<tr>
<td>50000</td>
<td>0.0730</td>
<td>0.0799</td>
<td>8.9979</td>
<td>0.0797</td>
<td>0.0930</td>
<td>8.9535</td>
<td>6.5609</td>
<td>6.6067</td>
<td>0.0358</td>
</tr>
</tbody>
</table>

Figure 9. Comparison of the theoretical and ANSYS results without disk speed. The period is the same as the natural period of the primary mass $M_t$ with a base excitation $Z = z_0 \cos(\omega_n t)$. ($z_0 = 0.001$ m and $\omega_n = 2.3682$ Hz).

Figure 10. Comparison of the theoretical and ANSYS results with optimum disk speed. The period is the same as the natural period of the primary mass $M_t$ with a base excitation $Z = z_0 \cos(\omega_n t)$. ($z_0 = 0.001$ m and $\omega_n = 2.3682$ Hz).

Figure 11. Comparison of the theoretical and ANSYS results without disk speed. The period is the same as the natural period of the primary mass $M_t$ with a base excitation $Z = z_0 \cos(\omega_n t)$. ($z_0 = 0.001$ m and $\omega_n = 2.3682$ Hz).

Figure 12. Comparison of the theoretical and ANSYS results with optimum disk speed. The period is the same as the natural period of the primary mass $M_t$ with a base excitation $Z = z_0 \cos(\omega_n t)$. ($z_0 = 0.001$ m, $\Omega_{opt} = 22500$ rpm and $\omega_n = 2.3682$ Hz).
In this section, the calculated results of Lagrange’s equations were compared with the results obtained from ANSYS Workbench to work out its displacements and mode frequencies. The simplified dynamic simulation model is given in Fig. 2. The theoretical and ANSYS results of the resonant frequencies with different disk speeds of gyro are given in Table 2. It can be deduced from the comparison that the tendencies of the mode frequencies agreed well with the ANSYS ones. However, the first mode frequencies \( \omega_1 \) contain minor errors due to the linearized equations. When rotor speed has higher values, which means more force in the horizontal axis, the mode frequencies of \( \omega_2 \) and \( \omega_3 \) increase while the first mode frequency \( \omega_1 \) decreases.

The time responses of displacements from the theoretical and ANSYS approaches are shown in Fig. 9–12. The comparison showed that the displacements of the theoretical and ANSYS results are almost the same for the value of the primary mass frequency \( \omega_n \). Meanwhile, the vibration mitigation performance of the gyro-pendulum increased using the optimum rotational speed of disk at the resonant frequency of primary mass.

5. CONCLUSIONS

This paper provides some theoretical equations to investigate a gyro-pendulum absorber which can be capable of producing unidirectional thrust along the forcing excitation axis while the disk is spinning. In order to verify the above results, the model is built strictly according to the size in Table 1. Therefore, the results of ANSYS are exactly in accord with theoretical equations due to the simplified model. The correctness of the theoretical results is verified by ANSYS simulations. From the consistency of the present study, the following
conclusions are found:

- It can be deduced from the comparison that the theoretical results agree well with the ANSYS ones. The resonant frequencies calculated by the proposed theoretical prediction match well with those obtained by ANSYS.

- This paper introduces the frequency equations to investigate the optimum disk speed of the gyro. The study clearly indicates that the relation between the gyro-pendulum and the resonant frequency of primary mass should be determined by Eq. (29), which is an approximation of the optimum angular momentum of the gyro-pendulum. This equation shows that, as much as disk speed is increased, the rotary inertia of the disk can be reduced. Since the rotary inertia of the disk is proportional to the mass of disk, the weight and volume of the system can be reduced. Additionally, the rotation of disk can be used in both directions, in which the pendulum rotates in direction of the disk when the rotary inertia of the disk is assumed to be positive.

The proposed gyro-pendulum has a practical advantage in comparison with conventional ones because, with an adjustable disk speed of gyroscope, the gyro-pendulum is efficient on a large frequency range from the pendulum resonance to the main system resonance.

REFERENCES


1. INTRODUCTION

The asphalt paver (AP) was one of the construction machines used to pave asphalt mixture on road surface construction rapidly and uniformly.\(^1,2\) Therefore, the vibration screed system of an asphalt paver (VSS-AP) was equipped with a vibrator screed and a couple of tamper mechanisms (compacting beams).\(^3-5\) The tamper was used to compress the asphalt mixture to become tighter and more uniform in density while the vibrator screed was used to improve smooth and the finish of the road surface construction. The paving performance of the AP was mainly assessed by three indexes of the compression efficiency, paving quality, and working stability.\(^1,6\)

The compression efficiency was affected not only by the operating parameters of the VSS but also by the asphalt materials and ground vibrations.\(^1,7-12\) The influence of density, temperature, and size of particles of the asphalt mixture on compression efficiency was studied.\(^9,13-15\) The studies showed that the temperature of the hot asphalt-mix greatly impacted on the asphalt density and compression efficiency. In order to achieve the desired density, the temperature of the hot asphalt-mix in the compression process was quickly analyzed by a fuzzy clustering technique.\(^10\) The errors of unequal compaction coverage, temperature, and compaction delay were then controlled based on the compaction monitoring system.\(^16\) The influence of the different temperature regions of the asphalt-mix on compression efficiency was analyzed by using a multi-sensor infrared temperature scanning bar system.\(^17\) Besides, the ground motions and vibrations could affect the performance of machines working on the ground,\(^11,18\) especially the elastoplastic ground soils.\(^19\) Additionally, with the operating parameters of the VSS, the influence of the compression forces,\(^20\) phase deflections, and excitation frequencies\(^7,21\) of the tamper on the smoothness of the pavement was also investigated. The results indicated that the vibration excitation of the tamper mainly affected the compression efficiency. However, in all the above research, only the vertical vibration with a quarter model of the VSS-AP was considered.

The paving quality and working stability of the AP were significantly affected by the excitation frequency of the tamper \(f_t\) and of vibration screed \(f_s\).\(^2,6,22\) Based on the 2D dynamic model of the VSS,\(^3\) the analysis results showed that the paving quality was better with the vibration excitation of \(f_t\) from 10 to 20 Hz and of \(f_s\) from 30 to 40 Hz. The optimal paving performance was found at the excitation frequency 15 Hz of \(f_t\) and 32 Hz of \(f_s\) on a type of asphalt-mix materials.\(^6\) Three different types of asphalt-mix materials of SMA-13, AC-20, and AC-25 were then expanded to fully analyze the influence of the excitation frequency of \(f_t\) and \(f_s\).\(^1,4,5\) All researches showed that compression efficiency was significantly improved, but paving quality and working stability were still low. To solve this problem, the mass of tamper and eccentric distance of the eccentric shaft were optimized to reduce the vibration amplitude on all nodes of the screed which helped to improve the compression surface quality.\(^23\) Both the excitation frequencies of \(f_t\) and \(f_s\) were also optimized based on a genetic algorithm to decrease the pitching vibration \(\phi\) of the screed.\(^5\) Besides, the angular deviation \(\gamma\) between the front/rear tampers was also optimized via ADAM.
and ABAQUS software to improve the paving performance. The studies concluded that the angular deviations of the tamper significantly influenced the working stability apart from the excitation frequencies. Additionally, the actual width of the VSS was usually 9000 or 1200 mm. Therefore, the rolling vibration (θ) of the screed, the angular deviations of excitation forces between the right and left sides (α) and between tampers (β) of the tamper could greatly impact the paving performance of the AP. However, this problem has not been investigated yet.

To evaluate the influence of vibration on the VSS as well as the paving performance of the AP, the root-mean-square (RMS) of the acceleration response was used as the evaluation index. In this study, the experimental method is used to evaluate the paving performance of the AP. A 3D dynamic model is also built to analyze the influence of the dynamic parameters of the VSS on the paving efficiency based on the RMS acceleration responses at the centre of gravity of the screed. The dynamic parameters are then controlled to further improve the paving performance of the AP.

The point of this new study is the evaluation of the impact of excitation vibrations on the screed shaking which mainly causes the unevenness of the paving density and paving surface, whilst the dynamic parameters are controlled to enhance the paving performance of the AP.

2. VIBRATION ANALYSIS OF THE VSS-AP

2.1. Experimental Model

The VSS has been equipped with eight couples of tamper mechanisms, in which, four couples of the right tamper mechanisms and four couples of the left tamper mechanisms are symmetrically designed. The angular deviation β between the excitation forces of tamper mechanisms is also symmetrically designed following the tamper mechanisms. With each couple of tampers, the angular deviation between the front and rear tampers is γ. Besides, the vibration excitation of the left and right tamper mechanisms is also deviated by a phase angle α. Additionally, the mass of the eccentric configuration of the vibrator screed is installed at the centre of gravity of screed.

To analyze the effect of vibration excitations on the paving performance, an asphalt paver with a screed width of 12000 mm, 3D accelerometers ICP®, and an analysis system of Belgium LMS have been used to measure the accelerations under screed floor. There are 24 sensors with their sampling frequency of 300 Hz installed at 12 points on the front (f) and rear screed (r), as shown in Fig. 1. The excitation of the tamper and screed are defined by the ratios of κ and δ as follows:

$$\kappa = \frac{\omega_t}{\omega_{t_{\max}}} = \frac{f_t}{f_{t_{\max}}};$$
$$\delta = \frac{\omega_s}{\omega_{s_{\max}}} = \frac{f_s}{f_{s_{\max}}};$$

(1)

where \{ω_t, ω_s\} and \{ω_{t_{\max}}, ω_{s_{\max}}\} are the rotation angular velocities and maximum rotation angular velocities of tamper and vibrator screed; \{f_t, f_s\} and \{f_{t_{\max}}, f_{s_{\max}}\} are their corresponding excitation frequencies, (f_{t_{\max}} = 20.83 and f_{s_{\max}} = 45 Hz).

A multi-point measurement method of vertical RMS accelerations under the screed floor has been applied and performed in two cases: (1) under excitation frequencies of tamper \(f_t = [10\%, 20\%, \ldots, 100\%] \times f_{t_{\max}}\) corresponding to κ = [0.1, 0.2, \ldots, 1.0] and without the excitation of vibrator screed δ = 0. (2), also under the same excitations of κ but adding 50% of the maximum excitation of vibrator screed δ = 0.5. Through the signal processor, the measured data of the acceleration responses and their RMS values at 24-test positions have been displayed.

This experimental method can simultaneously determine the acceleration responses at different points under the front/rear screed floors, while previous studies have only measured the acceleration response at each measurement point or at three different points under the screed floor. Therefore, this
method easily analyzes the stability and compressive efficiency as well as calculates the vibration shaking of the VSS in comparison with the previous experiments.

### 2.2. Analysis of Measurement Results

The measured RMS results under the front/rear screed floors at the different excitations of the tamper $\kappa = [0.3, 0.5, 0.7, 0.9]$ with $\delta = [0, 0.5]$ are shown in Figs. 2 and 3. Without the excitation of the vibrating screed $\delta = 0$, observing Figs. 2a and 2b, the RMS values at the measured points under the front/rear screed floors are relatively uniform and symmetrical with $\kappa = [0.3, 0.5]$. It implies that the paving quality is relatively stable, but the compression efficiency is low due to the RMS values being quite small. With $\kappa$ is 0.7 or 0.9, the RMS values are strongly increased, thus, the compression efficiency of the VSS is also enhanced. However, the RMS values at the measured points are remarkably skewed and asymmetrical, so the VSS works are unstable and the paving quality is low.

Under the actual paving condition of the AP, the excitation of the vibrating screed $f_s$ has been added to the VSS part from the main excitation $f_t$ of the tamper. With adding 50% the excitation of $f_s$ ($\delta = 0.5$), the results in Figs. 3a and 3b show that the RMS values at the measured points are not only greatly skewed but also higher than their results without the excitation of $f_s$, especially at $\kappa = 0.3$. This issue is due to the influence of the vibration excitation of $f_s$ apart from the main excitation of $f_t$. Similarly to the case without the excitation of the vibrating screed, the compression efficiency of the VSS is also increased while the paving quality and stability are reduced.

Consequently, it can be concluded that both vibration excitations of $f_t$ and $f_s$ greatly affect the paving performance of the AP.

Based on the experimental results, the RMS values at the points on the screed floor are greatly different. It means that the screed shaking is large under the excitation frequencies of $f_t$ and $f_s$. However, this issue has not yet been concerned in the previous studies. To determine the pitching and rolling vibrations of the screed, the RMS acceleration response of the vertical, pitching and rolling vibrations at the centre of gravity of the screed has been calculated based on the RMS value at the measured points and kinetic relationship of the screed. Assuming that the angular deformations of the screed are negligible, the RMS values at the centre of gravity of screed have been determined by:

\[
\ddot{z} = \frac{\ddot{z}_{fs}b_4 + \ddot{z}_{rs}b_3}{b_3 + b_4}; \quad \ddot{\phi} = \frac{\ddot{z}_{rs} - \ddot{z}_{fs}}{b_3 + b_4}; \quad \ddot{y} = \frac{(\ddot{z}_{r1} - \ddot{z}_{r11})d_3 + (\ddot{z}_{f4} - \ddot{z}_{f13})d_4}{(l_1 + l_2)(d_3 + d_4)}; \quad (2)
\]

where $\ddot{z}_k$ is the vertical RMS acceleration at the measured points of $k = f_1, f_6, f_{11}, r_1, r_6, r_{11}; b_3, 4$ and $l_1, 2$ are the distances from the centre of gravity of screed to the corresponding measurement points in the axis of $x$ and $y$.

The measured RMS results at the centre of gravity of screed under the excitations of $\kappa = [0.1, 0.2, \ldots, 1.0]$ with $\delta = [0, 0.5]$ are plotted in Figs. 4a–4c. The results show that the vertical, pitching and rolling RMS values of the screed are greatly affected by both ratios of $\kappa$ and $\delta$, especially the ratio of
been established as in Fig. 5. The dynamic model which can fully reflect the screed shaking has been developed to analyze the vibration of the VSS.

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3. MODELLING OF THE VSS

3.1. Mathematical Model

Based on the actual structure of the VSS-AP, a 3D nonlinear dynamic model which can fully reflect the screed shaking has been established as in Fig. 5.

In Fig. 5, the vertical, pitching, and rolling motions of the VSS-AP model in Fig. 5b as follows:

\[
m\ddot{x} = -\sum_{n=1}^{8} F_n + \sum_{i=1}^{8} F_{ti} + F_s;
\]

\[
I_g \ddot{\phi} = \sum_{n=1}^{4} (-1)^{n+1} F_n b_v + \sum_{i=1}^{8} F_{ti} b_2 - F_s b_1;
\]

\[
I_z \ddot{\theta} = \sum_{n=1}^{4} (-1)^{n} F_n l_u + \sum_{i=1}^{8} F_{ti} l_i - \sum_{i=4}^{8} F_{ti} l_i;
\]

where \( F_n \) is the vertical dynamic force of the vibrator screed-pavement interaction in the paving process; \( F_{ti} \) and \( F_s \) are the excitation forces of tamper and vibrator screed.

The vertical dynamic force \( F_n \) has been determined by:

\[
F_n = k_n (z + (-1)^n b_v \phi + (-1)^{n+1} l_u \theta) + c_n [\ddot{z} + (-1)^n b_v \ddot{\phi} + (-1)^{n+1} l_u \ddot{\theta}];
\]

where \( n = 1, 2, 3, 4 \) then \( u = 1 \) and \( v = n + 2 \) and \( v = n \).

The excitation force \( F_{ti} \) has been calculated based on the VSS-AP model in Fig. 5b as follows:

\[
F_{ti} = \sum_{x=f,r} (-m_{x,i} \ddot{x}_i + c_{x,i} \dddot{x}_i + k_{x,i} \ddot{x}_i);
\]

where \( i = 1, 2, 3, 4 \) then \( a = 0 \) and \( x = f \) then \( b = 0 \);

\[
z_{x,i} = e_{x,i} \sin(\omega_t t + \beta_i + \psi); \quad \psi = a \alpha + b \gamma;
\]

where \( i = 5, 6, 7, 8 \) then \( a = 1 \) and \( x = r \) then \( b = 1 \); \( \alpha \), \( \beta_i \), and \( \gamma \) are the angular deviations of excitation forces between the right and left sides, between tampers, and between the first/second tampers of the tamper mechanisms, respectively.

By replacing Eq. (6) into (5) and mathematical transformation, we have:

\[
F_{ti} = \sum_{x=f} \left\{ m_{x,i} e_{x,i} \omega^2 \sin(\omega t + \beta_i + \psi) + c_{x,i} e_{x,i} \omega^3 \cos(\omega t + \beta_i + \psi) + k_{x,i} e_{x,i} \sin(\omega t + \beta_i + \psi) \right\};
\]

The excitation force of the vibrator screed \( F_s \) has been determined by:

\[
F_s = m_s e_s \omega^2 \sin \omega_s t;
\]

where \( m_s \) and \( e_s \) are the mass and distance of the eccentric configuration.

By combining Eqs. (3), (4), (7) and (8), the vertical, pitching, and rolling accelerations at the center of the screed have then been determined.
Also, the vertical acceleration at points on the front/rear screed floors in Fig. 1c have been given by:
\[
\ddot{z}_\chi = \ddot{z} + (-1)^v b_y + (-1)^{v+1} l_y \ddot{\theta};
\]  
(9)

where subscript \( \chi \) denotes \( f_y \) or \( r_y \); when \( y = 0, 1, \ldots, 6 \) then \( v = 1 \), and when \( y = 7, 8, \ldots, 12 \) then \( v = 2 \).

To evaluate the influence of the vibration on the systems, the RMS acceleration response was used as the evaluation index. In this study, the paving performance of the AP is evaluated via the RMS values at points on the screed floor or the centre of gravity of screed as follows:

\[
RMS_x = \sqrt{\frac{1}{T} \int_0^T [\ddot{z}_x(t)]^2 \, dt}^\frac{1}{2} ;
\]  
(10)

\[
RMS_w = \sqrt{\frac{1}{T} \int_0^T \{a_w(t)\}^2 \, dt}^\frac{1}{2} ;
\]  
(11)

where \( \ddot{z}_x(t) \) is the vertical acceleration responses calculated by Eq. (9); subscript \( w \) refers to the vertical, pitching, and rolling motions at the centre of gravity of screed; \( a_w(t) \) is the acceleration at the motion of \( w \); and \( T \) is the duration of the simulation.

Therefore, the result of the RMS, values on the screed floor is high and uniform; or the result of the RMS, value is high and both the RMS, and RMS, values are low, it means that the compression efficiency, paving quality, and working stability of the VSS are better.

### 3.2. Influence of the Parameters of the VSS

To analyze the influence of the dynamic parameters of the VSS on the paving performance based on the dynamic model of the VSS-AP, the accuracy of the mathematical model should be verified by comparing the results between simulation and experiment methods under the same operating conditions of \( \kappa = [0.1, 0.2, \ldots, 1.0] \) with \( \delta = [0, 0.5] \). With the reference parameters of the VSS-AP, as listed in Table 1, the simulation results of RMS values at the centre of gravity of screed are compared with the measured results in the same Figs. 4a–4c. Observing the comparison results, it can see that the characteristic curves between simulation and measurement results are similar. Thus, VSS-AP’s mathematical model can be reliable in analysis of the influence of dynamic parameters.

The VSS’s vibration is greatly affected at \( \kappa = 0.7 \). Thus, to evaluate the influence of other dynamic parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha ), ( \beta ), ( \gamma )</td>
<td>( 0.163 ), ( 0.114 ), ( 0.0 )</td>
<td>( \delta ), ( \epsilon )</td>
<td>( 0.125 ), ( 0.3083 )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( 2700 )</td>
<td>( \kappa )</td>
<td>( 2.5 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

Table 1. The numerical values of the VSS-AP.
Influence of the angular deviation $\alpha$: The RMS results are also given in the same Figs. 6a–6c. It can see that the RMS values of the screed are insignificantly changed by $\alpha$. This is due to the angular deviations $\alpha$ between excitation forces of tampers being symmetrically designed following the tamper mechanisms.

Influence of the angular deviation $\gamma$: The simulation results are also plotted in Fig. 6. Observing both Figs. 6a and 6b, we can see that both the vertical and pitching RMS values of the screed are decreased with increasing the angular deviation of $\gamma$ from $0^\circ$ to $180^\circ$, while the rolling RMS value is increased, as shown in Fig. 6c. Both the vertical and pitching RMS values can reach a maximum at a range of $\gamma$ from $0^\circ$ to $60^\circ$. Therefore, the compression efficiency of the VSS is the maximum. With the width of 12000 mm of the screed, the rolling vibration thus strongly influences the paving quality and working stability of the VSS-AP. However, the rolling RMS value is the smallest in a range of $\gamma$ from $0^\circ$ to $60^\circ$. Accordingly, to improve the paving performance, the angular deviation of $\gamma$ from $0^\circ$ to $60^\circ$ should be controlled.

Based on the above analysis results, it is deduced that three dynamic parameters of $\delta$, $\alpha$, and $\gamma$ greatly influence the paving performance of the AP apart from the main excitation of $f_t$. The compression efficiency of the VSS is increased while both paving quality and working stability are reduced and vice versa. It is very difficult to satisfy all objective functions simultaneously. In addition, only excitation frequencies of $f_t$ and $f_s$ have been optimized. $\{\delta, \alpha, \gamma\}$, the numerical values of $\delta = [0.1, 0.2, \ldots, 1.0]$ and of $\{\alpha, \beta, \gamma\}$ from $0^\circ$ to $180^\circ$ are respectively simulated under the working condition of $\kappa = 0.7$. The RMS results of the screed are shown in Figs. 6a–6c.

The simulation results are also plotted in Fig. 6. Observing both Figs. 6a and 6b, we can see that both the vertical and pitching RMS values of the screed are decreased with increasing the angular deviation of $\alpha$ from $0^\circ$ to $180^\circ$, while the rolling RMS value is increased, as shown in Fig. 6c. Both the vertical and pitching RMS values can reach a maximum at a range of $\alpha$ from $0^\circ$ to $60^\circ$. Therefore, the compression efficiency of the VSS is the maximum. With the width of 12000 mm of the screed, the rolling vibration thus strongly influences the paving quality and working stability of the VSS-AP. However, the rolling RMS value is the smallest in a range of $\alpha$ from $0^\circ$ to $60^\circ$. Accordingly, to improve the paving performance, the angular deviation of $\alpha$ from $0^\circ$ to $60^\circ$ should be controlled.
Figure 7. Control system model of the VSS-AP.

Based on the characteristic curves of $\kappa$, $\alpha$, and $\gamma$ in Figs. 4 and 6, and analysis results of the influence of parameters on paving performance of the AP, the numerical values $\alpha$ and $\gamma$ of the control function in Eq. (6) $\psi = a\alpha + b\gamma$ have been expressed as:

$$\begin{align*}
30^\circ &\leq \alpha \leq 90^\circ; 0^\circ \leq \gamma \leq 60^\circ; &\text{if } \kappa \leq 0.7 \\
60^\circ &\leq \alpha \leq 120^\circ; 30^\circ \leq \gamma \leq 90^\circ; &\text{else}
\end{align*}$$

(12)

The numerical values of $\alpha$ and $\gamma$ in Eq. (12) are then controlled based on the FLC.

**Applying the FLC:** In order to control the vibration systems, the control methods as PID, FLC, or H were mainly applied. Further development was the combination of control methods such as PID-Fuzzy, PID-Neural, Skyhook-Fuzzy. The different control methods were applied depending on different control objectives. This research aim is to control numerical values of $\alpha$ and $\gamma$ to reach the goals of the maximum RMS value, minimum RMS, and RMS values. In the above control methods, the FLC is a controller that does not depend on the designed operation conditions, on the contrary, the FLC can control multi-objective based on its wide and fuzzy inference system. Thus, the FLC is suitable for controlling the vibration of the VSS.

**FLC’s design process:** The structure of FLC includes a fuzzification interface (FI), a fuzzy inference system (FIS), and a defuzzification interface (DI). The control principle of the FLC is that the input-numerical values in FI are firstly transformed into input-linguistic variables (LVs), the FIS is then formed into input-linguistic variables based on the membership function of control rules, and finally, the output-LVs are transformed back to output-numerical values via DI.

Based on the FLC model in Fig. 7, three input-numerical values $\{\kappa, \alpha, \gamma\}$ and two output-numerical values $\{\alpha, \gamma\}$ with their membership functions are defined as in Fig. 8. Herein, three input-LVs are defined by minimum (MIN, $\kappa \leq 0.7$), maximum (MAX, $\kappa > 0.7$), negative big (NB), negative medium (NM), negative small (NS), zero (Z), small (S), medium (M), and big (B). Besides, two output-LVs are also defined as very small (VS), small (S), small medium (SM), medium (M), and big (B). If-then rules in FIS are applied to describe the relationship of input- and output-numerical values based on the analysis results in Figs. 4 and 6 and the designer’s experience. There are ninety-eight rules in which forty-nine rules are given in Table 2 and forty-nine rules are given in Table 3 as follows, $(i = 1 – 49)$:

- If $\alpha$ is MIN, $\alpha$ is $A_i$, and $\alpha$ is $B_i$, then $\alpha$ and $\gamma$ are $C_i$;
- If $\kappa$ is MAX, $\alpha$ is $A_i$, and $\alpha$ is $B_i$, then $\alpha$ and $\gamma$ are $D_i$.

According to the minimum function and the centroid method of Mamdani and Assilian, the FIS of Mamdani has been selected to control the VSS model.

**4.2. Control Results**

Based on the control system model and the FLC method, the numerical simulation is then performed to control the vibration of the VSS. The control results of the acceleration responses at the centre of gravity of screed with $\kappa = 0.7$ are plotted in Fig. 9.

The control results show that the vertical acceleration re-
Figure 9. Acceleration responses of the screed with $\kappa = \delta = 0.7$.

Figure 10. Control results of the vertical, pitching, and rolling RMS accelerations at the center of gravity of the screed.

The RMS values are used to evaluate the control performance on the vertical, pitching, and rolling RMS accelerations at the center of gravity of the screed. The control results show that the vertical RMS values are increased while both the pitching and rolling RMS values are decreased in comparison with their RMS results without control. It implies that the paving performance of the AP is significantly improved under different excitation frequencies $f_t$ of tamper.

With the width of 12000 mm of the screed, the rolling acceleration of screed greatly affects paving quality and working stability. Therefore, the vertical RMS accelerations at points on the front/rear screed floors are also used to evaluate the control performance on the paving quality and working stability of the VSS under different working conditions of $\kappa = \delta = 0.5$ and $\kappa = \delta = 0.7$. The control results of RMS values are plotted in Fig. 11, and their maximum (Max-) and minimum (Min-) RMS values are also listed in Tables 4 and 5.

Table 4. Maximum and Minimum RMS values on the front screed floor.

<table>
<thead>
<tr>
<th>$\kappa = \delta$</th>
<th>Max-RMS</th>
<th>Min-RMS</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.313</td>
<td>5.195</td>
<td>2.22%</td>
</tr>
<tr>
<td>0.7</td>
<td>8.657</td>
<td>8.529</td>
<td>1.48%</td>
</tr>
</tbody>
</table>

Table 5. Maximum and Minimum RMS values on the rear screed floor.

<table>
<thead>
<tr>
<th>$\kappa = \delta$</th>
<th>Max-RMS</th>
<th>Min-RMS</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.953</td>
<td>4.886</td>
<td>1.35%</td>
</tr>
<tr>
<td>0.7</td>
<td>8.402</td>
<td>8.315</td>
<td>1.04%</td>
</tr>
</tbody>
</table>

Observing Fig. 11a, under both two excitation cases of $\kappa = \delta = 0.5$ and $\kappa = \delta = 0.7$, the vertical RMS values with control at points on the front screed floor are relatively uniform in comparison with their values without control. Besides, the Max- and Min-RMS values on the front screed floor in Table 4 also show that the deviations between Max- and Min-RMS values with control are smaller by 2.22% and 1.48%, while their deviations without control are 8.09% and 8.38% under both two excitation cases. Similarly, the vertical RMS values at points on the rear screed floor with control are also more uniform than without control, as shown in Fig. 11b. The results in Table 5 also show that the deviations between Max-
and Min-RMS values with control are smaller by 1.35% and 1.04%, while their deviations without control are 11.38% and 13.38% under both two excitation cases. Consequently, it can be concluded that the paving quality and working stability of the VSS are significantly improved with the controlled parameters.

5. CONCLUSIONS

The paving performance of the AP is studied via an experimental method. The influence of the dynamic parameters on the paving performance is evaluated via the numerical simulation method. The numerical values of $\alpha$ and $\gamma$ are then controlled to improve the compression efficiency, paving quality, and working stability of the VSS-AP. The research results are summarized as follows:

The excitation frequencies of $f_1$ and $f_2$, and the angular deviations of $\alpha$, $\beta$, and $\gamma$ greatly affect the paving performance of the AP, particularly the numerical values of $f_1$, $\alpha$, and $\gamma$.

Based on the input-numerical signals of the excitation frequency $f_1$ and screed shaking accelerations $a_{\phi}$ and $a_\theta$, the compression efficiency, paving quality, and working stability of the AP are clearly improved by controlling the angular deviations of $\alpha$ and $\gamma$ under different working conditions. Especially, based on the databases of the compaction monitoring system using the global positioning system technologies, the fuzzy clustering techniques apply to quickly analyze the hot mix asphalt compaction data, and the multi-sensor infrared temperature scanning bar system used to analyze the paving quality, the paving performance of the AP can be further improved by controlling the vibration of the VSS based on these databases.

The research results not only contribute to the existing body of knowledge on the asphalt pavers but also can provide an important reference for optimal design or control of the compression force of paving machines to further improve the paving efficiency. Besides, with the soil compactors, the excitation force of the vibratory drum is also used to compact the off-road deformable, however, the excitation force has not yet been interested and controlled. Thus, the research results can also be used as an important reference for controlling the excitation force of the vibratory drum to enhance the compaction efficiency of the soil compactors.

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A Novel Deep Belief Network Model Constructed by Improved Conditional RBMs and its Application in RUL Prediction for Hydraulic Pumps

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Residual Useful Life (RUL) prediction is a key step of Condition-Based Maintenance (CBM). Deep learning-based techniques have shown wonderful prospects on RUL prediction, although their performances depend on heavy structures and parameter tuning strategies of these deep-learning models. In this paper, we propose a novel Deep Belief Network (DBN) model constructed by improved conditional Restrict Boltzmann Machines (RBMs) and apply it in RUL prediction for hydraulic pumps. DBN is a deep probabilistic digraph neural network that consists of multiple layers of RBMs. Since RBM is an undirected graph model and there is no communication among the nodes of the same layer, the deep feature extraction capability of the original DBN model can hardly ensure the accuracy of modeling continuous data. To address this issue, the DBN model is improved by replacing RBM with the Improved Conditional RBM (ICRBM) that adds timing linkage factors and constraint variables among the nodes of the same layers on the basis of RBM. The proposed model is applied to RUL prediction of hydraulic pumps, and the results show that the prediction model proposed in this paper has higher prediction accuracy compared with traditional DBNs, BP networks, support vector machines and modified DBNs such as DEBN and GC-DBN.

1. INTRODUCTION

As the power source of the whole hydraulic system, hydraulic pump plays a pivotal role in the hydraulic system. The performance of the hydraulic pump directly affects the safe operation of the whole system, so it is necessary to search for a method to accurately realize residual useful life prediction (RUL) prediction of the hydraulic pump. By virtue of the remaining life prediction of the hydraulic pump, the system failure caused by hydraulic pump failure can be effectively avoided. At the same time, it can provide a favorable guarantee for the targeted performance inspection plan and the maintenance strategy.

Deep learning is a new research direction in the field of machine learning, which is developed on the basis of Artificial Neural Networks (ANN). Deep belief network (DBN) is a classic deep learning model proposed by Hinton, the founder of deep learning theory in 2006. DBN model is composed of multiple Restrict Boltzmann Machines (RBMs). RBM consists of two layers of neurons, a hidden layer and a visible layer, which are fully and symmetrically connected between layers, but not connected within layers. Using unsupervised learning, each RBM is trained to encode in its weight matrix a probability distribution that predicts the activity of the visible layer from the activity of the hidden layer. The advantage of DBN rests on the unsupervised layer-by-layer pre-training with the Contrastive Divergence (CD) algorithm, on which supervised learning and inference can be efficiently performed. Compared with the traditional shallow learning model, DBN has recently become a popular approach in machine learning for its promised advantages such as fast inference and the ability to encode richer and higher order network structures. With the development of machine learning, DBN has been successfully utilized in machine vision recognition, biometric detection, data prediction and so on. Roy et al. proposed a text recognition method based on DBN and recurrent neural network, which further improved the recognition accuracy. Mohamed et al. proposed a voice recognition method based on DBN, and used TIMIT data to verify that this method had bet-
DBN has achieved good application effect in the field of pattern recognition\(^\text{16}\) and RUL prediction.\(^\text{23–25}\) Zhao et al. proposed a fusion RUL prediction approach based on Deep Belief Network (DBN) and Relevance Vector Machine (RVM) where DBN is responsible for extracting degradation features of Lithium-ion batteries.\(^\text{23}\) Zhang et al. proposed a remaining useful life estimation model based on multi-objective deep belief networks ensemble.\(^\text{24}\) Sun et al. proposed a deep belief echo-state network (DBEN) model to address the issue of slow convergence and local optimum.\(^\text{25}\) Compared with the traditional shallow learning method, DBN can train the model through bottom-up layer by layer learning, and then apply top-down feedback learning to adjust the parameters of the model, so as to realize fast and effective independent learning of data. For high-dimensional degradation characteristics of the hydraulic pump, the DBN model can learn and extract the deep correlation information in the input characteristics, retain key features of information at the same time effectively and reduce the interference of high-dimensional degradation characteristics contained in the component. The complicated implicit function approximation has a very good effect, thus the DBN model is very suitable for application to the hydraulic pump fault prediction field. However, using data discretization method to extract feature data leads to the limited ability of DBN gradient descent, which limits the application of DBN to the prediction of temporal data to some extent. To solve this problem, Taylor\(^\text{22}\) put forward the Conditional Restrict Boltzmann Machine (CRBM) on the basis of RBM. CRBM can effectively utilize the temporal association information of the input data by adding the temporal join factor, so the DBN model can deal with the prediction of time series data stably.\(^\text{13}\) Zhang et al.\(^\text{14}\) applied CRBM to forecast stock data and get better prediction results. Chen et al.\(^\text{15}\) proposed a continuous RBM to predict water quality parameters in the Huai River and obtained good results.

Although CRBM has increased the application of temporal association information, there is still a lack of constraints between the CRBM hidden layers, which makes it difficult for the DBN model to extract the depth association features in the input data and affects the prediction results.\(^\text{16}\) To solve this problem, we added constraint variables between the hidden layer units on the basis of CRBM to adjust the activation probability between the hidden layers, and optimized the training process of CRBM. A novel RUL prediction model based on the ICRBM-DBN was established as a result. In order to verify the prediction accuracy of the ICRBM-DBN model for the time series data, performance of the ICRBM-DBN model is evaluated on the whole lifetime data of hydraulic pumps. The experimental results show that the proposed method is feasible and compared with the RBN-DBN, BP neural networks and Support Vector Machine (SVM), the predicting accuracy is satisfactory, which is able to meet the requirements of CBM.

The structure of this article is organized as follows. In Section 2, the basic theory and mathematical formulas of DBN are introduced in detail. In Section 3, the RUL prediction model based on ICRBM-DBN is proposed to improve the accuracy of prognostics. In Section 4, the performance of the proposed approach is shown on the whole lifetime vibration data of hydraulic pump. In Section 5, conclusions are noted.

\section*{2. DEEP BELIEF NETWORK}

DBN proposed by Hinton et al. is the most widely used in deep learning. DBN is composed of an input layer, a middle layer and an output layer, in which the middle layer is composed of multiple RBMs,\(^\text{17}\) the network structure of DBN is shown in Fig. 1. Figure 1 is a typical DBN model consisting of three layers of RBMs. RBM is a typical energy model consisting of visible and hidden layers.\(^\text{18}\) The first input layer \(v_1\) can be regarded as the visual layer of RBM\(_1\), and the hidden layer \(h_1\) together constitute the first layer RBM. RBM\(_1\) extracts the corresponding information from the visual layer and passes it to the hidden layer of RBM\(_1\). At this time, the hidden layer of RBM\(_1\) can be seen as the visual layer of the next RBM. Then, the data of the RBM\(_2\) visual layer can be further extracted and passed to the hidden layer of RBM\(_2\), and so on, the last hidden layer of RBM is the output data of the DBN model. Thus, a DBN model consisting of multi-layer RBMs is implemented.

RBM is the core of the DBN model. It is a kind of energy model that obtains the dependencies between input parameters by associating the input parameters with a suitable energy function. For the energy models, the magnitude of energy is inversely proportional to the probability of combinations of parameters. This means that if a combination of parameters is considered to have a greater probability of rationality, it should also have a smaller energy. Therefore, for the given set of parameter data, the configuration combinations of the parameters that minimize the corresponding energy values are obtained by training the model continuously.
Assuming that \( m \) and \( n \) are respectively the unit numbers of the hidden layer \( h \) and the visual layer \( v \), where \( h_i \) and \( v_j \) are the \( i \)th unit vectors of the hidden layer and the \( j \)th cell vectors of the visual layer respectively. The probability distribution of the hidden layer element \( h_i \) can be defined as:

\[
P(h_i = 1|v) = \text{sigmoid} \left( \sum_{j=0}^{vis} W_{ij}v_j + b_i \right); \tag{1}\]

where, \( \text{sigmoid} = \frac{1}{1+e^{-z}} \), \( W_{ij} \) is the weight matrix between the hidden layer element and the visual layer element, \( b_i \) is the offset of hidden layer elements. In Eq. (1), the activation probability of the hidden layer elements is modeled according to the \( S \) type by the sigmoid function, and \( W_{ij} \) is constantly updated based on the input data. The probability distribution of the visual layer element is determined by the hidden layer data, and its probability distribution is defined as:

\[
P(v_j = 1|h) = \text{sigmoid} \left( \sum_{i=0}^{hid} W_{ij}h_i + c_j \right); \tag{2}\]

where, \( c_j \) is the offset of the visual layer element. According to the above probability distribution, the joint probability distribution between the hidden layer and the visual layer can be defined as:

\[
p(v,h) = \exp(-E(v,h))/Z; \tag{3}\]

where, \( Z = \sum_{v,h} \exp(-E(v,h)) \) is a normalized function, the RBM energy function can be defined as:

\[
E(v,h) = -\sum_{i=1}^{vis} \sum_{j=1}^{hid} W_{ij}v_jh_i - \sum_{j=1}^{vis} c_jv_j - \sum_{i=1}^{hid} b_ih_i; \tag{4}\]

In Eq. (4), \( E(v,h) \) is the system energy of the RBM. According to the energy model theory, when the model energy is the minimum, the RBM is the most stable. Therefore, the optimal parameters of the RBM model can be obtained by solving the minimum value of \( E(v,h) \).

In training RBM, the method of calculating the minimum gradient for the log likelihood function is usually used to solve the DBN model parameters, the RBM weight update model is defined as:

\[
\Delta w_{ij} = E_{\text{data}}(v_jh_j) - E_{\text{model}}(v_jh_j); \tag{5}\]

where, \( E_{\text{data}}(v_jh_j) \) is the energy value expectation of training samples, \( E_{\text{model}}(v_jh_j) \) is the expectation defined by the model. In the process of solving \( \text{min}_{w}(-\log P(v,h)) \), due to the existence of normalization factor, its computational complexity is high, and it is difficult to accurately solve the DBN model parameters. Hinton proposed an approximate solution called the Contrastive divergence (CD) algorithm. The CD algorithm calculates the model parameters by single-step or multi-step Gibbs (Gibbs Chain) sampling, completes the above two expectations updates, and achieves the fast learning of RBM.

In summary, the DBN model is trained by greedy algorithm layer by layer, and the training process of DBN model is shown in Fig. 2. First of all, RBM in the first layer uses CD algorithm to train the input feature data, and the model parameters of RBM in the first layer are obtained. Then, the hidden layer of RBM in the first layer is used as the visual layer of the next RBM, and continues to train until the top of the DBN model. Finally, the DBN model uses the labeled input characteristic data to reverse trimming model parameters, and optimizes the model parameters by supervised training. When the model output error is less than a predetermined threshold, the model training is completed.

### 3. THE THEORY OF ICRBM-DBN

#### 3.1. ICRBM

Since RBM cannot effectively utilize the temporal association information in time-series data, the gradient descent ability of RBM is limited. This limits the application of the RBM-DBN model in the prediction of time-series data. On the basis of RBM, Montufar proposed a CRBM based on RBM.\(^{19}\) CRBM is an extension of RBM, which inherits many good features of RBM, including simple reasoning process and effective training process, and the structure of CRBM is shown in Fig. 3. In CRBM, two kinds of connections are added to share temporal association information in time-series data. One kind of connections are the autoregressive connections between the visual layer of the previous \( n \) time RBM and the current RBM visual layer. The other kind of connection is the autoregressive connections between the visual layers of the previous \( n \) time RBM and the current RBM hidden layer.

By adding the above two kinds of connections, CRBM can make full use of the temporal association of the input data and improves the ability to deal with the time-series data. However, there is a lack of connection between each CRBM hidden layer units, which results in an unstructured weight model for the DBN model. This problem affects the prediction accuracy of the DBN model. To overcome this problem, a constraint variable between the CRBM hidden layers is added to adjust the activation probability between the hidden layers and optimize the training process of the CRBM. The new improved CRBM
The input value of the previous visual layer and the hidden layer of the target frame via the autoregressive model (AR). The input value of the previous visual unit of historical frame can be connected to the visual layer of the same layer, change the weights of the variables of other units, while the variables will cross layer unit to transmit information to the visual units of the same layer. When the output of the hidden layer element is higher than the set threshold, the constraint variable of the unit is activated, and the information continues until the last cell is passed. In order to simplify the computation, the direction of signal transmission is set to one-way transmission. The update rules for the hidden layer unit is defined as:

\[ h_i = \text{sigmoid}(h_i' + a \times g_i); \]  

(8)

where, \( h_i \) is the updated output of hidden layer, \( g_i \) is a constraint variable, \( a \) is a tune parameter which requires manual settings to control the effect of horizontal variables on the output of the ICRBM. The larger \( a \) is the greater the impact is on the hidden layer, and vice versa. The original output \( h'_i \) of the hidden layer can be defined as:

\[ h'_i = \sum_j W_{ij} \nu_j + b_{now_i}; \]  

(9)

where \( W_{ij} \) is the weight matrix between the hidden layer unit and the visual layer unit, \( \nu_j \) is the state of the visual layer unit, and \( b_{now_i} \) is the dynamic offset of the hidden layer unit. In order to improve the training speed of ICRBM, this paper uses the activation probability of each unit as the output value directly. The constraint variable \( g_i \) is defined as:

\[ g_i(t) = \begin{cases} 1, (h'_i > \theta \cup g_{i-1}(t-1) = 1) \\ \beta g_i(t-1), \text{else}; \end{cases} \]  

(10)

where \( \theta \) is the threshold of hidden layer output and \( \beta \) is the attenuation factor. Once the output of the hidden layer reaches the set threshold or the previous constraint variable is activated, the constraint variable of the unit is activated, and the signal is passed to the next constraint variable. Otherwise, the constraint variable is not activated and the weight is gradually attenuated.

### 3.3. DBN Constructed by ICRBMs

DBN prediction methods are generally divided into static multi-step prediction method and dynamic multi-step prediction method. The static multi-step prediction method, after...
each prediction, only updates the predicted data according to the prediction results, and does not update the input data. It is easy to cause the superposition of prediction errors and affect the prediction accuracy. The dynamic multi-step prediction method not only predicts the data according to the prediction results, but also updates the input vectors of the prediction model by using the predicted data. This ensures the dynamic adaptation of the model and improves the prediction accuracy. Therefore, this paper uses dynamic multi-step prediction method to construct DBN prediction model, and the procedures are detailed in the following.

(1) Initialize connection weights $W$, learning speed $\epsilon$ and constraint variable $g$;

(2) According to the training rule of ICRBM, calculate the $E_{data}(\nu, h_1)$ and $p(\nu, h)$ until the gradient of $E_{data}(\nu, h_1)$ is less than the set threshold, and stop calculating;

(3) Take the hidden layer of ICRBM$_1$ as the visual layer of ICRBM$_2$ in the next layer and repeat the step (2) until the ICRBM training of the last layer is completed. The output of the last layer of ICRBM is the crude predictive value of the DBN model;

(4) Update the ICRBM weights until the MSE is less than the set threshold, and the ICRBM-DBN is trained; and

(5) Make a prediction by the trained ICRBM-DBN based upon the dynamic multi-steps strategy.

4. RUL PREDICTION BASED ON ICRBM-DBN

4.1. Prediction Process and Key Steps

There are mainly five steps in the process of RUL prediction for hydraulic pumps, including full life data collection, feature extraction and preprocessing, selection of training samples and testing samples, parameter optimization and ICRBM-DBN model training, ICRBM-DBN prediction and evaluation as follows.

(1) Full life data collection. Several full life tests of hydraulic pumps are conducted on our full life test platform and several sets of experimental data such as vibration, oil flow, volumetric efficiency.

(2) Feature extraction and preprocessing. The vibration data is analyzed by bispectrum and 15 bispectrum entropy features corresponding to 15 different frequency bands are extracted and normalized into $[0, 1]$.

(3) Sample Selection. One hundred sets of vibration data and volumetric efficiency in the slow degradation stage are taken as the training data and 40 sets of vibration data and volumetric efficiency in the rapid degradation stage are taken as the testing data.

(4) Parameter optimization and ICRBM-DBN model training. Three kinds of error indexes are applied to optimize the model parameters and full life data of hydraulic pumps are applied to train the ICRBM-DBN model. The stacked ICRBMs are learnt forward unsupervisedly and then the back fine tuning is conducted supervisedly.

(5) ICRBM-DBN prediction and evaluation. Fifteen bispectrum entropy features are taken as the input data. The volumetric efficiency is taken as the output label data. Forty data samples are tested to predict the next 40 groups of volumetric efficiency. The critical moment when the volume efficiency is less than 85% is taken as the failure point of hydraulic pumps. The time difference between the predicted failure point and the real failure point is taken as the prediction error.

4.2. Full Life Data Collection

In order to verify the effectiveness of the method proposed in this paper, the ICRBM-DBN is applied in the RUL prediction of hydraulic pumps. The whole lifetime data of hydraulic pump used in this paper comes from the hydraulic pump full life test platform, which is shown in Fig. 5. The hydraulic pump tested is L10VSO28DFR, which has 9 pistons. A new hydraulic pump is taken for full life degradation experiment under the accelerated condition where the settled pressure is 27 MPa and the speed is 2780 rpm-1. The signals are sampled and stored by the cDAQ-9171 system of NI Corporation. The sampling frequency is 10 KHz and the sampling time is 1 s. The interval time is 20 min. The volumetric efficiency $\eta$ is taken as the evaluation parameter of the hydraulic pump. When the volume efficiency is less than 85%, the hydraulic pump is judged to fall into the failure state, and the experimental platform is automatically stopped. In this paper, a new hydraulic pump is used to conduct full life test and obtain the whole lifetime data. When the operating time is 575 h, $\eta$ is less than 85%. The hydraulic pump is judged to be totally invalid by the control system and the experimental operating is shut down automatically. After the experiment, the tested pump is disassembled, and it is clear in Fig. 6 that the failure mode is severe loose slipper.

4.3. Feature Extraction and Preprocessing

In order to obtain the feature that can reflect the degradation state of the hydraulic pump accurately, bispectrum analysis of vibration signals is carried out in this paper, and the bispectrum entropy of different frequency bands is extracted as the prediction feature. Considering that the characteristics frequency of the vibration signal are mainly within 3 KHz, the bispectral entropy in the frequency bands $[0, 200), [200, 400), \ldots , [2800, 3000]$ are taken as the 15 prediction features in total. The DBN model constructed in this paper takes the 15 prediction features as the input layer vector of this model, and the hydraulic pump volume efficiency is the output layer vector. First of all, in order to reduce the prediction error of DBN prediction model, the 15 prediction features
need to be normalized, and the normalized formula is shown in Eq. (11):

\[ \tilde{x} = \frac{x - x_{\text{min}}}{x_{\text{max}} - x_{\text{min}}} \].

(11)

4.4. Parameter Optimization and ICRBM-DBN Model Training

In order to better evaluate the results of the model, the root mean square error (RMSE), the mean relative error (MRE) and the mean absolute error (MAE) are used as the accuracy evaluation indexes. These three evaluation indexes can accurately and comprehensively analyze the prediction accuracy and stability of the ICRBM-DBN prediction model.

In this paper, the depth of ICRBM-DBN prediction model is set to 3 layers, in which the visual layer unit is set to 15. At present, there are no explicit standards for the number of hidden layer units. We use RMSE, MRE and MAE as evaluation criteria to determine the number of hidden layer units through experiments. As can be seen from the Fig. 7, when the number of hidden layer units is 12, the model prediction error is the smallest. In Fig. 8, the effect of different previous frame number on prediction accuracy is analyzed. When the number of previous frames is 3, the model prediction error is the smallest. The experimental result shows that, when the previous frame number exceeds 3, the prediction error is not reduced but increased with the increase of previous frames. This is because the original vectors near the target frame have similar information, while the farther input vectors cannot effectively evaluate the current target vectors, resulting in error accumulation.

In the ICRBM-DBN model, the stability of the prediction model is often determined by the selection of learning rate. If the value is large, the prediction error of the model increases dramatically; if the value is small, the learning ability of the model is poor. In order to guarantee the stability of the prediction model, we calculate the prediction error of the model under different learning rates, as shown in Table 1. In the table, when the learning rate is less than 0.001, the prediction accuracy is very limited, while the training speed will decline a lot. Therefore, we will set the model learning rate as 0.001.
<table>
<thead>
<tr>
<th>Learning rates</th>
<th>RMSE</th>
<th>MAE</th>
<th>MRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0612</td>
<td>0.0536</td>
<td>0.1328</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0524</td>
<td>0.0467</td>
<td>0.1047</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0482</td>
<td>0.0421</td>
<td>0.0962</td>
</tr>
<tr>
<td>0.005</td>
<td>0.0421</td>
<td>0.0386</td>
<td>0.0812</td>
</tr>
<tr>
<td>0.001</td>
<td>0.0364</td>
<td>0.0316</td>
<td>0.0738</td>
</tr>
<tr>
<td>0.0005</td>
<td>0.0486</td>
<td>0.0412</td>
<td>0.1084</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.0401</td>
<td>0.0306</td>
<td>0.0753</td>
</tr>
</tbody>
</table>

Table 1. The prediction errors of different learning rates.

4.5. ICRBM-DBN Prediction and Evaluation

In this paper, the volumetric efficiency $\eta$ is taken as the evaluation parameter of the hydraulic pump and the threshold is settled as 85%. Considering the failure mechanism and the fluctuation tends to curve, the whole changing process of the volumetric efficiency has been divided into several statuses. During $\eta \geq 95\% (0 - 145 \text{ h})$, the part is the normal status ($F_1$); During $95\% > \eta \geq (145 \text{ h} - 312 \text{ h})$, the part is initial degradation stage ($F_2$); During $93\% > \eta \geq (322 \text{ h} - 510 \text{ h})$, the part is slow degradation stage ($F_3$); During $87\% > \eta \geq (510 \text{ h} - 575 \text{ h})$, the part is rapid degradation stage ($F_4$). In the experiment, the degradation of hydraulic pump has continuity and consistency in $F_3$ and $F_4$. However, there are more data samples in $F_3$ and the model training needs enough data samples, so the training samples are selected from the stage of $F_3$, and the testing samples are selected from the stage of $F_4$. The threshold is settled according to the 1619th sample, which is considered as the occurrence time of failure. The prediction results are shown in Fig. 9 and Fig. 10.

Fig. 9 shows the results predicted by ICRBM-DBN. The predicting series by the ICRBM-DBN coincide with the real data and the errors are relatively small. The predicting algorithm reaches the threshold in the 39th sample and the affirmation of failure is verified. Therefore, the error of the RUL prediction is 1 data point, which is 20 min. Fig. 10 shows the results predicted by RBM-DBN. The predicted curves are basically consistent with the actual values, and can basically reflect the actual trend of degradation. And after the 38th sample, the algorithm reaches the threshold. The error of RUL is 3 data points, which is 60 min., the prediction accuracy of RBM-DBN is lower compared with ICRBM-DBN. This is mainly due to the inability of RBM to utilize temporal association information when processing sequential data, resulting in limited gradient descent capability. ICRBM-DBN based on the CD model improves the gradient descent ability and gets higher prediction accuracy. For further indication of the advantages of the proposed method, based on the same training data and predicting data, the BP neural network and SVM method are taken for comparison. Results are shown in Fig. 11 and Fig. 12.

Fig. 11 and Fig. 12 show the results predicted by the BP neural network and SVM method. The shallow learning methods
cannot accurately extract the deep correlation information of input data, and has the problem of overfitting. Therefore, the errors between the predicted data and the real data are obvious, and the prediction curves cannot accurately fit the degradation process of the hydraulic pump. The prediction curves are almost a straight line, and the failure time of the hydraulic pump cannot be predicted accurately, especially after the 30th sample. To further demonstrate good performance of ICRBM-DBN, two modified DBN models are applied to the same RUL prediction task. One model is the deep belief echo-state network (DBEN) proposed by Sun et al. in 2017.25 Another model is a new deep belief network based on RBM with glial chains (GC-DBN) proposed by Geng et al. in 2018.27 The prediction results are shown in Figs. 13–14.

For further quantitative evaluation, the Root Mean Square Error (RMSE), the Mean Relative Error (MRE) and the Mean Absolute Error (MAE) are selected as the evaluation indexes. The results of the above 3 methods are shown in Table 2.

Table 2. The prediction results by different algorithms.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>RMSE</th>
<th>MAE</th>
<th>MRE</th>
<th>Error of RUL prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICRBM</td>
<td>0.023</td>
<td>0.046</td>
<td>0.078</td>
<td>20 min</td>
</tr>
<tr>
<td>RBM-DBN</td>
<td>0.032</td>
<td>0.062</td>
<td>0.107</td>
<td>40 min</td>
</tr>
<tr>
<td>BP neural network</td>
<td>0.136</td>
<td>0.104</td>
<td>0.173</td>
<td>120 min</td>
</tr>
<tr>
<td>SVM</td>
<td>0.186</td>
<td>0.1232</td>
<td>0.184</td>
<td>Failed</td>
</tr>
<tr>
<td>DBEN</td>
<td>0.027</td>
<td>0.042</td>
<td>0.084</td>
<td>60 min</td>
</tr>
<tr>
<td>GC-DBN</td>
<td>0.075</td>
<td>0.136</td>
<td>0.187</td>
<td>180 min</td>
</tr>
</tbody>
</table>

The full life time is 30630 min. The sampling period is 10 min, and there are 3063 sets of data in the whole life of the hydraulic pump. The starting point is set as the 3020th set of data, and the failure point the 3063rd set of data. The prediction result of the proposed method is shown in Fig. 17. The predicted trend is consistent with the real trend and the prediction error is only one data point. That means the RUL prediction error is only 10 min. The satisfactory result demonstrates good applicability of the proposed method.

5. CONCLUSIONS

This paper proposes a novel prediction method called ICRBM-DBN based on deep belief network theory for the RUL prediction of hydraulic pumps. In order to improve the capability of deep belief network and increase the prediction accuracy, the conditional RBMs, serving as the core of the improved DBN model, are added with timing linkage factors and constraint variables between the same layers. And the
The bispectrum entropy features of different frequency bands supply enough degradation information to reflect and predict the health state of the hydraulic pump which is indicated by the index of volumetric efficiency. The experiment results and the comparisons are concluded as follows:

(1) The proposed ICRBM model solves the problems of the traditional RBM model such as inability to extract deep features and limited gradient descent.

(2) A new prediction model, ICRBM-DBN, is constructed by multiple ICRBMs under the improved updating rule of CRBM. Experimental results and comparisons show that the proposed prediction model of ICRBM-DBN can achieve a better prediction accuracy compared with the original RBM-DBN, BP neural network, SVM and modified DBNs such as DEBN and GC-DBN when conducting RUL prediction of hydraulic pumps; and

(3) The proposed method provides a useful tool for residual useful life prediction and the prediction accuracy is acceptable for the requirements of CBM.

6. ACKNOWLEDGEMENTS

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Modeling, Vibration Analysis and Fabrication of Micropumps Based on Piezoelectric Transducers

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The parametric and vibrational characteristics of PZTs (Piezoelectric Transducers) with different diameters before and after coupling are discussed by finite element analysis. It is shown that the vibration stability of the piezoelectric transducer decreases with increasing driving frequency. The PZT’s variation of maximum displacement with frequency shows the same trend for different driving conditions according to vibration measurement under conditions of both free and forced vibration (before and after sealing with the pump body). The maximum displacement under forced vibration is less than that under free vibration. The maximum displacement is inversely proportional to the diameter of the transducer and directly proportional to the driving voltage under both free and forced vibration. Micropumps with diffuser/nozzle microvalves are designed and fabricated with different external diameters of the PZTs. Finally, the flow rate and pressure of the micropumps are measured, which are consistent with the vibrational results. Moreover, the maximum displacement is larger under a square-wave driving signal, followed by a sine-wave signal, and then a triangle-wave signal. For a PZT with an external diameter of 12 mm, the maximum flow rate and pressure value are 150 μl/min and 346 Pa, respectively, under sine-wave driving at 100 Vpp driving voltage.

1. INTRODUCTION

Piezoelectric transducers are the devices that use the piezoelectric properties of some crystals or other materials to convert one type of energy to another one. Moreover, piezoelectric transducers are often used as driving devices for actuating microfluidic systems, since flow inside a microfluidic device could be affected by vibration.1–3 Therefore, it is important to study the behavior of piezoelectric transducers, especially when they are coupled with microfluidic devices.4–6 Specifically, in practical applications, it is very important to understand the influence of transducer vibration on microfluidics performance.7–10 For example, Liu11 studied the coupled vibration of a sandwich piezoelectric transducer using an approximate analytical method. The results showed that the resonant frequencies obtained from the coupled resonant frequency equations are in good agreement with those from numerical methods, and super harmonic resonant frequencies can also be obtained. Zhu et al.12 simulated and analyzed the behavior of a piezoelectric flexible plate using MATLAB and ANSYS software. The vibration control experiments were performed based on an APDL (the design language of ANSYS) program. The designed controller had good vibration suppression performance according to the analysis and experimental results. Catarino et al.13 proposed a microfluidic mixing device with two different piezoelectric materials. The mixing-time reduction for both materials, above 90% for PZT and above 80% for β-PVDF, were tested according to numerical simulations and experimental studies. Huang et al.14 studied a piezoelectric harvester’s resonant frequencies and voltage output equation at various excitation frequencies according to numerically calculated results as well as experiments, which showed a high level of voltage output.

Piezoelectric micropumps are popular in microfluidic systems. They have been used in medicine, agriculture, and aviation, among other applications.15–21 Therefore, it is extremely important to study how to improve the performance of micropumps, and researchers have made a lot of efforts in studying the performance of a common piezoelectric material: Lead Zirconate Titanate (PZT). Aggarwal et al.22 fabricated and studied two different sizes of piezoelectrically actuated micromachined silicon valveless micropumps with a vertical diffuser/nozzle microchannel, and the flow rate and the back pressure were evaluated. Cheng et al.23 fabricated an impedance micropump with nickel electroforming components, a stainless steel vibration plate, and a piezoelectric actuator. The flow rate of 0.24 ml/min and back pressure of 2.35 kPa were demonstrated with a 200 Vpp driving voltage. He et al.24 presented a new type of valveless piezoelectric micropump with synthetic jet and Coanda effect to achieve larger and bidirectional flow rate. An optimal frequency of 50 Hz and a Reynolds number of 1000 was identified for a maximum flow rate of 6.8 ml/min. Wang et al.25 presented a Finite Element Analysis (FEA) micropump model actuated by a piezoelectric actuator. The maximum displacement of the PZT ceramic disk was found to vary along with the diameter ratio, membrane thickness and the diameter of the chamber. Kang and Auner26 designed a piezoelectrically actuated check valve diaphragm micropump and studied the average displacement of the diaphragm. The relationship between the stroke volume and backpressure were simulated with an FEA tool. Singh et al.27 fabricated and tested a piezoelectrically actuated polydimethylsiloxane (PDMS) planar valveless micropump. The predictions of the analytical model and numerical simulations in terms of flow rate versus frequency, voltage and pressure-flow matched with experimental data (within 20%). Zhang28 presented a self-sensing piezoelectric pump with a bimorph transducer. The output flow and pressure could be achieved through a single piezoelectric element, and the simultaneous function could be achieved by the other PZT disk.

We can see that previous research has concentrated on FEA modeling of piezoelectric transducers and new structures of...
piezoelectric micropumps, with leads no studies on the vibrational performance of the piezoelectric transducer, especially the comparison between before and after the transducers being coupled with micropumps. To the best of our knowledge, the factors that influence the vibration of PZT before and after coupling to a micropump as well as the influence on the pumping performance have not been studied. Note that, as the main power supply device of the micropump, the vibration characteristics of PZT has an important influence on the output performance of micropump, such as the flow rate, pressure and efficiency. Through the vibrational analysis of the PZT before coupling, the optimal working conditions of PZT are obtained for further guiding the influence of PZT after coupling on the working effect of the micropump. Meanwhile, the optimal parameters combination for the micropump performance obtained during the numerical analysis realize the optimization output of the micropump performance and reduce the economic loss and simplify the operation. Therefore, in this paper, we study the vibrational performance of the piezoelectric transducer and discuss thoroughly the working principles and the influence on the performance of the micropump. The optimal PZT structures, the flow rate and pressure of the micropump are achieved according to the vibrational analysis and experiments.

2. DESIGN AND FABRICATION

2.1. Structural Analysis of The Piezoelectric Transducer

Ideally, the material of a piezoelectric transducer needs to have high reliability, a wide frequency response range, a linear response to the applied voltage, and reasonably low cost.\textsuperscript{29–31} PZT-5H piezoelectric ceramic plates with different diameters were adopted in this study.\textsuperscript{32,33} The three-dimensional structure of the piezoelectric transducer is shown in Fig. 1(a). It consists of three parts: a PZT piezoelectric ceramic, a connecting layer, and a copper substrate. Depending on the pump chamber dimensions, three different piezoelectric actuators were used, with external diameters (i.e., the diameter of the copper substrate) of 12, 15, and 20 mm. The parameters of each of these transducers are shown in Fig. 1(b) and Table 1.

2.2. Structural Design and Fabrication of The Micropump

The working principle of the micropump with a diffuser/nozzle microchannel is schematically represented in Fig. 2. First of all, there is no force on the PZT without voltage applied as shown in Fig. 2(a). In the “supply mode,” the piezoelectric transducer vibrates vertically upwards, increasing the chamber volume and thereby reducing the chamber pressure. The asymmetric arrangement of the inlet/outlet microchannel of the pump chamber enables a pressure difference to form between the pump chamber and the inlet/outlet. Then, the working fluid is sucked into the pump chamber from both the inlet (diffuser direction) and the outlet (nozzle direction), as shown in Fig. 2(b). However, the rate of the flow in the diffuser direction is larger than that of the nozzle direction as shown with red arrows of different sizes in Fig. 2(b). For the “pump mode,” the reverse phenomenon occurs, as shown in Fig. 2(c). The effectiveness of the flow rectification of the micropump can be produced based on the net flow of the fluid from the inlet to the outlet if a reasonable design is adopted.\textsuperscript{34–37} The force F1 of the PZT from the transformation of electromechanical energy by the piezoelectric transducer is transmitted to the fluid in the pump chamber. At the same time, the force F2 from the fluid has a counter effect in the opposite direction. As far as the present investigation is concerned, the pump body and the piezoelectric actuator are the most important components.

The design of the piezoelectric micropump with a diffuser/nozzle microchannel is displayed in Fig. 3. As shown in Fig. 3(a), the five-layered structure of the micropump consists of, from top to bottom, an upper jig, an upper PDMS layer with inlet and outlet pipes, a glass wafer, a silicon wafer, a piezoelectric actuator, a lower PDMS layer, and a lower jig. A plan view of the nozzle/diffuser microchannel with a 7° diffuser angle is shown in Fig. 3(b). A minimum section width of 0.04 mm and a length of 1.093 mm were obtained according to the advanced silicon deep reactive ion etching (DRIE) technique on 450 µm-thick silicon wafers.\textsuperscript{34,38–40} In Fig. 4, the depths of the microchannel and inlet/outlet holes were 0.1 mm, and the depth of the center pump chamber was 0.45 mm.

The dimensions of the pump chamber and the silicon layer are shown in Fig. 3(b) together with a scanning electron microscope (SEM) image of part of the pump containing a microchannel in Fig. 3(c). A high-temperature bonding technique was used to produce an irreversible seal between the Pyrex 7740 glass wafer and the silicon wafer. The piezoelectric transducer was fixed to the lower side of the silicon wafer with 3M glue for total sealing. Finally, all five layers were sealed irreversibly together by ultraviolet light irradiation through the lower PDMS layer as shown in Fig. 3(d). In order to prevent
Table 1. Parameters of the piezoelectric transducers.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piezoelectric plate diameter</td>
<td>11.3 mm</td>
</tr>
<tr>
<td>Piezoelectric plate thickness</td>
<td>0.20 mm</td>
</tr>
<tr>
<td>Piezoelectric coefficient matrix (C/m²)</td>
<td>$egin{bmatrix} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 17 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 17 &amp; 0 &amp; 0 \ -6.5 &amp; -6.5 &amp; 23.3 &amp; 0 &amp; 0 &amp; 0 \end{bmatrix}^T$</td>
</tr>
<tr>
<td>Permittivity ε (F/m)</td>
<td>$1.7 \begin{bmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1.58 \end{bmatrix} \times 10^{11}$</td>
</tr>
<tr>
<td>Young’s modulus (Pa)</td>
<td>$1.7 \begin{bmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1.58 \end{bmatrix} \times 10^{11}$</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\lambda_{12} = \lambda_{13} = \lambda_{23} = 0.3$</td>
</tr>
<tr>
<td>Shear modulus (Pa)</td>
<td>$0.23 \begin{bmatrix} 0 &amp; 2.3 &amp; 0 \ 0 &amp; 0 &amp; 2.3 \end{bmatrix} \times 10^{10}$</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
<td>7500</td>
</tr>
<tr>
<td>Copper substrate diameter</td>
<td>12, 15, 20 mm</td>
</tr>
<tr>
<td>Copper substrate thickness</td>
<td>0.12, 0.37, 0.47 mm</td>
</tr>
<tr>
<td>Copper Young’s modulus (Pa)</td>
<td>$0.9 \times 10^{11}$</td>
</tr>
<tr>
<td>Copper Poisson’s ratio</td>
<td>0.32</td>
</tr>
<tr>
<td>Copper density (kg/m³)</td>
<td>8500</td>
</tr>
<tr>
<td>Resonant frequency</td>
<td>$5.3 \pm 0.5$ kHz, $10.0 \pm 0.5$ kHz, $7.2 \pm 0.5$ kHz</td>
</tr>
<tr>
<td>Impedance</td>
<td>500, 300, 300 Ω</td>
</tr>
<tr>
<td>Capacitance</td>
<td>$15 \text{nF} \pm 30%, 9 \text{nF} \pm 30%, 12 \text{nF} \pm 30%$</td>
</tr>
</tbody>
</table>

Leakage, the upper and lower jigs were added to fix the micropump body. A photograph of the complete pump assembly is shown in Fig. 3(e).

3. SIMULATION AND VIBRATIONAL ANALYSIS OF THE MICROPUMP

Numerical modeling has been often adopted for solving three-dimensional fluid problems using CFD-ACE+ software based on the finite volume analysis, which include solid, fluid, and mechanical/electric field coupling calculations. However, the core of numerical analysis is solving partial differential equations. The forces on the piezoelectric transducer during micropump operation are shown in Fig. 2. The piezoelectric transducer has transverse contraction under the sine-wave alternating voltage. The force $F_1$ from the transformation of electromechanical energy by the piezoelectric transducer will be transmitted to the fluid in the pump chamber. At the same time, the force $F_2$ from the fluid has a counter effect in the opposite direction. The physical parameters of the piezoelectric transducer used in calculation are listed in Table 1.

The flow into the micropump is considered as incompressible and laminar. Therefore, for numerical modeling, the electromechanical-coupled equation, piezoelectric transducer control equation, Navier-Stokes equation, and continuity equation are as follows: \[ \sigma = C \varepsilon - \varepsilon E; \]
\begin{equation}
D \nabla^4 W + \rho h \frac{\delta^2 W}{\delta t^2} = f - F_1; \tag{2}
\end{equation}

\begin{equation}
\rho D \frac{\delta \tilde{v}}{\delta t} = \rho \tilde{g} + \mu \nabla^2 \tilde{v} - F_2; \tag{3}
\end{equation}

\begin{equation}
\frac{\delta \rho}{\delta t} + (\tilde{v} \cdot \nabla)\rho = 0. \tag{4}
\end{equation}

The boundary conditions of the piezoelectric ceramic sheet at a round clamp are as follows:

\begin{equation}
\frac{\delta^2 W}{\delta x^2} = \frac{\delta^2 W}{\delta y^2} = 0; \tag{5}
\end{equation}

\begin{equation}
\frac{\delta W}{\delta t} = 0; \tag{6}
\end{equation}

where \( \varepsilon \) is the mechanical stretch tensor, \( \sigma \) is the mechanical pressure tensor, \( E \) is the electric vector, \( c \) is the piezoelectric continuum tensor, \( C \) is the elastic stiffness constant tensor, \( \rho \) is the fluid density, \( \rho = 1 \times 10^3 \) kg/m\(^3\), \( v \) is the velocity vector, \( \mu \) is kinematic viscosity, and \( \mu = 1.4 \times 10^{-3} \) Pa s.

The boundary conditions of the piezoelectric transducer are the fixed periphery, and no slip occurs at the interface between the fluid and solid and the free surface for the inlet/outlet. Hence, the boundary control equations are the same as Eqs. (5),(6). By solving Eqs. (1),(2),(3),(4),(5),(6) the PZT-Membrane-Fluid coupled analysis can be obtained. In addition, considering that the initial stress (\( \sigma \)) has a great influence on the electromechanical coupling constant of PZT,\(^{42} \) and when the value of initial stress exceeds 150 MPa according to literature,\(^{42} \) the coupling constant will change abruptly due to the non-dimensional wavenumber, so the value of initial stress is set to 150 MPa in the simulation.

The displacement curves of the piezoelectric vibrator before (free-vibration) and after (forced-vibration) coupling with the micropump in the \( \pm Z \) directions under three driving frequencies (100 Hz, 1000 Hz and 5000 Hz, respectively) are shown in Fig. 6. The 12 mm of external diameter and the related parameters are adopted during finite element analysis. Note that, free-vibration means the vibration of the piezoelectric transducer occurs uncoupled or unsealed with the micropump (marked as PZT-100 Hz/1000Hz/5000Hz in Fig. 5), and forced-vibration means the vibration of the piezoelectric transducer occurs coupled and sealed with the micropump (marked as pump-100Hz/1000Hz/5000Hz in Fig. 5); and the latter is the real working situation. It can be seen that the displacement of the piezoelectric vibrator shows the same sinusoidal trend before and after coupling with the pump with 100 Hz, 1000 Hz and 5000 Hz of sine wave driving frequencies under the same time step (100 time steps as shown in Fig. 5 with X label). Comparing the \( \pm Z \) directions displacements (both free- and forced-vibration modes) between 100 Hz, 1000 Hz, and 5000 Hz of driving frequencies, the coincidence with sinusoidal variation is very good under the lowest driving frequency (100 Hz) because F2 has little effect on the vibration of the piezoelectric transducer in Fig. 2. Additionally, the displacements in the \( + Z \) and \( - Z \) directions are the same. However, when the frequency increases and reaches a certain value, like 5000 kHz (the eigenfrequency of PZT-5H is 5.3 \( \pm \) 0.5 kHz as shown in Table 1), the displacements of \( \pm Z \) directions distort seriously. Because the force F2 that liquid exerts on the piezoelectric vibrator acts at a disadvantage with respect to the driving force of the piezoelectric vibrator F1 in Fig. 2. Moreover, the liquid force F2 plays a significant role when the frequencies become larger, which increases the deformation of the piezoelectric transducer and the volume flow of the fluid inside the pump (It doesn’t mean the net flow of the micropump in Fig. 1) also become larger. The average deformation value of the piezoelectric transducer after coupling is smaller than that before coupling. This is because the lower surface of the piezoelectric vibrator is a free vibration surface before coupling, that is, zero loads. The force F2 that liquid exerts on the piezoelectric surface blocks the piezoelectric transducer vibration after coupling (Fig. 5), so the induced deformation value decreases. For example, comparing the displacements of PZT-5000Hz with Pump-5000Hz, the high frequency (5000 Hz) is closer to the mechanical resonance frequency of the piezoelectric actuator (5.3 \( \pm \) 0.5 kHz), the displacement of the PZT after fixed with the pump (Pump-5000Hz) greatly reduced (close to that of the low frequency, such as 100 Hz) from that of the free-vibration mode (PZT-5000Hz).

The contour maps of the pressure on the central cross section chamber under driving frequencies of 100 Hz, 1000 Hz, and 5000 Hz are shown in Fig. 6 (For convenience, the quarter cycle maps of the whole period is selected because the maximum displacement occurred at that point with sine wave driving). The pressures (from 0 to 1.453 N/m\(^2\)) are positive because the maximum displacement occurred in the positive direction of quarter cycle under the sine-wave signal driving (Fig. 5) with the 100 Hz driving frequency (Fig. 6(a)), and there are small amounts of negative pressures (from \(-14.67\) to 0 N/m\(^2\)) under the 1000 Hz driving frequency (Fig. 6(b)) because of the uneven vibration is appeared with high driving frequency (Fig. 5). However, under the 5000 Hz driving frequency (Fig. 6(c)), a large area of negative pressure (from \(-1335\) to 0 N/m\(^2\)) dominates in the center of the chamber, which results in an insufficient fluid-filled micro-pump, and the pressure of the chamber accelerates with messy, inconsistent directions. It further proves that the higher frequency is not suitable as the working frequency for the micropump, i.e. near the mechanical resonance frequency. The velocity contours of the \( + Z \)-direction (or \(+ W\)-direction in Fig. 6d) should be upward (from 0 to +0.03774 m/s), but some contours are downward (from \(-0.1892\) m/s to 0 m/s during the filling process in Fig. 6(d). This indicates the decreased efficiency of the micropump. The velocity contours of the X-direction (parallel with the fluids flow) below 5000 Hz frequencies are shown in Fig. 6(e). The \(+ Z \)-direction deformation maps of the PZT with quarter cycle under 100 Hz and 5000 Hz driving frequencies are shown in Fig. 6(f) and Fig. 6(g), respectively. The deformation of the piezoelectric transducer appears hemispherical, the displacement in the central point is the largest, and the displacement in the perimeter is the smallest; this result is consistent with the boundary conditions. The deformation of every point is uniformly distributed in the plane at 100 Hz. However,
the uniformity is broken below 5000 Hz, and uneven deformation occurs in the perimeter in Fig. 6(g). The reason is the vibration at high frequency is not sinusoidal and the deformation occurred as shown in Fig. 5.

4. EXPERIMENT AND ANALYSIS

4.1. Experimental Investigation

The vibrational performance of the piezoelectric transducers incorporated into the micropump was tested under conditions of both free and forced vibration using a PSV400 Scanning Vibrometer in Fig. 7(a). The experimental setup for performing these tests included a signal generator, a power amplifier, and an oscillograph, among other equipment, and is shown in Fig. 7(a). The flow rate and pressure of the micropump were tested using weighing and water-column methods, respectively in Fig. 7(b) and Fig. 7(c). The schematic of the measurements is shown in Fig. 7(d) that includes driving module, vibrational measurement module and flow rate, pressure measurement module. As shown in Fig. 7(e), nine measurement points were specified on the center plane (the diameter is 1 mm from the center to external circle) of the piezoelectric transducer in order to determine the distribution of vibrational displacements. The dynamic vibrational displacements of the transducer with 1 mm diameter of the PZT are shown in Fig. 7(f) on the display screen with the PSV 400 scanning vibrometer. Here, the three different transducers with diameters of 12, 15, and 20 mm were studied under both uncoupled and coupled with the micropump conditions.

4.2. Vibrational Displacement Measurement of the PZT

The vibrational displacements of the nine measurement points were tested under both free-vibration and forced-vibration conditions. The maximum vibrational displacements of nine points are collected for transducers with 12 mm, 15 mm and 20 mm external diameter at a driving voltage of 40 Vpp, 70 Vpp and 100 Vpp of a sine-wave driving signal, respectively. The curve shapes of the maximum vibrational displacement are similar for all points from 1–9 under sine-wave driving according to analysis. For convenience, the average maximum displacements of nine points were calculated under every driving condition.

In order to examine the effect of driving voltage, Fig. 8 shows the maximum displacement versus frequency the piezoelectric transducers with external diameters of 12, 15, and 20 mm driven by a sine-wave signal at 100 Vpp and three driving voltages 40, 70 and 100 Vpp of a sine-wave signal with 12 mm diameter. It can be seen that the average maximum displacements in Fig. 8(a) and Fig. 8(c) first increases, and then decreases under free-vibration as the frequency increases. The maximum displacement attains its greatest values near 60 Hz,
which is the “resonant” frequency of the PZT at low driving frequency. At this frequency, the vibration displacement of PZT reaches the maximum value because the vibration is 1st modal. Under 1st modal vibration, the piezoelectric vibration in the Z direction is towards to the same direction, which is more suitable for the generation of vibration. At the same time the volume efficiency of the micropump should be the maximum value. Moreover, it can be seen from Fig. 8(c) that when the frequency is smaller (lower than 60 Hz), the maximum displacement produced by PZT with larger diameter (20 mm) is larger than that with smaller diameter, which indicates that the resonant frequency of PZT decreases with the increase of diameter under low frequency driving. However, the average maximum displacements decrease as the frequency increases in Fig. 8(b) and Fig. 8(d). The maximum displacement is proportional to the voltage and inversely proportional to the diameter at the same frequency. The higher the voltage is, the greater the displacement is. The larger the diameter is, the greater the displacement is. Comparing with the simulation and experiment results under 40 Vpp of driving voltage in Fig. 5 and Fig. 8(a), Fig. 8(b), the maximum displacement from experiment results emerged near 60 Hz or below 150 Hz and the value is smaller than the simulation one because the PZT vibration is obstructed after sealing with the lower PDMS and jig. Moreover, the maximum displacement is no longer increased along with the increase in the frequency (from 150 Hz to 800 Hz (even 5000 Hz)). That is because the displacement in the experiment is an average value (nine points in the diameter of 1mm circle as shown in Fig. 7(e)) due to the limitation of experiment setup. However, the experiment results further prove the high frequency is not suitable as the working frequencies of the micropump. So lower frequencies (below 1000 Hz) are adopted in the later experiments.

Comparing the free-vibration with the forced-vibration in Fig. 8(a), (c) and Fig. 8(b), (d), the behavior of the maximum displacements are different under the 100 Vpp driving voltages with free-vibration and forced-vibration situations. The vibrodynamic displacement under free vibration is higher than that under forced vibration over the entire frequency range. Because of the existence of the clamping boundary condition with the forced-vibration situation, the resonance frequency disappeared below a 100 Hz driving frequency. That is because the forces F1 and F2 from solid and fluid act as the dampers during the micropump operation according to Fig. 2. The displacements in the X and Y directions are close to zero under forced-vibration conditions once the piezoelectric transducer has been incorporated into the micropump. The nearer the measurement point is to the center of the transducer, the greater the vibrational displacement is. Thus, the displacements of the nine points under free-vibration are greater than those under forced-vibration in Fig. 8. Furthermore, it can be seen from Fig. 8 that the changing trend of the PZT displacement is basically the same before and after coupling, that is free-vibration and forced-vibration respectively. For example, the vibration performance with 100 Vpp of voltage and 12 mm of diameter before coupling is the best, after coupling this situation is the same. This fully shows that the performance of PZT after coupling can be inferred from that before coupling, so as to simplify the detection steps, reduce the cost of testing, and achieve the optimization performance of the micropump for the future research.

4.3. Flow Rate and Pressure Measurement of the Micropump

Performance testing of the micropumps included measurements of the flow rate and pressure with different driving signals in Fig. 7(b). The flow rates (in practice, the maximum flow rates) of the micropumps with three diameters of piezoelectric transducer were measured by the same procedure. First, the inlet and outlet pipe were placed on the same plane, then the micropumps were actuated for $\Delta t = 2$ min without any difference in height between the inlet and outlet reservoirs (zero pressure head). Finally, the mass of liquid in the outlet reservoir was determined using an analytical balance, and the flow rate was calculated as:

$$\phi = \frac{W \times 10^6}{\rho \Delta t} \mu L/min; \quad (7)$$

where $\phi$ is the flow rate, $W$ is the mass of liquid in the outlet reservoir (in g), and $\rho$ is the liquid density.

The pressures (in practice, the maximum pressure heads) of both micropumps were measured by placing the outlet pipe
in a vertical position and measuring the liquid height, $\Delta Z$, in Fig. 7(b), from which the pressure was calculated as:

$$\Delta P = \Delta Z \rho g; \quad (8)$$

where $\Delta P$ is the pressure head at zero flow rate, and $g = 9.80 \text{ m/s}^2$ is the acceleration due to gravity.

The effects of driving voltage, transducer diameter and driving signal on flow rate and pressure are shown in Fig. 9. It can be seen that the behavior of the flow rate in Fig. 9(a), (c) and (e) and the pressure in Fig. 9(b), (d) and (f) are the same with the frequency. Both of them show a trend with first increasing, then decreasing, and then increasing and decreasing again as the frequency increases. There are two maximum flow rates and pressure values at 60 Hz and 600 Hz along the whole driving frequency. It further proves that the resonance frequency of the piezoelectric transducer before coupling will be changed after coupling. The optimal working condition of the micropump needs to be measured in detail and then conclusion can be deduced. The flow rate and pressure are proportional to the driving voltage. Both increase with increasing driving voltage in Fig. 9(a) and (b). The flow rate and pressure are higher for the transducer of $12 \text{ mm}$ external diameter than for the $15–20–\text{mm}$ transducers in Fig. 9(c) and (d); that means the flow rate and pressure are inversely proportional to the PZT diameters. This is in accordance with the values of the maximum displacement in Fig. 8(c) and (d). From Fig. 9(e) and (f), we can see that both flow rate and pressure are higher for the square-wave driving signal than for the sine-wave and triangle-wave signals. Furthermore, the $12 \text{ mm}$ diameter, square driving signal and $100 \text{ Vpp}$ driving voltage within the range of all measured diameters, driving signals and voltages show the best performance. These results are highly coincident with the numerical and vibrational results. At $100 \text{ Vpp}$ for the $12 \text{ mm}$ diameter under sine-wave driving, the maximum flow rate and pressure are $150 \mu l/\text{min}$ and $346 \text{ Pa}$ according to the experiment, respectively.

5. CONCLUSIONS

Three piezoelectric transducers with external diameters of 12, 15, and 20 mm were studied and designed for driving micropumps, and three kinds of micropumps with sandwich structures containing these transducers were fabricated using an advanced DRIE method on a silicon wafer. The optimal driving parameters of the piezoelectric transducer and micropump were determined by investigating their influence on the maximum vibrational displacement before and after coupling and under free- and forced-vibration conditions. Finally, experimental measurements of the three micropumps with different driving waves, driving voltages, and driving frequencies were carried out. The conclusions can be summarized as follows:

(1) The average displacements of the piezoelectric transducer after coupling with the pump body are smaller than those before coupling because of the counter force of the liquid. The vibration is steady and the deformation of the PZT is close to a sinusoidal trend under low frequencies (i.e., below $1000 \text{ Hz}$).

(2) The trend of variation with frequency of the maximum displacement of the nine measurement points at the same driving voltage is the same for different driving waves for all three piezoelectric transducers. The maximum displacement under forced vibration is less than that under free vibration because of the fixed boundary conditions in the former case. The vibrational displacement near the center of the piezoelectric transducer is larger than elsewhere.

(3) Under both free- and forced-vibration conditions, the maximum displacement is inversely proportional to the diameters of the piezoelectric transducer and directly proportional to the driving voltage. The displacement reaches the maximum value under the square-wave driving signal, followed by the sine-wave signal, and then the triangle-wave signal. The optimal diameter of the PZT is $12 \text{ mm}$. The performance of the micropump shows a good efficiency with square driving signal and $100 \text{ Vpp}$ driving voltage.

(4) The flow rate and pressure of the micropump show the same trends as the maximum displacement under the same driving conditions. Also, the trends of the experiment results coincide with the vibrational measurement results under the same driving conditions. There are two peak driving frequency values: $60 \text{ Hz}$ and $600 \text{ Hz}$. For the piezoelectric transducer of $12 \text{ mm}$ diameter, under sine-wave driving at $100 \text{ Vpp}$, the maximum flow rate and pressure are $150 \mu l/\text{min}$ and $346 \text{ Pa}$, respectively.

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Figure 9. The flow rate and pressure of the micropump with different driving voltages, diameters of the PZT and driving signals. (a) Flow rate and (b) pressure versus frequency for the 12 mm diameter transducer with 40 Vpp, 70 Vpp and 100 Vpp different driving voltages. (c) Flow rate and (d) pressure versus frequency for the 12, 15, and 20 mm diameter transducers at different driving voltages. (e) Flow rate and (f) pressure versus frequency for the 12 mm diameter transducer at different driving signals.

Identification of Vibration Modes of Quartz Crystal Plates with Proportion of Strain and Kinetic Energies

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For the design of quartz crystal resonators, finding and determining the vibration modes have always been very important and cumbersome. Vibration modes are usually identified through plotting displacement patterns of each coupled modes and making comparisons. Over the years, there is not much improvement in the identification procedure while tremendous efforts have been made in refining the equations of the Mindlin plate theory to obtain more accurate results, such as the adoption of the Finite Element Method (FEM) by implementing the high-order Mindlin plate equations for efficient analysis. However, due to the old fashioned mode identification method, the FEM application is still inadequate and cannot be fully automated for this purpose. To have this situation improved, a method using the proportions of strain and kinetic energies to characterize the energy level of each vibration mode is proposed. With solutions of displacements, the energy distribution of each vibration mode is calculated and the mode with the highest energy concentration at a specific frequency is designated as the dominant mode. The results have been validated with the traditional approach by plotting mode shapes at each frequency. Clearly, this energy approach will be advantageous with the FEM analysis for vibration mode identification automatically.

1. INTRODUCTION

In the design of a quartz crystal resonator, it is important to make full and accurate analysis of its vibrations and to identify the functioning mode through the detailed results like frequency and mode shapes.1–4 The precise identification of vibration modes is essential because the optimal functioning mode of the resonator in the designated vibration frequency should be dominant and needs to be well isolated from so-called spurious modes of negative effects. Consequently, the functioning mode is extremely important to be identified, which is the thickness-shear mode usually, to immunize from couplings to other modes, or the notorious spurious modes. As there is no suitable way to eliminate the infinite couplings of vibration modes in a finite quartz crystal plate, it is practical to find the least coupled thickness-shear vibrations to enhance the performance characteristics of a quartz crystal resonator. In other words, the challenge is to determine the optimal parameters of the resonator structure, avoid the strong coupling of modes, and let the primary working mode be enhanced.3 In a typical resonator, the primary working mode mainly refers to the thickness-shear mode and the strongly coupled one is the flexural mode.5 For the accurate analysis of vibrations within a quartz crystal plate in the vicinity of the thickness-shear mode, the most effective method we have chosen is based on the Mindlin plate theory which can easily include more vibration modes.

The strong couplings happen just between the neighboring vibration modes, so only a few strongly coupled vibration modes are needed for the analysis to obtain the accurate results such as frequency and displacements which can be used for mode identification. Usually the analysis is done with the straight-crested waves for applications, and currently the Mindlin plate equations have been implemented in the finite element analysis for more precise considerations of structural complications. The identification of vibration modes, as from the beginning, is done with the plotting of mode shapes which show the dominance of specific mode at each frequency by picking out the largest displacement.4,5

Initially, the analysis of plate vibrations was presented as a three-dimensional problem which inevitably demanded complicated calculations in solving the differential equations of motion. Fortunately, with the Mindlin plate theory, we can analyze the vibrations from a two-dimensional perspective and this greatly simplifies the solution procedure. Accordingly, accurate solutions from the Mindlin plate equations with analytical and numerical procedures have made considerable progress recently corresponding with methods to enhance the mode identification, and in conjunction with the finite element method, it is highly regarded as a major improvement. As a result, the mode shapes can be plotted and identified with the displacements.

In the early beginning, the Mindlin plate theory was presented with only equations of zeroth- and first-order mechanical displacements, then Mindlin and Tiersten expanded the two-dimensional equations of the plate under forced vibrations to higher-order equations, and introduced the electric field considerations. Later Mindlin and Spencer presented the normalized wavenumber and dispersion relations of the infinite plate. Furthermore, equations of AT-cut quartz rectangular plates in free vibrations were derived and the frequency spectra were calculated. The normalized dispersion relations and frequency spectra, as effective representations of the vibration analysis results, were widely used for the optimum selection of plate parameters. Of course, other researchers also proposed similar or novel two-dimensional theory in resonator vibration analysis, like Lee, et al. and Sekimoto, et al.

In recent years, applications of Mindlin plate theory have been further improved. Based on the high-order Mindlin plate theory, Yong and Zhang and Wang, et al. developed the finite element method in analyzing high-frequency vibrations of quartz crystal resonators. Then Yong, et al. studied vibration modes with the first- and third-order plate equations and found out that the higher-order theory results are more accurate than lower-order ones for some vibration modes. Therefore, more solutions of vibration analysis prefer higher-order plate theory and finite element methods regarding design tool for development. However, using such solutions usually means solving a very large matrix problem in order to obtain the eigenvalues as vibration frequencies. In the traditional linear algebra problems, this means a lot of computer memory and computing time which greatly affect the computational efficiency. Lately, Wang, et al. have developed a finite element program with parallel computing capabilities on Linux clusters to quickly handle sparse matrices and algebraic operations with software components from public domains and this effectively improved the computational efficiency of high frequency vibration problems. Subsequently, they also implemented nonlinear finite element analysis with the program.

Currently, the Mindlin plate equations with truncation and corrections for parallel finite element analysis are up to the fifth-order or even higher which also takes into account the potential and overtone displacement variations. At the same time, the analysis of nonlinear vibrations is also included. Generally speaking, with the higher-order plate theory, considerations of nonlinearity, and other factors to improve the accuracy and parallel computing technology to improve the computation efficiency, we can effectively obtain the vibration solutions with great accuracy. For the design of the quartz crystal resonators, we still need the exact mode from the massive calculation results by plotting the mode shapes. There is a lack of efficiency because it requires manual interventions which will affect the automation of the parallel computing.

Based on the fact that the vibration energies are stored in each vibration mode, the energy method can be used for vibration analysis, as shown in modal strain energy approach. For piezoelectric plate vibrations, the energy calculations were often used to describe the energy trapping and harvest effect. Through the research of Yong, et al., the ratios of the trapped energies can be calculated to evaluate the quality of resonators.

Previously, the use of energy methods in vibrations has been common in structural dynamics and the calculated energy has been used for the identification of vibration modes in a complex structure. The statistical energy analysis (SEA) is a mature method for the efficient calculation of the response of structural vibration. In a different approach from the traditional solutions of mechanical vibrations, the statistical method uses the energy of each subsystem as the basic parameters to solve the vibration problem.

Besides SEA, there are also other methods which utilize the energy contributions of vibration modes to make more accurate approximations to vibrations such as the detection of dominant modes of shell structures under external excitations like wind or ground motion and using dominating energy distribution to characterize the vibration signals. In this paper, we focus on the high frequency vibration analysis of the quartz crystal resonator and target locating the pure thickness-shear mode from the coupled vibrations with energy formulation at each frequency with multiple vibration modes or degree-of-freedoms (DOFs).

To serve the purpose of identifying the dominant mode of a quartz crystal resonator, the strain and kinetic energies of vibrations with multiple DOFs are presented with formulations by vibration solutions at each frequency. Using the strain energy of each DOF to distinguish the dominant mode of a gear pair determined effective as described by Wang, et al. Unlike using the time-averaged kinetic energy, in this research, the strain and kinetic energies in each vibration cycle are calculated because the quartz crystal resonator exhibits perfectly steady vibrations. As a result, both the kinetic and strain energy distributions in a vibration cycle are used to identify the dominant functioning mode with the highest energy concentration.

To demonstrate the use of the energy-based vibration mode identification method, we chose the well-known Mindlin first-order equations of a rectangular plate for calculations. First, we adopted the first-order plate equations and calculated the displacement expressions of vibration modes. Then, we established the theoretical formulation of the kinetic and strain energy densities and integrated them over the plate for total energy. Next, we divided the total energy according to specific mode types and calculated the energy distributions of each vibration mode. We calculated the energies in three different material types, which were two different sets of mode couplings with an AT-cut and a full set of mode couplings with an SC-cut quartz crystal plate. Finally, we successfully observed the functioning thickness-shear mode from the highest concen-
trations of kinetic and strain energies. The results were also confirmed by mode shapes at each resonant frequency by the traditional mode plotting method. This energy approach can be effective and easy to use in the vibration mode identification and is capable of improving the features of finite element analysis.

2. THE FIRST-ORDER EQUATIONS OF THE MINDLIN PLATE THEORY

For the analysis of thickness-shear vibrations of a quartz crystal plate, the first-order Mindlin plate equations with limited couplings of modes are adequate for accurate results. We chose a rectangular plate with large length to thickness ratio for the calculations, and the analytical solutions based on the straight-crested wave assumptions are useful in the design and optimization of quartz crystal resonators.

As known in fundamental thickness-shear vibrations of quartz crystal resonators, there are six modes to be analyzed based on the Mindlin first-order plate theory, which are the extension (E) $u^{(0)}_1$, flexural (F) $u^{(0)}_2$, face-shear (FS) $u^{(0)}_3$, thickness-shear (TSh) $u^{(1)}_1$, thickness-stretch (TS) $u^{(2)}_1$, and thickness-twist (TT) $u^{(3)}_3$. However, in different material types the coupling modes are different with the number of modes in the solutions. The simple model of a plate for the crystal resonator we used here is a rectangular AT-cut quartz crystal plate with four free sides as shown in Fig. 1 with $2b$, $2a$, and $2c$ representing its thickness, length, and width, respectively.

To demonstrate our energy approach, we utilized the equations without electrodes, with and without external forces, and with temperature at $25^\circ C$ to avoid frequency-temperature effect. Furthermore, only the straight-crested waves traveling along the $x_3$ direction are considered, making the equations greatly simplified.

With the given plate model and boundary conditions in Fig. 1, we have the stress-displacement equations for quartz crystal plate with the full set of elastic constants and modifica-

\[
T^{(0)}_j = 2b\kappa^{(0)}_j \left[ c_{11}u^{(0)}_1 + c_{12}\kappa^{(0)}_2 u^{(1)}_2 + c_{13}u^{(0)}_3 \right] + c_{44}\kappa^{(0)}_4 \left[ u^{(0)}_{2,3} + u^{(1)}_3 \right] + c_{66}\kappa^{(0)}_6 \left[ u^{(0)}_{2,1} + u^{(1)}_1 \right];
\]

\[
T^{(1)}_j = \frac{2b^2}{3}\kappa^{(1)}_j \left[ c_{11}u^{(0)}_1 + c_{13}\kappa^{(1)}_4 u^{(1)}_{2,3} + c_{13}\kappa^{(1)}_4 u^{(1)}_{2,3} \right] + \tau_{15} \left[ u^{(1)}_{3,1} + \tau_{15}\kappa^{(1)}_{13} u^{(1)}_{2,3} \right] + \tau_{15} \left[ u^{(1)}_{3,1} + \tau_{15}\kappa^{(1)}_{13} u^{(1)}_{2,3} \right];
\]

\[
T^{(2)}_2 = 0, \tau_{ij} = c_0 - \frac{c_{ij}c_{2j}}{c_{22}}, \; i,j = 1, 2, 3, 4, 5, 6; \tag{1c}
\]

where $T^{(n)}_j (n = 0; j = 1, 2, \ldots 6)$ are the nth-order stresses, and $c_0$ are the elastic constants of quartz crystal. To correct the discrepancy due to the truncation of higher-order flexural mode, elastic constants $c_0$ have been modified. The constant $\kappa^{(n)}_j$ are the correction factors to compensate the truncation and their value are

\[
\kappa^{(0)}_p = \frac{\pi}{2\sqrt{3}} \kappa^{(1)}_p = 1, p = 2, 4, 6; \tag{2a}
\]

\[
\kappa^{(0)}_q = 1, \kappa^{(1)}_q = 1, q = 1, 3, 5. \tag{2b}
\]

The corresponding two-dimensional equations of motion of the six vibration modes are

\[
\frac{\partial T^{(0)}_1}{\partial x_1} + \frac{\partial T^{(0)}_5}{\partial x_3} = 2b\rho\ddot{u}^{(0)}_1; \tag{3a}
\]

\[
\frac{\partial T^{(0)}_6}{\partial x_1} + \frac{\partial T^{(0)}_3}{\partial x_3} = 2b\rho\ddot{u}^{(0)}_3; \tag{3b}
\]

\[
\frac{\partial T^{(0)}_5}{\partial x_1} + \frac{\partial T^{(0)}_3}{\partial x_3} = 2b\rho\ddot{u}^{(0)}_3; \tag{3c}
\]

\[
\frac{\partial T^{(1)}_4}{\partial x_1} + \frac{\partial T^{(1)}_5}{\partial x_3} - T^{(0)}_6 = \frac{2b^3}{3} \rho\ddot{u}^{(1)}_1; \tag{3d}
\]

\[
\frac{\partial T^{(1)}_6}{\partial x_1} + \frac{\partial T^{(1)}_4}{\partial x_3} - T^{(0)}_2 = \frac{2b^3}{3} \rho\ddot{u}^{(1)}_2; \tag{3e}
\]

\[
\frac{\partial T^{(1)}_5}{\partial x_1} + \frac{\partial T^{(1)}_4}{\partial x_3} - T^{(0)}_4 = \frac{2b^3}{3} \rho\ddot{u}^{(1)}_3; \tag{3f}
\]

where $\rho$ is the density of quartz crystal.

The six displacements of straight-crested waves travelling along the $x_1$ direction are defined as

\[
u^{(0)}_j = A_j \sin \xi x_1 e^{i\omega t}; \tag{4a}
\]

\[
u^{(1)}_j = \frac{A_{j+3}}{b} \cos \xi x_1 e^{i\omega t}, \; j = 1, 2, 3; \tag{4b}
\]
Figure 2. Dispersion relations of coupled thickness-shear and flexural vibrations of an AT-cut quartz crystal plate.

Figure 3. Dispersion relations of coupled thickness-shear, flexural, and face-shear vibrations of an AT-cut quartz crystal plate.

Figure 4. Dispersion relations of coupled thickness-shear, thickness-stretch, thickness-twist, extension, flexural, and face-shear vibrations of an SC-cut quartz crystal plate.

By solving Eq. (6), we can obtain the dispersion relations of a quartz crystal plate as shown in Figs. 2-4. The dispersion relations validated the equations and also revealed the characteristics of waves in the plate.

With the above results of normalized wavenumbers at each frequency, we can rewrite the displacements of Eq. (4) as

\[ u_j^{(0)} = \sum_{r=1}^{6} \left[ \alpha_{jr} \beta_r A_{44} \frac{b}{\xi} \sin \left( \frac{\pi \xi r x_1}{2} \right) \right] e^{i\omega t}; \quad (7a) \]

\[ u_j^{(1)} = \sum_{r=1}^{6} \left[ \alpha_{(j+3)r} \beta_r A_{44} \frac{b}{\xi} \cos \left( \frac{\pi \xi r x_1}{2} \right) \right] e^{i\omega t}, j = 1, 2, 3; \quad (7b) \]

where \( Z_r \) are the normalized wavenumbers, \( A_{4r} \) stands for the amplitude \( A_4 \) at the fourth wavenumber and \( \alpha_{jr} \) are the corresponding amplitude ratios defined as:

\[ \alpha_{jr} = \frac{A_{jr}}{A_{4r}}, \quad j, r = \begin{cases} 2, 4, & \text{AT - cut, two modes;} \\ 2, 3, 4, & \text{AT - cut, three modes;} \\ 1, 2, 3, 4, 5, 6, & \text{SC - cut, six modes;} \end{cases} \quad (8a) \]

\[ \beta_r = \frac{A_{4r}}{A_{44}}, \quad r = \begin{cases} 2, 4, & \text{AT - cut, two modes;} \\ 2, 3, 4, & \text{AT - cut, three modes;} \\ 1, 2, 3, 4, 5, 6, & \text{SC - cut, six modes;} \end{cases} \quad (8b) \]

where \( A_{jr} \) is the amplitude \( A_j \) at the \( r \)th wavenumber.

Using the traction-free boundary conditions of the plate shown in Fig. 1, the corresponding stress boundary conditions at \( x_1 = \pm a \) are:

\[ T_5^{(0)} = T_6^{(0)} = T_1^{(0)} = T_5^{(1)} = T_6^{(1)} = T_1^{(1)} = 0. \quad (9) \]

By substituting Eq. (7) to Eq. (1) then to Eq. (9), we can solve the frequency equations

\[ |N(\beta_r, Z_r)| = 0; \quad (10) \]
where \( N \) stands for the coefficient matrix of frequency equation of the coupled modes and we give the details in the Appendix.

Let the coefficient determinant of Eq. (10) vanishing, we can obtain the frequency spectra as frequency versus thickness ratio. Then, from the dispersion relations, we can get a set of wavenumbers \( Z_r \) at a specific frequency \( \Omega \). By substituting \( Z_r \) back into Eq. (10), we then obtain \( \beta_r \).

With displacements solved in Eq. (7) and for the calculation of strain energies later, we also give the expressions of nontrivial strains with displacements of the \( x_1 \)-propagating straight-crested waves as:

\[
S_1^{(0)} = u_{1,1}^{(0)}; \quad S_4^{(0)} = u_4^{(1)}; \quad S_2^{(0)} = u_2^{(1)};
\]
\[
S_5^{(0)} = u_{3,1}^{(0)}; \quad S_3^{(0)} = 0; \quad S_6^{(0)} = u_{2,1}^{(0)} + u_1^{(1)}; \quad \text{(11a)}
\]
\[
S_1^{(1)} = u_{1,1}^{(1)}; \quad S_4^{(1)} = 0; \quad S_2^{(1)} = 2u_2^{(2)};
\]
\[
S_5^{(1)} = u_{3,1}^{(1)}; \quad S_3^{(1)} = 0; \quad S_6^{(1)} = u_{2,1}^{(1)}. \quad \text{(11b)}
\]

All the derivation above is primarily based on the first-order Mindlin plate theory. After all the displacements and strains in Eqs. (7) and (8) are obtained, the calculation of the strain and kinetic energies of the plate can be carried out without any difficulty.

To obtain the dominant vibration modes, it needs the vibration mode shapes from Eq. (7) to find out the largest amplitude for each frequency in the frequency spectra. With this process, all the curves in the spectra will be labeled by the dominant vibration mode and the optimal selection of length, or the length to thickness ratio, can also be made from frequency spectra. Clearly, it is a lengthy and tedious procedure requiring the calculation and plotting of vibration modes with a visual check by an experienced researcher. In case of numerical analysis with a larger set of data and solutions, the plotting of vibration modes is the last choice of the vibration mode identification procedure we can consider.

### 3. THE CALCULATION OF ENERGIES OF VIBRATIONS OF A PLATE

To enable a formulation and calculation with displacement solutions for the vibration mode identification, it is natural to turn to the energy of vibrations for a possible solution. The idea is that the largest vibration mode should be the one with the largest proportion of the strain energy or kinetic energy. Then the identification of vibration modes can be made by checking the energy proportion from vibration solutions. Clearly, it is simpler to calculate and compare the energy proportion in both analytical and numerical solutions for this purpose. To this objective, we first need to obtain the energy formulation with vibration solutions.

#### 3.1. The Strain Energy

First, the strain energy is the energy stored in the plate when it undergoes deformation in vibrations. By analyzing strain energy of the plate, the relations of stresses, strains, and displacements will be merged, implying a simple result for the evaluation of the state of vibrations. We start with the strain energy density which can be used to represent the energy in the plate through integration. With the first-order Mindlin plate theory of selected modes, the strain energy density is:

\[
\bar{U} = \frac{1}{2} c_{pq} \sum_{m,n} B_{mn} S_p^{(m)} S_q^{(n)} = \frac{1}{2} c_{pq} \left( B_{00} S_p^{(0)} S_q^{(0)} + B_{11} S_p^{(1)} S_q^{(1)} \right); \quad \text{(12)}
\]

where \( B_{mn} (m,n = 0,1) \) and \( S_p^{(n)} (p = 1, 2, 3, 4, 5, 6; n = 0, 1) \) are the integral constants and strains of the \( n \)-th order as given in Eq. (12), respectively. The energy varies with different cases of mode coupling as we show in the following.

#### 3.1.1. AT-cut quartz crystal plates with two vibration modes

As the case of two modes \( u_2^{(0)} \) and \( u_1^{(1)} \), Eq. (11) is simplified with zero displacements and the Eq. (12) becomes:

\[
\bar{U} = \frac{1}{2} c_{66} B_{00} S_6^{(0)} c_6^{(0)} + \frac{1}{2} c_{11} B_{11} S_1^{(1)} c_1^{(1)}
\]
\[
+ \frac{1}{2} c_{22} B_{11} S_2^{(1)} c_2^{(1)} + c_{12} B_{12} S_1^{(1)} c_1^{(1)}
\]
\[
= \frac{1}{2} B_{00} c_{66} \cdot \left( u_{2,1}^{(0)} \right)^2 + \frac{1}{2} B_{00} c_{66} \cdot \left( u_1^{(1)} \right)^2
\]
\[
+ \frac{1}{2} B_{11} c_{11} \cdot \left( u_{1,1}^{(1)} \right)^2 + B_{00} c_{66} \cdot u_{2,1}^{(0)} u_1^{(1)}. \quad \text{(13)}
\]

Next, we classify the energy densities according to the vibration modes in Eq. (13) as follows:

\[
\bar{U}_{\text{TSH}} = \frac{1}{2} B_{00} c_{66} \left( u_{1,1}^{(1)} \right)^2 + \frac{1}{2} B_{11} c_{11} \left( u_{1,1}^{(1)} \right)^2; \quad \text{(14a)}
\]
\[
\bar{U}_{\text{F}} = \frac{1}{2} B_{00} c_{66} \left( u_{2,1}^{(0)} \right)^2; \quad \text{(14b)}
\]
\[
\bar{U} = \frac{1}{2} B_{00} c_{66} u_{2,1}^{(0)} u_1^{(1)}; \quad \text{(14c)}
\]

where \( \bar{U}_{\text{TSH}} \) and \( \bar{U}_{\text{F}} \) are strain energy densities of single thickness-shear and flexural mode, while \( \bar{U}_{\text{F-TSH}} \) is of the coupled ones.

Finally, the total strain energies of the quartz crystal plate are obtained by integrating the energy densities of Eq. (14) in...
the entire plate in one vibration cycle as:

\[
U_X = \int_a^b \int_c^d \int_0^T U_X \, dx_1 \, dx_3 \, dt, \quad X = \text{TSh}, \text{F, F-TSh};
\]

(15a)

\[
U_{\text{TSh}} = \frac{2\pi c}{\omega} \left\{ \sum_{i=1}^2 A_{4i}^2 \alpha_{4i} V_{c2} + \sum_{i=1,j \neq i}^2 A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} V_{cc} \right\}
\]

\[+ \frac{\pi^3 c}{6\omega} \left\{ \sum_{i=1}^2 A_{4i}^2 \alpha_{4i}^2 Z_i^2 V_{c2} \right\};
\]

(15b)

\[
U_F = \frac{\pi^3 c}{\omega} \left\{ \sum_{i=1}^2 A_{4i}^2 \alpha_{4i}^2 Z_i^2 V_{c2} \right\};
\]

(15c)

\[
U_{F-Tsh} = \frac{4\pi^3 c}{\omega} \left\{ \sum_{i=1}^2 A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} Z_i V_{c2} \right\}
\]

\[+ \sum_{i=1,j \neq i}^2 A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} Z_i V_{cc} \right\};
\]

(15d)

\[
U = \int_a^b \int_c^d \int_0^T U_X \, dx_1 \, dx_3 \, dt = U_{\text{TSh}} + U_F + U_{\text{F-TSh}}, \quad T = \frac{2\pi}{\omega};
\]

(15e)

where \(T\) is the period of vibration, \(U\) is the total strain energy, \(U_{\text{TSh}}\) and \(U_F\) are the strain energy of thickness-shear and flexural modes, and \(U_{\text{F-TSh}}\) is the coupled energy, respectively. And the coefficients \(V_{c2}, V_{cc}, V_{s2}, \text{and } V_s\) in Eq. (15) are given as:

\[
V_{c2} = \frac{1}{b} \int_a^b \cos^2 \left( \frac{\pi Z_i x_1}{2 b} \right) \, dx_1 = \frac{a}{b} + \frac{1}{\pi Z_i} \sin \left( \frac{a}{b} \pi Z_i \right);
\]

(16a)

\[
V_{cc} = \frac{1}{b} \int_a^b \cos \left( \frac{\pi Z_i x_1}{2 b} \right) \cos \left( \frac{\pi Z_j x_1}{2 b} \right) \, dx_1
\]

\[= \frac{2}{\pi} \left[ \sin \left( \frac{\pi}{2} (Z_i - Z_j) \right) + \sin \left( \frac{\pi}{2} (Z_i + Z_j) \right) \right];
\]

(16b)

\[
V_{s2} = \frac{1}{b} \int_a^b \sin^2 \left( \frac{\pi Z_i x_1}{2 b} \right) \, dx_1 = \frac{a}{b} - \frac{1}{\pi Z_i} \sin \left( \frac{a}{b} \pi Z_i \right);
\]

(16c)

\[
V_s = \frac{1}{b} \int_a^b \sin \left( \frac{\pi Z_i x_1}{2 b} \right) \sin \left( \frac{\pi Z_j x_1}{2 b} \right) \, dx_1
\]

\[= \frac{2}{\pi} \left[ \sin \left( \frac{\pi}{2} (Z_i - Z_j) \right) + \sin \left( \frac{\pi}{2} (Z_i + Z_j) \right) \right].
\]

(16d)

3.1.2. AT-cut quartz crystal plate with three vibration modes

With the same procedure, we can obtain the strain energy density of the case of three modes as:

\[
U = \frac{1}{2} c_{55} B_{00} S_5^2 (S_0^0)^2 + \frac{1}{2} c_{60} B_{00} S_6^2 (S_0^0)^2 + \frac{1}{2} c_{66} B_{00} S_6^2 (S_0^0)^2
\]

\[+ \frac{1}{2} c_{11} B_1 S_1^1 (S_1^1)^2 + \frac{1}{2} c_{22} B_2 S_2^1 (S_1^1)^2 + \frac{1}{2} c_{12} B_{11} S_1^1 (S_1^1)^2
\]

\[= \frac{1}{2} B_{00} c_{66} \left( \frac{u_{2,1}^1}{u_{2,1}^1} \right)^2 + \frac{1}{2} B_{00} c_{55} \left( \frac{u_{3,1}^1}{u_{3,1}^1} \right)^2 + \frac{1}{2} B_{00} c_{66} \left( \frac{u_1^1}{u_1^1} \right)^2
\]

\[+ \frac{1}{2} B_{11} c_{11} \left( \frac{u_{1,1}^1}{u_{1,1}^1} \right)^2 + B_{00} c_{66} \cdot \frac{u_{2,1}^1}{u_{2,1}^1} \cdot \frac{u_{1,1}^1}{u_{1,1}^1}
\]

\[+ B_{00} c_{55} \cdot \frac{u_{2,1}^1}{u_{2,1}^1} \cdot \frac{u_{3,1}^1}{u_{3,1}^1} + B_{00} c_{66} \cdot \frac{u_1^1}{u_1^1} \cdot \frac{u_{3,1}^1}{u_{3,1}^1};
\]

(17)

Next, we classify the energy densities according to the vibration modes in Eq. (17) as follows:

\[
U_{\text{TSh}} = \frac{1}{2} B_{00} c_{66} \left( \frac{u_{1,1}^1}{u_{1,1}^1} \right)^2 + \frac{1}{2} B_{11} c_{11} \left( \frac{u_{1,1}^1}{u_{1,1}^1} \right)^2;
\]

(18a)

\[
U_F = \frac{1}{2} B_{00} c_{55} \left( \frac{u_{2,1}^1}{u_{2,1}^1} \right)^2;
\]

(18b)

\[
U_{\text{FS}} = \frac{1}{2} B_{00} c_{66} \left( \frac{u_{3,1}^1}{u_{3,1}^1} \right)^2;
\]

(18c)

\[
U_{\text{F-TSh}} = B_{00} c_{66} \cdot \frac{u_{2,1}^1}{u_{2,1}^1} \cdot \frac{u_{3,1}^1}{u_{3,1}^1};
\]

(18d)

\[
U_{\text{F-FS}} = B_{00} c_{55} \cdot \frac{u_{2,1}^1}{u_{2,1}^1} \cdot \frac{u_{3,1}^1}{u_{3,1}^1};
\]

(18e)

\[
U_{\text{TSh-FS}} = B_{00} c_{66} \cdot \frac{u_1^1}{u_1^1} \cdot \frac{u_{3,1}^1}{u_{3,1}^1};
\]

(18f)

where \(U_X (X = \text{TSh, F, FS, F-TSh, F-FS, TSh - FS})\) are the strain energies of different modes. Since for AT-cut quartz crystal \(c_{66}\) is much smaller, it is clear that the coupled energies should be much smaller in comparison with the energies of single modes. Thus, these coupled energies are neglected in following calculations.

Finally, the total strain energies of the quartz crystal plate are obtained by integrating the energy densities of Eq. (18) in the entire plate in one vibration cycle as:

\[
U_X = \int_a^b \int_c^d \int_0^T U_X \, dx_1 \, dx_3 \, dt, \quad X = \text{TSh, F, FS, F-TSh, F-FS, TSh - FS};
\]

(19a)

\[
U_{\text{TSh}} = \frac{2\pi c}{\omega} \left\{ \sum_{i=1}^3 A_{4i}^2 \alpha_{4i}^2 V_{c2} \right\}
\]

\[+ \sum_{i=1,j \neq i}^3 A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} V_{cc} \right\};
\]

\[
+ \frac{\pi^3 c}{6\omega} \left\{ \sum_{i=1}^3 A_{4i}^2 \alpha_{4i}^2 Z_i^2 V_{c2} \right\};
\]


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The SC-cut quartz crystal is one of the most complicated material types in vibration analysis with the first-order Mindlin plate equations as the elastic constant matrix is full. Due to the material characteristics of this special orientation, all the six vibration modes are coupled and need to be calculated in the energy solutions. First, we can obtain the strain energy density of six modes as:

\[
U = \frac{1}{2} \sum_{p=1,2,5,6} \sum_{q=1,2,4,5,6} c_{pq} B_{pq} S_p^{(0)} S_q^{(0)}
\]

\[
+ \frac{1}{2} \sum_{p=1,2,5,6} \sum_{q=1,2,4,5,6} c_{pq} B_{pq} S_p^{(1)} S_q^{(1)}
\]

\[
= \frac{1}{2} B_{00} \left[ c_{11} (u_{11}^{(0)})^2 + c_{22} (u_{12}^{(0)})^2 + c_{44} (u_{44}^{(0)})^2 + c_{55} (u_{55}^{(0)})^2 + c_{66} (u_{66}^{(0)})^2 + c_{12} (u_{12}^{(0)})^2 + c_{14} (u_{14}^{(0)})^2 + c_{15} (u_{15}^{(0)})^2 + c_{34} (u_{34}^{(0)})^2 + c_{56} (u_{56}^{(0)})^2 + c_{65} (u_{65}^{(0)})^2 \right]
\]

Observing the Eq. (20), though there are many terms of coupled energy, but many elastic constants associated with them are smaller in comparison to the single mode terms. As a result, the coupled energies are also neglected here, and only the larger energy density terms are presented as:

\[
U_{TSH} = \frac{1}{2} B_{00} c_{00} (u_{11}^{(0)})^2 + \frac{1}{2} B_{11} c_{11} (u_{11}^{(1)})^2 ;
\]

\[
U_{TSH} = \frac{1}{2} B_{00} c_{12} (u_{12}^{(0)})^2 + \frac{1}{2} B_{11} c_{12} (u_{12}^{(1)})^2 ;
\]

\[
U_{TT} = \frac{1}{2} B_{00} c_{44} (u_{44}^{(0)})^2 + \frac{1}{2} B_{11} c_{44} (u_{44}^{(1)})^2 ;
\]

\[
U_{TR} = \frac{1}{2} B_{00} c_{11} (u_{11}^{(0)})^2 ;
\]

\[
U_{T} = \frac{1}{2} B_{00} c_{00} (u_{11}^{(0)})^2 ;
\]

\[
U_{FS} = \frac{1}{2} B_{00} c_{55} (u_{55}^{(0)})^2 .
\]

We also integrated the energy densities of Eq. (20) as:

\[
U_X = \int_{-a}^{a} \int_{-c}^{c} \int_{0}^{T} U_X dx_1 dx_3 dt,
\]

\[
X = TSH, TST, TT, E, F, FS ;
\]

\[
U_{TSH} = \frac{2\pi c}{\omega} c_{00} \left\{ \sum_{i=1}^{6} A_{2i}^2 c_{4i}^2 V_{c2} \right\}
\]

\[
+ \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} c_{4i} c_{4j} V_{cc}
\]

\[
+ \frac{\pi^3 c}{6\omega} \left\{ \sum_{i=1}^{6} A_{2i}^2 c_{4i}^2 Z_i^2 V_{s2} \right\}
\]

\[
+ \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} c_{4i} c_{4j} Z_i Z_j V_{s4} \}
\]

\[
; (22b)
\]
of each vibration modes. In other words, we define the energy percentages of each vibration mode as

\[ P_X = \frac{U_X}{\tilde{U}} \times 100\% \]

where \( \tilde{U} \) is total of the strain energy of the plate which excludes the coupling energy parts:

\[ \tilde{U} = \left\{ \begin{array}{ll} U_{\text{TSh}} + U_F, & \text{AT - cut, two modes,} \\ U_{\text{TSh}} + U_F + U_{\text{FS}}, & \text{AT - cut, three modes,} \\ U_{\text{TSh}} + U_{\text{TSh}} + U_{TT} + U_{E} + U_F + U_{\text{FS}}, & \text{SC - cut, six modes;} \end{array} \right. \tag{23} \]

while \( P_X \) are the percentage of strain energy of the X mode. The strain energies of coupled modes are neglected due to their insignificance. Such description will be enough for the measurement of dominance of strain energies of each vibration mode.

Since the kinetic energy in vibration also spread to all modes just as the strain energy, and its equivalence to strain energy also implies the possibility to use its proportions for vibration mode identification. With the successful utilization of the strain energy, we also want to explore the possibility and applicability of kinetic energy in a similar manner.

### 3.2. The Kinetic Energy

With known vibrations of the plate, we can calculate the kinetic energy and their proportions as we have demonstrated in the calculation of strain energy. We are expecting that the proportions of kinetic energy are consistent with the strain energies for the determination of dominance of specific mode shapes.

The kinetic energy of the first-order Mindlin plate is

\[ K = \frac{1}{2} \sum_m \sum_n \rho B_{mn} U_{ij} (m) \dot{U}_{ij} (n) \]

\[ = \frac{1}{2} \rho B_{00} \sum_{i=1}^{3} \dot{u}_i (0) \dot{u}_i (0) + \frac{1}{2} \rho B_{11} \sum_{i=1}^{3} \dot{u}_i (1) \dot{u}_i (1). \tag{24} \]

By separating the vibration displacements as for the calculation of the strain energy, Eq. (24) can also be rewritten by the vibration modes as:

\[ K_{\text{TSh}} = \frac{1}{2} \rho B_{11} \dot{u}_1 (1) \dot{u}_1 (1); \tag{25a} \]

\[ K_{\text{TSh}} = \frac{1}{2} \rho B_{00} \dot{u}_1 (0) \dot{u}_1 (0); \tag{25b} \]

\[ K_{\text{TT}} = \frac{1}{2} \rho B_{11} \dot{u}_3 (1) \dot{u}_3 (1); \tag{25c} \]

\[ K_E = \frac{1}{2} \rho B_{00} \dot{u}_3 (0) \dot{u}_3 (0); \tag{25d} \]

\[ K_{\text{FS}} = \frac{1}{2} \rho B_{00} \dot{u}_3 (0) \dot{u}_3 (0); \tag{25f} \]

### 3.1.4. Energy percentages of each vibration modes

With all the expressions of strain energies of the plate obtained, we can calculate the energy proportions of each vibration mode. In this study, in order to develop a procedure based on the energy for mode identification, we focus not only on the
where \( \bar{u}_j^{(n)} (n = 0, 1; j = 1, 2, 3) \) are the velocities of each mode in vibrations. We can see from Eq. (25), in comparison with the strain energy, the kinetic energy density equations are not coupled. This is certainly advantageous over the strain energies in their complexity for calculation.

With displacement solutions in Eq. (7), we also integrate Eq. (25) over the plate for the three cases of couplings.

### 3.2.1. AT-cut quartz crystal plate with two vibration modes

For two modes only thickness-shear and flexural mode are calculated here as:

\[
K_X = \int_a^a \int_a^c \int_0^T K_X dx_1 dx_3 dt, \quad X = \text{TSh, F}; \quad (26a)
\]

\[
K_{\text{TSh}} = \frac{4\pi b^2 \rho c}{3\omega} \left\{ \sum_{i=1}^{2} A_i^2 \alpha_i^2 V_{c2} + \sum_{i=1, j \neq i}^{2} A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} V_{cc} \right\}; \quad (26b)
\]

\[
K_F = \frac{4\pi b^2 \rho c}{\omega} \left\{ \sum_{i=1}^{2} A_i^2 \alpha_i^2 V_{s2} + \sum_{i=1, j \neq i}^{2} A_{4i} A_{4j} \alpha_{2i} \alpha_{2j} V_{ss} \right\}; \quad (26c)
\]

\[
K = \int_a^a \int_a^c \int_0^T K dx_1 dx_3 dt = K_{\text{TSh}} + K_F. \quad (26d)
\]

### 3.2.2. AT-cut quartz crystal plate with three vibration modes

The kinetic energy of thickness-shear, flexural, and face-shear mode are as:

\[
K_X = \int_a^a \int_a^c \int_0^T K_X dx_1 dx_3 dt, \quad X = \text{TSh, F, FS}; \quad (27a)
\]

\[
K_{\text{TSh}} = \frac{4\pi b^2 \rho c}{3\omega} \left\{ \sum_{i=1}^{3} A_i^2 \alpha_i^2 V_{c2} + \sum_{i=1, j \neq i}^{3} A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} V_{cc} \right\}; \quad (27b)
\]

\[
K_F = \frac{4\pi b^2 \rho c}{\omega} \left\{ \sum_{i=1}^{3} A_i^2 \alpha_i^2 V_{s2} + \sum_{i=1, j \neq i}^{3} A_{4i} A_{4j} \alpha_{2i} \alpha_{2j} V_{ss} \right\}; \quad (27c)
\]

\[
K_{\text{FS}} = \frac{4\pi b^2 \rho c}{\omega} \left\{ \sum_{i=1}^{3} A_i^2 \alpha_i^2 V_{s2} + \sum_{i=1, j \neq i}^{3} A_{4i} A_{4j} \alpha_{3i} \alpha_{3j} V_{ss} \right\}; \quad (27d)
\]

\[
K = \int_a^a \int_a^c \int_0^T K dx_1 dx_3 dt = K_{\text{TSh}} + K_F + K_{\text{FS}}. \quad (27e)
\]

### 3.2.3. SC-cut quartz crystal plate with six vibration modes

In an SC-cut plate all the kinetic energies of the six modes are considered as:

\[
K_X = \int_a^a \int_a^c \int_0^T K_X dx_1 dx_3 dt, \quad X = \text{TSh,TSt, TT, E, F, FS}; \quad (28a)
\]

\[
K_{\text{TSh}} = \frac{4\pi b^2 \rho c}{3\omega} \left\{ \sum_{i=1}^{6} A_i^2 \alpha_i^2 V_{c2} + \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} \alpha_{4i} \alpha_{4j} V_{cc} \right\}; \quad (28b)
\]

\[
K_{\text{TSt}} = \frac{4\pi b^2 \rho c}{3\omega} \left\{ \sum_{i=1}^{6} A_i^2 \alpha_i^2 V_{c2} + \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} \alpha_{5i} \alpha_{5j} V_{cc} \right\}; \quad (28c)
\]

\[
K_{\text{TT}} = \frac{4\pi b^2 \rho c}{3\omega} \left\{ \sum_{i=1}^{6} A_i^2 \alpha_i^2 V_{c2} + \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} \alpha_{6i} \alpha_{6j} V_{cc} \right\}; \quad (28d)
\]

\[
K_E = \frac{4\pi b^2 \rho c}{\omega} \left\{ \sum_{i=1}^{6} A_i^2 \alpha_i^2 V_{s2} + \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} \alpha_{1i} \alpha_{1j} V_{ss} \right\}; \quad (28e)
\]

\[
K_F = \frac{4\pi b^2 \rho c}{\omega} \left\{ \sum_{i=1}^{6} A_i^2 \alpha_i^2 V_{s2} + \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} \alpha_{2i} \alpha_{2j} V_{ss} \right\}; \quad (28f)
\]

\[
K_{\text{FS}} = \frac{4\pi b^2 \rho c}{\omega} \left\{ \sum_{i=1}^{6} A_i^2 \alpha_i^2 V_{s2} + \sum_{i=1, j \neq i}^{6} A_{4i} A_{4j} \alpha_{3i} \alpha_{3j} V_{ss} \right\}; \quad (28g)
\]

\[
K = \int_a^a \int_a^c \int_0^T K dx_1 dx_3 dt = K_{\text{TSh}} + K_{\text{TSt}} + K_{TT} + K_E + K_F + K_{\text{FS}}. \quad (28h)
\]

### 3.2.4. Energy percentages of each vibration modes

Following the same procedure, the distributions of kinetic energies are also calculated as:

\[
\bar{\eta}_X = \frac{K_X}{K} \times 100\%, \quad X = \begin{cases} T\text{Sh}, F, & \text{AT - cut, two modes}, \\ T\text{Sh,F,FS}, & \text{AT - cut, three modes}, \\ T\text{Sh,TT, TSt, E, F, FS}, & \text{SC - cut, six modes}; \end{cases} \quad (29)
\]
Table 1. Strain/Kinetic(S/K) energy distributions of the crystal plate at each resonance with two vibration modes.

<table>
<thead>
<tr>
<th>Serial Number</th>
<th>Normalized Frequency</th>
<th>Flexural (S/K %)</th>
<th>Thickness-shear (S/K %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9213</td>
<td>86.76/88.87</td>
<td>13.24/11.13</td>
</tr>
<tr>
<td>2</td>
<td>0.9764</td>
<td>85.59/84.17</td>
<td>14.41/15.83</td>
</tr>
<tr>
<td>3</td>
<td>1.0017</td>
<td>2.45/1.05</td>
<td>97.55/98.95</td>
</tr>
<tr>
<td>4</td>
<td>1.0141</td>
<td>25.04/12.46</td>
<td>74.96/87.54</td>
</tr>
<tr>
<td>5</td>
<td>1.0285</td>
<td>69.67/52.86</td>
<td>30.33/47.14</td>
</tr>
<tr>
<td>6</td>
<td>1.0461</td>
<td>47.71/28.26</td>
<td>52.29/71.74</td>
</tr>
<tr>
<td>7</td>
<td>1.0733</td>
<td>57.04/37.54</td>
<td>42.96/62.46</td>
</tr>
<tr>
<td>8</td>
<td>1.0953</td>
<td>67.3/49.13</td>
<td>32.7/50.87</td>
</tr>
<tr>
<td>9</td>
<td>1.1237</td>
<td>61.75/42.49</td>
<td>38.25/57.51</td>
</tr>
<tr>
<td>10</td>
<td>1.1509</td>
<td>68.49/50.48</td>
<td>31.51/49.52</td>
</tr>
<tr>
<td>11</td>
<td>1.1800</td>
<td>68.57/50.49</td>
<td>31.43/49.51</td>
</tr>
</tbody>
</table>

where $\tilde{K}$ is the sum of the kinetic energy of all vibration modes as:

$$\tilde{K} = \begin{cases} 
K_{TSh} + K_{F}, & \text{AT - cut, two modes,} \\
K_{TSh} + K_{F} + K_{FS}, & \text{AT - cut, three modes,} \\
K_{TSh} + K_{TSt} + K_{TT} + K_{E} + K_{F} + K_{FS}, & \text{SC - cut, six modes,} 
\end{cases}$$

and $T_X^L$ are the proportions of kinetic energies of $X$ mode. The clear advantage of the kinetic energies is that there is no coupling and the calculation can be simpler.

4. NUMERICAL RESULTS

With formulations of energy proportions in terms of displacements, we presented a simple model for energy-based method for the identification of vibration modes in a quartz crystal plate in the vicinity of thickness-shear vibrations. With solutions of displacements and frequencies, the calculations of both strain and kinetic energies are straightforward, and the identification of vibration modes are also done with the numerical value corresponding to each displacement.
4.2. AT-cut quartz crystal plate with three vibration modes

The results have been tabulated in Table 2 for an AT-cut quartz crystal plate with three modes and the model is same as the case of two modes. We also chose the length to thickness ratio \(a/b\) to 32.9148 for demonstration purposes. With the given aspect ratio, 14 frequencies are in the frequency spectra and the identification techniques based on energy will be applied to these points shown in Fig. 8.

Table 2. Strain/Kinetic(S/K) energy distributions of the crystal plate at each resonance with three vibration modes.

<table>
<thead>
<tr>
<th>Serial Number</th>
<th>Normalized Frequency</th>
<th>Flexural (S/K %)</th>
<th>Face-shear (S/K %)</th>
<th>Thickness-shear (S/K %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9200</td>
<td>76.4/88.6</td>
<td>23.2/11.0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.9745</td>
<td>72.0/78.9</td>
<td>24.1/14.6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.9818</td>
<td>9.2/5.5</td>
<td>0.4/0.4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.0017</td>
<td>2.0/1.6</td>
<td>0.0/0.0</td>
<td>99.4/98.4</td>
</tr>
<tr>
<td>5</td>
<td>1.0139</td>
<td>17.7/13.7</td>
<td>82.2/86.2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.0276</td>
<td>53.9/53.4</td>
<td>45.7/46.2</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.0457</td>
<td>29.1/26.1</td>
<td>70.7/73.8</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.0723</td>
<td>39.3/36.0</td>
<td>56.6/57.3</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.0752</td>
<td>5.0/3.9</td>
<td>17.3/14.8</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1.0946</td>
<td>49.9/48.1</td>
<td>49.5/51.2</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.1230</td>
<td>44.9/42.7</td>
<td>55.0/57.2</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>1.1499</td>
<td>51.5/49.7</td>
<td>47.4/49.3</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1.1685</td>
<td>1.5/1.0</td>
<td>97.5/98.5</td>
<td>1.0/0.5</td>
</tr>
<tr>
<td>14</td>
<td>1.1791</td>
<td>51.7/50.0</td>
<td>47.9/49.7</td>
<td></td>
</tr>
</tbody>
</table>

Similarly, the energy distributions of each frequency are calculated and listed in the Table 2.

From Table 2, it is clear that the energy distribution has fluctuations and the results of strain energy are similar to the kinetic energy in a systematic trend with only some minor deviations which are mainly caused by the coupling parts of the strain energy. To see the details, we take a look at the 4th frequency with the mode feature which is obviously simple with energy distributions. At the 4th frequency where the normalized frequency is 1.0017, and the thickness-shear vibrations are considered as the dominant mode because of its highest energy proportion with the percentage value of 99%, almost 100%, implying this frequency represents the optimal functioning mode. For the 5th column, we can see that the strain energy distribution is varying from the lowest 1.0 percent to the highest 99.4 percent and we can quickly locate the mode according to these values. Thus, it demonstrated that the energy method can help us quickly and effectively implement the mode identification technique and find the optimal functioning mode.

For further validation, we took the displacement data of each frequency from the above model and plot mode shapes individ-
The mode shapes above clearly show that they are consistent with the findings in Table 2. Taking Fig. 9 as an example, the amplitude of thickness-shear mode has the clear dominance and obviously it is the functioning mode of the thickness-shear vibrations, when its energy distribution reaches 99.4% in strain and 98.4% in kinetic energies, respectively. This correlation also applies to other modes and their mode shapes, proving the energy identification method is correct.

4.3. SC-cut quartz crystal plate with six vibration modes

The results have been presented in Table 3 for the SC-cut quartz crystal plate and we also chose the length to thickness ratio \((a/b)\) to 34.4097 for demonstration purposes. Meanwhile for the SC-cut plate the frequency of working modes of thickness-shear vibrations are lower than the AT-cut and we chose a different frequency range for the examination. With the given aspect ratio, we chose 12 modes in the frequency spectra and the identification method will be applied to these frequency points as shown in Fig. 11.

By examining the Table 3, the domain modes are also clearly identified by both strain and kinetic energy solutions. Different from the AT-cut quartz crystal plates above, the concentration of energy of the functioning thickness-shear mode is about 91% due to more couplings. Some deviations occur when the flexural mode couples with other modes like as in the 4th or 11th frequencies. Other than this, predictions of dominant modes can be trusted based on energy distributions. The further validation of mode shapes are given with Figs. 12 and 13.

From Fig. 12, we find the prediction of mode shapes with kinetic energy pointing to the flexural mode which is clearly the dominant mode. As we know, the SC-cut plate is more likely to be affected by the spurious modes due to stronger couplings. The strain energy solutions are more sensitive to stronger mode couplings as we chose to neglect the coupled terms in strain energies. We can see the kinetic energy is better in determining the feature of strong couplings of vibration modes. In general, both methods based on energy are reliable in identifying the dominant vibration modes as shown in Fig. 13.

4.4. Energy analysis with experimental data

Since the energy method is accurate with the theoretical analysis, we used the experimental data by Yamashita et al.\textsuperscript{31} for comparison. The models of the experiment are the rectangle AT-cut quartz plates with length-to-thickness aspects from 10.33 to 19.40 and width-to-thickness aspect as 3.78. We compared the same frequency spectra with mode identification as shown in Fig. 14.

The experimental data marked with “x” are the thickness-shear modes. The selected four points A, B, C, and D are near the middle of the curves and the dominant thickness-shear modes from the theoretical analysis.\textsuperscript{15} By taking the plate information of the four points, we calculated energies of flexural and thickness-shear modes and list them in Table 4.

In Table 4, the predictions from energy of the four points are consistent with the experimental data. The energy values are closer to the 90% level. It shows the
The objective of this study was to identify the vibration modes from the coupled vibrations with the Mindlin plate equations at high frequency by means of the energy proportion from displacement solutions. The formulations of kinetic and strain energy densities of the first-order Mindlin plate have been introduced and the total energy is obtained. The vibrations of a simple rectangular AT-cut quartz crystal plate with four free parallel sides were considered without including the electrodes and frequency-temperature effect. The energy distributions of each vibration mode were calculated with the given plate configurations. The comparison between the energy distributions and the mode shape patterns confirmed that the proposed approach actually identified the vibration modes accurately and was able to single out the optimal working mode without using mode shape plots. More importantly, the method was purely numerical and is well suited with the FEM and other numerical solutions. Particularly, using the kinetic energy for the energy distributions of each mode will be a simpler calculation.

Calculating the kinetic and strain energies of the plate within a given frequency range and obtaining the energy distribution of each vibration mode were demonstrated to be simple and easy as exhibited in this study. The conclusion from the energy distortion were consistent with mode shape plots. Although there were some small differences between kinetic and strain energies with the strongly coupled vibration modes, such method of mode identification was simple and reliable. In general, the energy based method for mode identification led to the accurate characterization of vibration modes without difference. Overall this was a convenient and efficient method for the identification of vibration modes with strongly coupled modes and high frequency in structures of anisotropic materials and complex configurations. Furthermore, this energy approach will be particularly preferential in finite element analysis because the energy calculation and characterization can be easily done without utilizing complicated visualization functions and tools. In addition, it was easy to be automated with data flow and the numerical procedure. It is expected that the energy based approach for vibration mode identification can be adopted to meet similar requirements in the finite element analysis of general structures.

5. CONCLUSIONS

Table 3. Strain/Kinetic(S/K) energy distributions of the crystal plate at each resonance with six vibration modes.

<table>
<thead>
<tr>
<th>Serial Number</th>
<th>Normalized Frequency</th>
<th>Extension (S/K %)</th>
<th>Flexural (S/K %)</th>
<th>Face-shear (S/K %)</th>
<th>Thickness-stretch (S/K %)</th>
<th>Thickness-twist (S/K %)</th>
<th>Thickness-shear (S/K %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8669</td>
<td>0.36/2.89</td>
<td>90.39/80.54</td>
<td>0.21/0.86</td>
<td>0.58/1.89</td>
<td>0.15/0.32</td>
<td>8.31/13.48</td>
</tr>
<tr>
<td>2</td>
<td>0.9052</td>
<td>11.18/12.88</td>
<td>3.43/0.55</td>
<td>0.01/0.4</td>
<td>0.30/0.45</td>
<td>0.60/0.16</td>
<td>0.37/0.11</td>
</tr>
<tr>
<td>3</td>
<td>0.9142</td>
<td>1.72/15.45</td>
<td>87.79/64.57</td>
<td>0.03/0.45</td>
<td>0.65/1.96</td>
<td>0.15/0.34</td>
<td>9.17/14.72</td>
</tr>
<tr>
<td>4</td>
<td>0.9266</td>
<td>21.56/69</td>
<td>65.01/15.27</td>
<td>3.55/10.15</td>
<td>3.37/10.15</td>
<td>0.20/0.22</td>
<td>9.13/14.87</td>
</tr>
<tr>
<td>5</td>
<td>0.9408</td>
<td>0.01/0.01</td>
<td>1.87/0.55</td>
<td>0.00/0.00</td>
<td>0.00/0.00</td>
<td>0.00/0.00</td>
<td>0.00/0.00</td>
</tr>
<tr>
<td>6</td>
<td>0.9504</td>
<td>0.09/0.27</td>
<td>20.21/6.58</td>
<td>1.85/0.55</td>
<td>2.25/0.55</td>
<td>2.00/0.22</td>
<td>9.25/14.72</td>
</tr>
<tr>
<td>7</td>
<td>0.9626</td>
<td>0.38/2.16</td>
<td>74.93/45.42</td>
<td>0.21/0.6</td>
<td>0.22/0.6</td>
<td>0.08/0.22</td>
<td>22.64/44.49</td>
</tr>
<tr>
<td>8</td>
<td>0.9717</td>
<td>11.2/12.63</td>
<td>0.94/0.02</td>
<td>0.16/0.2</td>
<td>1.25/0.2</td>
<td>0.08/0.22</td>
<td>9.13/14.87</td>
</tr>
<tr>
<td>9</td>
<td>0.9743</td>
<td>0.26/0.96</td>
<td>55.3/24.29</td>
<td>0.31/0.76</td>
<td>1.57/0.55</td>
<td>1.95/0.46</td>
<td>1.13/0.63</td>
</tr>
<tr>
<td>10</td>
<td>0.9963</td>
<td>0.18/0.06</td>
<td>48.95/20.2</td>
<td>0.14/0.24</td>
<td>1.95/0.71</td>
<td>4.16/0.98</td>
<td>4.66/0.94</td>
</tr>
<tr>
<td>11</td>
<td>1.0099</td>
<td>14.16/51.71</td>
<td>62.62/18.43</td>
<td>2.42/0.72</td>
<td>0.97/1.78</td>
<td>1.74/0.87</td>
<td>22.7/29.95</td>
</tr>
<tr>
<td>12</td>
<td>1.0181</td>
<td>5.99/28.08</td>
<td>64.33/24.44</td>
<td>1.28/4.58</td>
<td>1.23/3.31</td>
<td>4.42/9.64</td>
<td>22.7/29.95</td>
</tr>
</tbody>
</table>

Table 4. Strain/Kinetic(S/K) energy distributions of the crystal plate at each resonance with experimental data.

<table>
<thead>
<tr>
<th>Experiment Points</th>
<th>Normalized Frequency</th>
<th>Length thickness ratio (S/K %)</th>
<th>Flexural (S/K %)</th>
<th>Thickness-stretch (S/K %)</th>
<th>Dominant Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.0115</td>
<td>12.0867</td>
<td>14.41/6.84</td>
<td>85.59/83.16</td>
<td>Thickness-shear</td>
</tr>
<tr>
<td>B</td>
<td>1.0083</td>
<td>13.5978</td>
<td>11.75/5.29</td>
<td>88.25/84.71</td>
<td>Thickness-shear</td>
</tr>
<tr>
<td>C</td>
<td>1.0069</td>
<td>15.4778</td>
<td>3.79/0.3</td>
<td>96.21/91.70</td>
<td>Thickness-shear</td>
</tr>
<tr>
<td>D</td>
<td>1.0054</td>
<td>17.0067</td>
<td>8.65/3.9</td>
<td>91.35/96.1</td>
<td>Thickness-shear</td>
</tr>
</tbody>
</table>

Figure 14. Frequency spectra of the experimental and theoretical data.

The material parameters of quartz crystal and its AT- and SC-cut can be found in Refs. 3, 5, 8, 11, 14.

5.1. Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grants 11672142 and 11772163) and the K. C. Wong Magna Fund through Ningbo University. Additional funding is from the National Key R&D Program of China (Grant 2017YFB102900) and the Technology Innovation 2025 Program of the City of Ningbo (Grant 2019B10122).

APPENDIX A

Equations and parameters for the vibration equations

The material parameters of quartz crystal and its AT- and SC-cut can be found in Refs. 3, 5, 8, 11, 14.
The matrix \( \mathbf{M} \) and \( \mathbf{N} \) in Eq. (6) and Eq. (10) are given for three cases.

1) AT-cut quartz crystal plates with two vibration modes

The dispersion relation from Eq. (6) is given as:
\[
Z^2 \pi \{ 2 - 12 \Omega^2 \} \frac{2 Z \pi}{Z \pi} = 0. \quad (A1)
\]

The frequency equation in Eq. (10) is given as:
\[
\left( \alpha_{22} \frac{2 \pi}{Z} + 1 \right) \cos \left( \frac{2 \pi}{Z} \right) \left( \alpha_{24} \frac{2 \pi}{Z} + 1 \right) \cos \left( \frac{2 \pi}{Z} \right) = 0. \quad (A2)
\]

2) AT-cut quartz crystal plates with three vibration modes

The dispersion relation from Eq. (6) is given as:
\[
\left( \frac{2 \pi}{Z} Z^2 - \Omega^2 \right) \cos \frac{2 \pi}{Z} Z^2 - \Omega^2 \left( \frac{2 \pi}{Z} \right) \left( \frac{2 \pi}{Z} \right) = 0. \quad (A3)
\]

The frequency equation from Eq. (10) is given as:
\[
\left[ \alpha_{32} \frac{2 \pi}{Z} + \kappa_0 \cos \left( \frac{2 \pi}{Z} \right) \left( \alpha_{22} \frac{2 \pi}{Z} + 1 \right) \cos \left( \frac{2 \pi}{Z} \right) \right] \left[ \alpha_{33} \frac{2 \pi}{Z} + \kappa_0 \cos \left( \frac{2 \pi}{Z} \right) \left( \alpha_{22} \frac{2 \pi}{Z} + 1 \right) \cos \left( \frac{2 \pi}{Z} \right) \right] \cdots \left[ \alpha_{34} \frac{2 \pi}{Z} + \kappa_0 \cos \left( \frac{2 \pi}{Z} \right) \left( \alpha_{22} \frac{2 \pi}{Z} + 1 \right) \cos \left( \frac{2 \pi}{Z} \right) \right] \left[ \alpha_{35} \frac{2 \pi}{Z} + \kappa_0 \cos \left( \frac{2 \pi}{Z} \right) \left( \alpha_{22} \frac{2 \pi}{Z} + 1 \right) \cos \left( \frac{2 \pi}{Z} \right) \right] = 0. \quad (A4)
\]

3) SC-cut quartz crystal plates with six vibration modes

The dispersion relation from Eq. (6) is given as:
\[
| \mathbf{M}_{ij} | = 0, i, j = 1, 2, \ldots, 6;
\]
\[
M_{11} = \kappa_0 (c_{11} \Omega^2 - c_{66} \Omega^2), M_{12} = \kappa_0 (c_{16} \Omega^2), M_{13} = \kappa_0 (c_{16} \Omega^2);
\]
\[
M_{14} = \kappa_0 (c_{16} \Omega^2), M_{15} = \kappa_0 (c_{12} \Omega^2), M_{16} = \kappa_0 (c_{12} \Omega^2);
\]
\[
M_{21} = \kappa_0 (c_{66} \Omega^2), M_{22} = \kappa_0 (c_{66} \Omega^2), M_{23} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{24} = \kappa_0 (c_{66} \Omega^2), M_{25} = \kappa_0 (c_{66} \Omega^2), M_{26} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{31} = \kappa_0 (c_{66} \Omega^2), M_{32} = \kappa_0 (c_{66} \Omega^2), M_{33} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{34} = \kappa_0 (c_{66} \Omega^2), M_{35} = \kappa_0 (c_{66} \Omega^2), M_{36} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{41} = \kappa_0 (c_{66} \Omega^2), M_{42} = \kappa_0 (c_{66} \Omega^2), M_{43} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{44} = \kappa_0 (c_{66} \Omega^2), M_{45} = \kappa_0 (c_{66} \Omega^2), M_{46} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{51} = \kappa_0 (c_{66} \Omega^2), M_{52} = \kappa_0 (c_{66} \Omega^2), M_{53} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{54} = \kappa_0 (c_{66} \Omega^2), M_{55} = \kappa_0 (c_{66} \Omega^2), M_{56} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{61} = \kappa_0 (c_{66} \Omega^2), M_{62} = \kappa_0 (c_{66} \Omega^2), M_{63} = \kappa_0 (c_{66} \Omega^2);
\]
\[
M_{64} = \kappa_0 (c_{66} \Omega^2), M_{65} = \kappa_0 (c_{66} \Omega^2), M_{66} = \kappa_0 (c_{66} \Omega^2);
\]
\[
| \mathbf{N}_{ij} | = \Omega, j, r = 1, 2, \ldots, 6;
\]
\[
N_{1r} = \left[ \left( c_{61} \Omega^2 + c_{65} \Omega^2 \right) + \kappa_0 (c_{66} \Omega^2) \right] \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right);
\]
\[
N_{2r} = \left[ \left( c_{61} \Omega^2 + c_{65} \Omega^2 \right) + \kappa_0 (c_{66} \Omega^2) \right] \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right);
\]
\[
N_{3r} = \left[ \left( c_{61} \Omega^2 + c_{65} \Omega^2 \right) + \kappa_0 (c_{66} \Omega^2) \right] \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right);
\]
\[
N_{4r} = \left[ \left( c_{61} \Omega^2 + c_{65} \Omega^2 \right) + \kappa_0 (c_{66} \Omega^2) \right] \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right);
\]
\[
N_{5r} = \left[ \left( c_{61} \Omega^2 + c_{65} \Omega^2 \right) + \kappa_0 (c_{66} \Omega^2) \right] \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right) \left( \frac{\pi}{2} \Omega^2 \right);
\]
\[ N_{6r} = \left( \tilde{c}_{65} \alpha_{6r} + \tilde{c}_{66} \kappa^{(1)}_{6r} + \tilde{c}_{61} \alpha_{4r} \right) Z_r \sin \left( \frac{\pi Z_r}{2 - b} \right). \]  

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Q. Huang, et al.: IDENTIFICATION OF VIBRATION MODES OF QUARTZ CRYSTAL PLATES WITH PROPORTION OF STRAIN AND KINETIC


Response of Duffing’s Oscillator to Harmonic Base Excitation and Significance of First Order Term

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Responses of systems with nonlinear stiffness subjected to base harmonic excitation are determined. An expression to estimate the amplitude in the fundamental frequency of oscillation is derived from first principles using Lindstedt’s method. It is observed that the amplitude determined using the zero order approximation is in error at low frequencies. Therefore, an expression for the first order approximation of the amplitude of response at the fundamental frequency is derived. Zero order and first order approximation terms together form the response. Characteristics showing the variation of the amplitude with the excitation frequency for various nonlinear spring parameters are presented. The issue at low frequencies is resolved by the incorporation of the first order term. An expression for the phase difference and the expression of the asymptote where the responses converge are also derived.

NOMENCLATURE

- \( k \) : stiffness
- \( m \) : mass
- \( c \) : coefficient of viscous damping
- \( f_{\text{spring}} \) : spring restoring force
- \( x \) : absolute displacement of the system
- \( \dot{x} \) : velocity of the system
- \( \ddot{x} \) : acceleration of the system
- \( \epsilon \) : small parameter, \( \epsilon << 1 \)
- \( \alpha, \beta \) : parameters related to the nonlinear spring
- \( \epsilon \zeta \) : damping factor
- \( \omega \) : parameter related to the natural frequency
- \( \epsilon Y \) : amplitude of the base excitation
- \( \Omega \) : frequency of excitation
- \( t \) : time
- \( T \) : time period of oscillation
- \( \tau \) : modified time variable
- \( \varphi \) : phase angle
- \( \omega_0 \) : linearised natural frequency
- \( x_i \) : \( i \)th order term of the displacement, \( i = 0, 1, 2... \)
- \( \varphi_i \) : \( i \)th order term of the phase angle, \( i = 0, 1, 2... \)
- \( A_i \) : \( i \)th order term of the amplitude of the displacement, \( i = 0, 1, 2... \)

1. INTRODUCTION

Many systems encountered in practice exhibit nonlinear behavior\(^1,2\) due to the change in their stiffness with deformation. This may be because of its geometrical construction or the characteristics of the material itself. Bi-linear and cubic are a couple of simple models that are used for representing such stiffness characteristics. A single degree-of-freedom (SDOF) system with a spring having cubic non-linearity is called a Duffing’s oscillator. Many equipment, machines, buildings etc. are provided with isolators to reduce their vibrations or the vibrations transmitted. Behavior of all such isolators can be represented by this model. Vaiana et. al.,\(^3\) Sudhir Kaul,\(^4\) Losanno et. al.\(^5\) and Madera Sierra et. al.\(^6\) used such models to represent the behavior of isolators, which show that such models are generally used for representing the nonlinear stiffness of isolators. These models find place even in the analysis of non-linear dampers\(^7-9\) and other devices.\(^10\)

Determining the response of a Duffing’s oscillator is a classic example of a nonlinear problem and a large amount of work is carried out on this subject. The text book by Meirovitch\(^11\) gives a good insight into this problem. Considering the nonlinearity to be weak, perturbation techniques are widely used for arriving at the solution. Lindstedt’s method, method of averaging, harmonic balance method, method of multiple scales are the most commonly used perturbation techniques.\(^12,13\)

Using the above mentioned techniques, the free vibration characteristics and the response to harmonic force excitation are studied extensively. These studies have provided expressions for determining the frequency of the free vibration of the Duffing’s oscillator. Expressions were also presented for the amplitude of the response when excited by a harmonic force.\(^11\) The responses show ‘jump phenomenon’ when they are subjected to force excitation. There are several works that discuss on the jump phenomenon.\(^14-18\) Malatkar and Nayfeh\(^16\) obtained the minimum force that needs to be applied to make the system to have jumps in their responses. Brennan et.al.\(^17\) obtained expressions for determining the frequencies at which the jumps in the responses occur. Works are also reported in determining the parameters of the nonlinear system from its response characteristics using methods like nonlinear subspace identification\(^18\) etc. All these works are related to force excitation.
Many of such nonlinear systems exhibit hysteresis. A suitable model that represents this type of hysteresis is Bouc-Wen model. This model was proposed initially by Bouc and later generalized by Wen. In this model the restoring force is related to the displacement through a first order nonlinear differential equation. By assigning suitable values for the parameters, the required hysteresis loop is represented. Vaiana et al. developed several models that provide significant computational advantages while representing the hysteresis behavior.

Though there are several works carried out on the response of nonlinear systems when subjected to force excitation as well as complex models that represent their hysteresis behavior, it is seen that the works reported on the response to base excitation are very few. Hundal determined the response of systems to base excitation with nonlinear damping. Xiaojuan Sun and Jianrun Zhang worked on the response of isolators with nonlinear damping. In these works, the stiffness was considered to be linear. Responses are also obtained for modulated sinusoidal excitation using method of multiple scales, but it was for a force excitation and not for a base excitation. In another study the response to base excitation of an oscillator is determined with a nonlinearity in the secondary spring. The main spring was still linear and the nonlinear spring connects only the damper.

Ravindra and Mallik obtained the response of a nonlinear system when subjected to base excitation. They used harmonic balance method for determining the solution. Zarko Milovanovic et.al. derived an expression for determining the amplitude of the response of a vibration isolator subjected to base excitation. Bahareh Zaghari et.al. determined the response to base excitation along with force excitation. In all the above works the base excitation problem is written in the form of force excitation problems and the relative responses are determined using the expressions for force excitation. The absolute displacements are then determined from the relative displacements. While doing the above, the phase of the absolute displacement is assumed, thus it does not give an analytically correct solution. Other works reported on base excitation adopt numerical techniques to determine the solution and hence no expressions are derived. A few works on response to base excitation, deal with cantilevered beam with tip mass. In those works, numerical techniques are used to determine the solution and no expressions are derived. Also, the nonlinearities present in those investigations are not of the cubic type.

Though there are expressions reported for determining the amplitude of the response of Duffing’s oscillator to base sine excitation, they are derived with certain assumptions on the phase difference which is analytically not exact. In this work, the absolute response is derived from the first principles (not from the relative response) without any such assumption. Here, Lindstedt’s perturbation technique which is mathematically rigorous is chosen to derive the expression for the amplitude. As in the previous studies the nonlinearity is assumed to be small. In previous works, only the zero order approximation was considered. It is shown that the first order approximation terms are also essential and expressions for these terms are derived. Using these expressions, characteristic curves for the base sine excitation are obtained. These results are particularly useful for Isolators used in sensitive equipment / optical elements mounted in spacecraft / aircraft, isolators used in gyro, isolators in building etc.

2. ZERO ORDER APPROXIMATION OF RESPONSE

Lindstedt’s method well established for determining the response to force excitation is used to obtain the response to base excitation.

Let us consider a quasi-harmonic system as shown in Fig. 1 subjected to a base harmonic excitation. It consists of a mass, a spring and a damper.

2.1. Equation of Motion

The spring force is given by:

\[ f_{spring}(x) = kx + c(x + \beta x^3); \]  

where \( x \) is the absolute displacement. The first derivative of \( f_{spring}(x) \) with respect to the displacement gives the stiffness of the spring and in general it is a function of the displacement. If \( \beta \) is +ve, stiffness increases with the displacement. Such springs are called hardening spring. In a softening spring the stiffness reduces with the displacement and \( \beta \) is –ve. If \( \beta = 0 \), the system will be linear and the corresponding stiffness is \( k(1 + \epsilon \alpha) \).

The second-order ordinary differential equation of motion of a linear SDOF system having mass \( m \), damping coefficient \( c \) and stiffness \( k \), when subjected to a base displacement \( y \), is:

\[ m\ddot{x} + c\dot{x} + kx = cy + ky. \]  

When the spring force is as given in Eq. (1), the differential equation of motion becomes:

\[ m\ddot{x} + c\dot{x} + k(1 + \epsilon \alpha) x + \epsilon k\beta x^3 = cy + k(1 + \epsilon \alpha) y + \epsilon k\beta y^3. \]  

The right hand side of the above equation has several terms with one nonlinear term whereas in the case of force excitation there is only one term and it is linear. Dividing Eq. (3) by the mass, we get:

\[ \ddot{x} + \omega^2 x = \epsilon \left[ -\omega^2 (\alpha x + \beta x^3) - 2\zeta \omega \dot{x} \right] + 2\epsilon (\omega \dot{y} + \omega^2 y + \epsilon \omega^2 (\alpha y + \beta y^3)); \]  

![Figure 1. Duffing’s oscillator with base excitation.](image)
where, $\omega^2 = k/m$ and $\epsilon \zeta = c/(2m\omega)$ is the damping factor.

For a harmonic base excitation of $y = \epsilon Y \cos \omega t$, the equation of motion in terms of absolute displacement can be shown to be:

$$\ddot{x} + \omega^2 x = \epsilon \left[ -\omega^2 (\alpha x + \beta x^3) - 2\omega \dot{x} \right] - 2\epsilon \omega \Omega Y \sin (\tau + \varphi) + \omega^2 \epsilon Y \cos (\tau + \varphi) + \epsilon \omega^2 [\epsilon \alpha Y \cos (\tau + \varphi) + \beta \epsilon^3 Y^3 \cos^3 (\tau + \varphi)].$$

Equation (5) represents the differential equation of motion of the Duffing’s oscillator when subjected to base excitation.

### 2.2. Expression for Amplitude of the Absolute Displacement

Our aim is to obtain a periodic solution of Eq. (5). The period of excitation is $T = \frac{2\pi}{\Omega}$. Shifting the time scale $t$ to $\tau$, which is the new time variable so that the period of oscillation is $2\pi$ (and not $\frac{2\pi}{\Omega}$), the equation of motion changes to:

$$\Omega^2 x'' + \omega^2 x = \epsilon \left[ \omega^2 (\alpha x + \beta x^3) - 2\omega \Omega \dot{x} \right] - 2\epsilon \omega \Omega Y \sin (\tau + \varphi) + \omega^2 \epsilon Y \cos (\tau + \varphi) + \epsilon \omega^2 [\epsilon \alpha Y \cos (\tau + \varphi) + \beta \epsilon^3 Y^3 \cos^3 (\tau + \varphi)].$$

where $\varphi$ is the phase angle. While doing so we employed the transformations $\tau = \Omega \tau - \varphi$ and $\frac{dx}{d\tau} = \Omega \frac{dx}{dt}$.

For a solution as given below:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots;$$

$$\varphi = \varphi_0 + \epsilon \varphi_1 + \epsilon^2 \varphi_2 + \ldots;$$

the equation of motion becomes (taking $x = x_0 + \epsilon x_1$):

$$\Omega^2 (x_0 + \epsilon x_1)'' + \omega^2 (x_0 + \epsilon x_1) =$$

$$\epsilon \left[ -\omega^2 (\alpha x_0 + \beta x_0^3) - 2\omega \Omega (x_0 + \epsilon x_1) \right] - 2\epsilon \omega \Omega Y \sin (\tau + \varphi) + \omega^2 \epsilon Y \cos (\tau + \varphi) +$$

$$+ \epsilon \omega^2 [\epsilon \alpha Y \cos (\tau + \varphi) + \beta \epsilon^3 Y^3 \cos^3 (\tau + \varphi)].$$

This can be further written as:

$$\left( \Omega^2 x_0'' + \omega^2 x_0 \right) + \epsilon \left( \Omega^2 x_1'' + \omega^2 x_1 \right) =$$

$$\epsilon \left[ -\omega^2 (\alpha x_0 + \beta x_0^3) - 2\omega \Omega \dot{x}_0 + \omega^2 Y \cos (\tau + \varphi) \right]$$

$$+ \epsilon \omega^2 \left[ (\alpha x_0 + 3\beta x_0^3) - 2\omega \Omega \dot{x}_1 \right] +$$

$$+ 2\omega \Omega Y \sin (\tau + \varphi) + \omega^2 \epsilon Y \cos (\tau + \varphi)$$

$$+ \epsilon \omega^2 [\epsilon \alpha Y \cos (\tau + \varphi) + \beta \epsilon^3 Y^3 \cos^3 (\tau + \varphi)] + \epsilon^2 x_2.$$  

(9)

It can be shown that with $\cos (\epsilon \varphi_1) = 1$ and $\sin (\epsilon \varphi_1) = \epsilon \varphi_1$ for small values of $\epsilon$ (for $\varphi_1 < 0.3$, the error in the trigonometric terms will be less than 5%):

$$\cos (\tau + \varphi) = \cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0$$

$$- \epsilon \varphi_1 \{ \cos \tau \sin \varphi_0 + \sin \tau \cos \varphi_0 \};$$

$$\sin (\tau + \varphi) = \sin \tau \cos \varphi_0 + \cos \tau \sin \varphi_0$$

$$- \epsilon \varphi_1 \{ \sin \tau \sin \varphi_0 - \cos \tau \cos \varphi_0 \}.$$  

(11)

Using the above relations in Eq. (10) and equating the coefficients of terms having the same powers of $\epsilon$, we get:

$$\Omega^2 x_0'' + \omega^2 x_0 = 0;$$

$$\Omega^2 x_1'' + \omega^2 x_1 = -2\epsilon \omega \Omega \dot{x}_1 - \omega^2 (\alpha x_0 + \beta x_0^3)$$

$$+ \omega^2 Y \cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0.$$  

(13)

(14)

The $x_i (\tau), i = 0, 1, 2, \ldots$ are solved sequentially, subject to the periodicity condition $x_i (\tau + 2\pi) = x_i (\tau)$ and the initial conditions $x_i (0) = 0$.

The solution of the first equation is:

$$x_0 = A_0 \cos \left( \frac{\omega}{\Omega} \right) \tau.$$  

(15)

To have a periodic solution, $\omega = \Omega$ and hence the zero order solution becomes:

$$x_0 = A_0 \cos \tau.$$  

(16)

$A_0$ is determined from the second differential equation, i.e. Eq. (14).

$$x_1'' + x_1 = 2\zeta A_0 \sin \tau$$

$$- \left[ \alpha A_0 \cos \tau + \beta (A_0 \cos \tau)^3 \right] + Y \cos (\tau + \varphi_0).$$  

(17)

Recalling that $\cos^3 \tau = \frac{1}{4} (3 \cos \tau + \cos 3\tau)$ we get:

$$x_1'' + x_1 = (2\zeta A_0 - Y \sin \varphi_0) \sin \tau$$

$$+ \left[ -\alpha A_0 - \frac{3}{4} \beta A_0^3 + Y \cos \varphi_0 \right] \cos \tau - \frac{1}{4} \beta A_0^3 \cos 3\tau.$$  

(18)

The solution must satisfy the periodicity condition to prevent the occurrence of secular term.$^{11}$ Therefore, coefficients of $\sin \tau$ and $\cos \tau$ should vanish. This leads to:

$$2\zeta A_0 = Y \sin \varphi_0;$$

$$\alpha A_0 + \frac{3}{4} \beta A_0^3 = Y \cos \varphi_0.$$  

(19)

(20)

To determine $A_0$, multiply Eqs. (19) and (20) by $\epsilon$. Squaring both sides of Eqs. (19) and (20) and then adding them we get:

$$\left( 2\epsilon \zeta A_0 \right)^2 + \left( \epsilon \alpha A_0 + \frac{3}{4} \epsilon \beta A_0^3 \right)^2 = (\epsilon Y)^2.$$  

(21)

By solving Eq. (21) the amplitude of the response can be obtained. The parameter $\epsilon\alpha$ is not an independent parameter and hence needs to be eliminated. This can be done by defining $\omega_0^2 = \Omega^2 (1 + \epsilon\alpha)$ where $\omega_0$ is the natural frequency of the associated linear system, that is with $\beta = 0$. Using the above
definition and re-arranging the terms, the zero order frequency-response function becomes:

$$(2\epsilon\zeta A_0\Omega^2)^2 + \left(\omega_0^2 \left[ A_0 + \frac{3}{4} \beta A_0^3 \right] - \Omega^2 A_0 \right)^2 = (\epsilon Y)^2 \Omega^4. \quad (22)$$

The above equation can be expanded to form a polynomial equation as given below, and then can be solved for $A_0$.

$$\frac{9}{16} (\epsilon\beta)^2 \omega_0^6 A_0^6 + \frac{3}{2} (\epsilon\beta) \omega_0^4 \left( \omega_0^2 - \Omega^2 \right) A_0^4 + \left( \omega_0^2 - \Omega^2 \right)^2 A_0^2 - (\epsilon Y)^2 \Omega^4 = 0. \quad (23)$$

Taking $S = A_0^2$, it becomes a cubic equation as:

$$\frac{9}{16} (\epsilon\beta)^2 \omega_0^4 S^3 + \frac{3}{2} (\epsilon\beta) \omega_0^2 \left( \omega_0^2 - \Omega^2 \right) S^2 + \left( \omega_0^2 - \Omega^2 \right)^2 S - (\epsilon Y)^2 \Omega^4 = 0. \quad (24)$$

Equation (24) gives the zero order frequency-response function for the absolute displacement of a damped nonlinear system for a harmonic base excitation. The zero order solution is given by Eq. (16) and $A_0$ can be determined by solving Eq. (24).

The phase difference, which is the difference with respect to the phase of the excitation, can be shown to be:

$$\tan \varphi_0 = \frac{2\epsilon\zeta}{\alpha + \frac{3}{4} \epsilon\beta A_0^2}. \quad (25)$$

3. RESPONSE FOR VARIOUS PARAMETERS OF THE SYSTEM

The responses of the system for its various parameters are determined through solving Eq. (24) and the results are given in the following figures. The responses shown are the amplitudes of the zero order approximation ($A_0$).

The response characteristics are shown in Figs. 2–6. The damping factor considered is 0.05. In all these figures the amplitudes of the absolute displacement are plotted against the frequency of excitation for a system having a particular value of linearised natural frequency. The characteristics are given for five different linearised natural frequencies ($\omega_0$) 200, 400, 600, 800, 1000 rad/s. For a particular value of linearised natural frequency of the oscillator, the responses are shown for 6 values of amplitudes of base excitation. The amplitudes of excitation are 0.1 mm, 0.5 mm, 1 mm, 1.5 mm, 2 mm and 2.5 mm. All the above curves are presented in one figure. All the curves in the same figure are drawn for a particular value of nonlinear spring parameter $\epsilon\beta$ in N/mm² and it is indicated in the title of the figure. The amplitudes of responses are given in millimeters.

Figures 2–7 provide the above characteristics for different values of nonlinear spring parameter $\epsilon\beta$. Figures 2–4 give the responses respectively for $\epsilon\beta = 0.001$, $\epsilon\beta = 0.005$ and $\epsilon\beta = 0.01$, when the spring is a hardening type. The results of systems with a softening type are shown in Figs. 5 and 6. Fig. 5 is for $\epsilon\beta = 0.005$ and Fig. 6 is for $\epsilon\beta = 0.01$.

The jump phenomenon in the response is seen as in forced excitation. The curves showing the variation of amplitude with the excitation frequency converge to the curve corresponding to $\epsilon Y = 0$. In a linear system, they asymptotically approach $\Omega = \omega_0$, when the damping is absent. In a nonlinear system, that is $\epsilon\beta \neq 0$, they asymptotically approach a parabola, the equation of which can be derived as:

$$A_0^2 = \frac{4}{3\epsilon\beta} \left\{ \left( \frac{\Omega}{\omega_0} \right)^2 - 1 \right\}. \quad (26)$$

The asymptote intersects the $\Omega$-axis at $\Omega = \omega_0$.

It is important to note that the responses estimated using the expression derived in this work show very low values of responses when the frequency of excitation is low. The amplitude of response should have been approximately equal to the amplitude of excitation. The amplitude of response estimated is the zero order approximation. This issue is resolved by incorporating the first order approximation terms. This is presented subsequently.

4. FIRST ORDER APPROXIMATION OF RESPONSE

The issue seen in the results at lower frequencies is investigated here.

As discussed previously the displacement response can be expanded as per Eq. (7) and the phase of the response can be expanded as per Eq. (8). The zero order approximation to the solution is $x_0 = A_0 \cos \tau$. Solving the differential equation

$$x''_1 + x_1 = -\frac{1}{4} \beta A_0^3 \cos 3\tau; \quad (27)$$

the first order approximation to the solution can be obtained as:

$$x_1 = A_1 \cos \tau + \frac{1}{32} \beta A_0^3 \cos 3\tau. \quad (28)$$

Combining both, the solution becomes:

$$x(\tau) = (A_0 + \epsilon A_1) \cos \tau + \frac{1}{32} \epsilon\beta A_0^3 \cos 3\tau + \ldots. \quad (29)$$

The value of $A_0$ can be determined using Eq. (24). The response estimated using $A_0$ alone, that is without $A_1$, is found to be not correct in the low frequencies. Therefore, $A_1$ is determined and investigated whether incorporation of it improves the results.

4.1. Expression for $A_1$

The differential equation of motion is:

$$\Omega^2 x'' + \omega^2 x = \epsilon \left[ -\omega^2 (\alpha x + \beta x^3) - 2\zeta \Omega x \right] - 2\epsilon \omega \Omega \sin (\tau + \varphi) + \omega^2 \epsilon Y \cos (\tau + \varphi) + \epsilon \omega^2 (\alpha Y \cos (\tau + \varphi) + \beta \epsilon^3 Y^3 \cos^3 (\tau + \varphi)). \quad (30)$$
Figure 2. Frequency response of hard spring ($\epsilon\beta = 0.001$) for various values of excitations.

Figure 3. Frequency response of hard spring ($\epsilon\beta = 0.005$) for various values of excitations.
Figure 4. Frequency response of hard spring ($\epsilon \beta = 0.01$) for various values of excitations.

Figure 5. Frequency response of soft spring ($\epsilon \beta = 0.005$) for various values of excitations.
Figure 6. Frequency response of soft spring ($\epsilon \beta = 0.01$) for various values of excitations.

Figure 7. Frequency response ($\epsilon A_1$) of hard spring ($\epsilon \beta = 0.001$).
Using the solution for \( x \) in terms of \( x_1, x_2 \) etc., the differential equation becomes:

\[
\Omega^2 \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 \right)'' + \omega^2 \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 \right) = \\
\epsilon \left[ -\omega^2 \left( \alpha [x_0 + \epsilon x_1 + \epsilon^2 x_2] + \beta [x_0 + \epsilon x_1 + \epsilon^2 x_2] \right) \right] \\
-2\zeta\omega\Omega \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 \right)' + \omega^2 Y \cos (\tau + \varphi) \\
+ \epsilon^2 \left[ -2\zeta\omega\Omega' \sin (\tau + \varphi) + \omega^2 Y' \cos (\tau + \varphi) \right] \\
+ \epsilon^4\omega^2 Y^2 \cos^3 (\tau + \varphi). \quad (31)
\]

For small values of \( \epsilon \), the values of higher powers of \( \epsilon \) are negligible and hence:

\[
[x_0 + \epsilon x_1 + \epsilon^2 x_2]^3 = (x_0^3 + 3x_0^2x_1). \quad (32)
\]

Using Eq. (32) the differential equation becomes:

\[
\Omega^2 \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 \right)'' + \omega^2 \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 \right) = \\
\epsilon \left[ -\omega^2 \left( \alpha [x_0 + \epsilon x_1 + \epsilon^2 x_2] + \beta [x_0 + \epsilon x_1 + \epsilon^2 x_2] \right) \right] \\
-2\zeta\omega\Omega \left( x_0 + \epsilon x_1 + \epsilon^2 x_2 \right)' + \omega^2 Y \cos (\tau + \varphi) \\
+ \epsilon^2 \left[ -2\zeta\omega\Omega' \sin (\tau + \varphi) + \omega^2 Y' \cos (\tau + \varphi) \right] \\
+ \epsilon^4\omega^2 Y^2 \cos^3 (\tau + \varphi). \quad (33)
\]

Expanding the differentials, we get:

\[
\left( \Omega^2 x_0'' + \omega^2 x_0 \right) + \epsilon \left( \Omega^2 x_1'' + \omega^2 x_1 \right) \\
+ \epsilon^2 \left( \Omega^2 x_2'' + \omega^2 x_2 \right) = \\
\epsilon \left[ -\omega^2 \left( \alpha x_0 + \beta x_0^2 \right) - 2\zeta\omega\Omega x_0 \right] + \omega^2 Y \cos (\tau + \varphi) \\
+ \epsilon^2 \left[ -\omega^2 \left( \alpha x_1 + 3\beta x_1^2 + \right) - 2\zeta\omega\Omega x_1 \right] \\
+ \omega^2 Y \sin (\tau + \varphi). \quad (34)
\]

Expanding \( \varphi \) as given by Eq. (8) and using the trigonometric expansions we get:

\[
\cos (\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) = \\
\cos (\tau + \varphi_0) - \epsilon\varphi_1 \cos (\tau + \varphi_0) - \epsilon^2\varphi_2 \cos (\tau + \varphi_0) \\
- \epsilon\varphi_1 \ast \epsilon^2\varphi_2 \cos (\tau + \varphi_0); \quad (35)
\]

\[
\sin (\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) = \\
\sin (\tau + \varphi_0) + \epsilon\varphi_1 \sin (\tau + \varphi_0) + \epsilon^2\varphi_2 \sin (\tau + \varphi_0) \\
- \epsilon\varphi_1 \ast \epsilon^2\varphi_2 \sin (\tau + \varphi_0). \quad (36)
\]

by approximating for the sin and cos of small angles. As value of \( \epsilon \) is very small, the values of \( \epsilon^2 \) and \( \epsilon^3 \) can be neglected. Therefore, one can approximate the cosine and sine expansions in Eq. (35) and (36) as:

\[
\cos (\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) \approx \cos (\tau + \varphi_0) - \epsilon\varphi_1 \sin (\tau + \varphi_0); \quad (37)
\]

\[
\sin (\tau + \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2) \approx \sin (\tau + \varphi_0) + \epsilon\varphi_1 \cos (\tau + \varphi_0). \quad (38)
\]

Applying Eq. (37) and (38) the differential equation becomes:

\[
\left( \Omega^2 x_0'' + \omega^2 x_0 \right) + \epsilon \left( \Omega^2 x_1'' + \omega^2 x_1 \right) \\
+ \epsilon^2 \left( \Omega^2 x_2'' + \omega^2 x_2 \right) = \\
\epsilon \left[ -\omega^2 \left( \alpha x_0 + \beta x_0^2 \right) - 2\zeta\omega\Omega x_0 \right] \\
+ \omega^2 Y \cos (\tau + \varphi_0) - \epsilon\varphi_1 \sin (\tau + \varphi_0) - \epsilon^2\varphi_2 \sin (\tau + \varphi_0) \\
+ \epsilon^2 \left[ -\omega^2 \left( \alpha x_1 + 3\beta x_1^2 + \right) - 2\zeta\omega\Omega x_1 \right] \\
+ \omega^2 Y \varphi_1 \cos (\tau + \varphi_0) - \epsilon\varphi_1 \sin (\tau + \varphi_0) - \epsilon^2\varphi_2 \sin (\tau + \varphi_0). \quad (39)
\]

On neglecting higher powers of \( \epsilon \):

\[
\left( \Omega^2 x_0'' + \omega^2 x_0 \right) + \epsilon \left( \Omega^2 x_1'' + \omega^2 x_1 \right) \\
+ \epsilon^2 \left( \Omega^2 x_2'' + \omega^2 x_2 \right) = \\
\epsilon \left[ -\omega^2 \left( \alpha x_0 + \beta x_0^2 \right) - 2\zeta\omega\Omega x_0 \right] - \omega^2 Y \varphi_1 \sin (\tau + \varphi_0) - \epsilon\varphi_1 \sin (\tau + \varphi_0) - \epsilon^2\varphi_2 \sin (\tau + \varphi_0). \quad (40)
\]

Equating the coefficients of \( \epsilon^0 \), the zero order approximation of the differential equation becomes:

\[
\Omega^2 x_0'' + \omega^2 x_0 = 0. \quad (41)
\]

Equating the coefficients of \( \epsilon^1 \) we get:

\[
\Omega^2 x_1'' + \omega^2 x_1 = \\
-\omega^2 \left( \alpha x_0 + \beta x_0^2 \right) - 2\zeta\omega\Omega x_0 \right] - \omega^2 Y \varphi_1 \sin (\tau + \varphi_0) - \epsilon\varphi_1 \sin (\tau + \varphi_0) \quad (42)
\]

Equating the coefficients of \( \epsilon^2 \), the differential equation becomes:

\[
\Omega^2 x_2'' + \omega^2 x_2 = -\omega^2 \left( \alpha x_1 + 3\beta x_1^2 + \right) - 2\zeta\omega\Omega x_1 \right] \\
- \omega^2 Y \varphi_1 \sin (\tau + \varphi_0) - \omega^2 Y \sin (\tau + \varphi_0) + \omega^2 \varphi_1 \cos (\tau + \varphi_0). \quad (43)
\]

The zero order approximation to the solution is as given by Eq. (16) and the first order approximation is as given by Eq. (28). Second order approximation can be obtained from Eq. (43), which can be written as:

\[
\Omega^2 x_2'' + \omega^2 x_2 = -\omega^2 \left( \alpha x_1 + 3\beta x_1^2 + \right) - 2\zeta\omega\Omega x_1 \right] \\
- \{\omega^2 Y \varphi_1 + 2\zeta\omega\Omega Y \} \cos (\tau + \varphi_0) \sin (\tau + \varphi_0) \sin (\tau + \varphi_0) \sin (\tau + \varphi_0). \quad (44)
\]

Using the solutions for \( x_0 \) and \( x_1 \) and dividing Eq. (44) by
\[ x'' + x_0 = -\left( \alpha A_1 \cos \tau + \frac{1}{32} \beta A_0^2 \cos 3\tau \right) \]

Using the trigonometric relations \( \cos 3\tau \cos^2 \tau = \frac{1}{4}(\cos \tau + 2\cos 3\tau + \cos 5\tau) \) and \( \cos^3 \tau = \frac{1}{4}(3\cos \tau + 3\cos 3\tau) \) the differential equation changes to:

\[
x'' + x_0 = -\left( \alpha A_1 \cos \tau + \frac{1}{32} \alpha \beta A_0^2 \cos 3\tau + \right. \]

\[
+ \frac{3}{32} \beta^2 A_0^3 \cos^3 \tau - (-2\zeta A_1 \sin \tau - \frac{3}{16} \zeta A_0^3 \sin 3\tau) \]

\[
- \{Y \varphi_1 + 2\varphi_0\} (\cos \varphi_0 + \cos \varphi_0 \sin \tau) + \alpha Y (\cos \tau \cos \varphi_0 - \sin \tau \sin \varphi_0). \] (46)

Collecting all \( \cos \tau, \cos 3\tau \) and neglecting \( \cos 5\tau \) terms, we get:

\[
x'' + x_0 = \cos \tau \left\{ - \alpha A_1 - \frac{9}{128} \beta^2 A_0^3 \right. \]

\[
- \{Y \varphi_1 + 2\varphi_0\} \cos \varphi_0 + \alpha Y \cos \varphi_0 \}

\[
+ \sin \{2\zeta A_1 - (Y \varphi_1 + 2\varphi_0) \cos \varphi_0 - \alpha Y \cos \varphi_0\}

\[
+ \cos 3\tau \left( \frac{1}{32} \alpha \beta A_0^2 + \frac{3}{4} \beta \beta A_0^3 \right)

\[
- \alpha A_1 - \frac{9}{128} \beta^2 A_0^3 \right) \]

\[
- \{Y \varphi_1 + 2\varphi_0\} \sin \varphi_0 + \alpha Y \cos \varphi_0 = 0; \] (49)

The aim now is to solve for \( A_1 \). Re-arranging the terms and removing the dependence on the phase angle:

\[
(Y \varphi_1 + 2\varphi_0)^2 + (\alpha Y)^2 = \]

\[
(\alpha A_1 + \frac{9}{4} \beta A_0^2 A_1 + \frac{3}{128} \beta^2 A_0^3)^2 + (2\zeta A_1)^2. \] (51)

Introducing \( \epsilon \) and considering \( \epsilon \varphi_1 \ll 2\epsilon \zeta \), we get:

\[
(2\epsilon \zeta A_1)^2 + (\epsilon \alpha Y)^2 = \]

\[
(\epsilon \alpha \epsilon A_1 + \frac{9}{4} \epsilon \beta A_0^2 \epsilon A_1 + \frac{3}{128} (\epsilon \beta)^2 A_0^3)^2 + (2\epsilon \zeta A_1)^2. \] (52)

As \( \frac{\omega_0^2 - \Omega^2}{\Omega^2} = \epsilon \alpha \), Eq. (52) gets modified as:

\[
(2\epsilon \zeta A_1)^2 + \left( \frac{\omega_0^2 - \Omega^2}{\Omega^2} \right) Y^2 = \]

\[
\left( \frac{\omega_0^2 - \Omega^2}{\Omega^2} \right) \epsilon A_1 + \frac{9}{4} \epsilon \beta A_0^2 \epsilon A_1 + \frac{3}{128} (\epsilon \beta)^2 A_0^3 \] (53)

Expanding the square of the expression and collecting the terms involving \( \epsilon A_1 \) and \( (\epsilon A_1)^2 \) we get:

\[
(2\epsilon \zeta A_1)^2 + \left( \frac{\omega_0^2 - \Omega^2}{\Omega^2} \right) Y^2 = \]

\[
\left( \frac{\omega_0^2 - \Omega^2}{\Omega^2} \right)^2 + \frac{9}{4} \epsilon \beta A_0^2 \] (54)

Simplifying the coefficient of \( (\epsilon A_1)^2 \), and re-arranging the terms:

\[
\left( \frac{\omega_0^2 - \Omega^2}{\Omega^2} \right)^2 + \frac{9}{4} \epsilon \beta A_0^2 \] (55)

The above relation is a quadratic equation in \( \epsilon A_1 \) as given be-
4.2. Results of $\varepsilon A_1$

The values of $\varepsilon A_1$ for various values of nonlinear spring parameter, damping and excitation etc., are shown in Figs. 7–10 for hard and soft springs with linearized natural frequency of 500 rad/s. The results show that the amplitude of response at low frequency is approximately the same as the amplitude of excitation. The problem in the low frequency can be resolved by incorporating the first order term.

5. RESPONSE WITH ZERO AND FIRST ORDER APPROXIMATIONS

It was discussed previously that the responses estimated using the expression derived in this work show very low value of response when the frequency of excitation is low. The amplitude of response should have been approximately equal to the amplitude of excitation. The amplitude of response estimated was with the zero order approximation and the first order approximation term was not included.

Here the amplitude of the response in the fundamental frequency of oscillation considering both zero order as well as the first order approximations ($A_0 + \varepsilon A_1$), is plotted at various excitation frequencies and various parameters of the system. They are shown in Figs. 11–18 and are presented in the same way as before.

Figure 11 gives the responses for the nonlinear spring parameter $\varepsilon \beta = 0.001$, Fig. 12 gives the results for $\varepsilon \beta = 0.005$, Fig. 13 gives the results for $\varepsilon \beta = 0.01$ and Fig. 14 shows the results for $\varepsilon \beta = 0.05$ when the spring is hardening type. Figures 15–18 show similar result for a system with softening spring.

It can be seen that the problems encountered at low frequencies are resolved by incorporating the first order approximation term and the responses are on expected lines. The response near resonance is contributed by zero order term and the response at low frequencies are from the first order term. The values of $A_0$ can be determined by solving Eq. (24) and $\varepsilon A_1$ can be determined by solving Eq. (56).

Following points can be noted from all the above response curves:

- As the frequency of excitation is increased or decreased, a jump is seen in the magnitude of the response. As the excitation frequency is being increased the jump occurs at a higher frequency and when the excitation frequency is being decreased the jump occurs at a lower frequency.

  - In the case of a hard spring with the increase in the magnitude of excitation ($\varepsilon Y$) the frequency at which the response is the maximum increases (for constant $\varepsilon \beta$ and $\varepsilon \zeta$). For the soft spring, the characteristics are reversed.

  - In the case of a hard spring, the frequency at which the response is the maximum increases considerably with the increase in the non-linearity ($\varepsilon \beta$).

  - For small values of non-linearities, the amplification remains approximately the same for different values of excitations. Consequently, the maximum response will be proportional to the excitation and will be independent of nonlinear spring parameters. The behavior of a soft spring is identical to that of a hard spring with reverse characteristics.

  - In the absence of damping, the response asymptotically converges to a parabola. This parabola intersects the $\Omega$-axis at $\Omega = \Omega_0$.

  - The response at the fundamental frequency leads the excitation by 90 degrees. This is considering the zero order approximation. The phase difference is zero if damping is absent, as in linear systems.

6. CONCLUSIONS

Responses of a system with nonlinear stiffness subjected to base harmonic excitation are determined using Lindstedt’s method. An expression to estimate the amplitude of the fundamental frequency of oscillation is derived. It exhibits the jump phenomena as shown for forced excitations. The amplitudes of responses at various values of excitation frequencies are obtained for various nonlinear parameters of the oscillator. The existing expression for the amplitude of response for base excitation is derived from relative displacement making certain assumptions on phase. But the present expression is derived from first principles without these assumptions. Additionally, the expression for the phase difference and the expression for the asymptote where the responses converge are also derived.

The responses estimated using the expression derived in this work show very low value of response when the frequency of excitation is low. At low frequencies, amplitude of response should have been approximately equal to the amplitude of excitation. The amplitude of response estimated is the zero order approximation and the first order approximation was not included. Therefore, an expression for the first order approximation is derived. Its characteristics for various parameters of the system are obtained. By incorporating the first order term, the problem in the low frequency is resolved. Variation of the amplitude, incorporating both zero order term as well as the first order term, with the excitation frequency are presented for various nonlinear spring parameters of the oscillator.
Figure 8. Frequency response ($\epsilon A_1$) of hard spring ($\epsilon \beta = 0.01$).

Figure 9. Frequency response ($\epsilon A_1$) of soft spring ($\epsilon \beta = 0.001$).
Figure 10. Frequency response ($\epsilon A_1$) of soft spring ($\epsilon \beta = 0.01$).

Figure 11. Frequency response ($A_0 + \epsilon A_1$) of hard spring ($\epsilon \beta = 0.001$).
Figure 12. Frequency response \( (A_0 + \epsilon A_1) \) of hard spring \( (\epsilon \beta = 0.005) \).

Figure 13. Frequency response \( (A_0 + \epsilon A_1) \) of hard spring \( (\epsilon \beta = 0.01) \).
Figure 14. Frequency response ($A_0 + \epsilon A_1$) of hard spring ($\epsilon \beta = 0.05$).

Figure 15. Frequency response ($A_0 + \epsilon A_1$) of soft spring ($\epsilon \beta = 0.001$).
Figure 16. Frequency response \((A_0 + \epsilon A_1)\) of soft spring \((\epsilon \beta = 0.005)\).

Figure 17. Frequency response \((A_0 + \epsilon A_1)\) of soft spring \((\epsilon \beta = 0.01)\).
REFERENCES


Seismic Optimization of Concrete Gravity Dams Using a Rubber Damper

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One of the major factors in an economic project of new concrete dams and safety valuation of available dams in seismic areas is the control and dissipation of the induced hydrodynamic pressure induced by dam and reservoir interaction. As one of the main control functions, dissipating the induced hydrodynamic pressure on the upstream face of the dam is considered in the evaluations. In this paper, the effects of a rubber damper as an isolation layer on the dam’s seismic control have been investigated. For optimization of the rubber damper thickness and height, the Monte Carlo probabilistic analysis is used. The ANSYS program on the basis of finite element technique is applied for modeling and analysis. The Pine Flat dam in California, due to components of El Centro, San Fernando and North Ridge earthquake is modeled as a case study to evaluate the effect of upstream isolation layer on seismic control and optimization. The effect of the thickness and height of the rubber damper on reducing the responses is investigated and the optimum thickness and height are selected using sensitivity analysis for safe and economic design. The obtained results show the capability of the rubber damper in the seismic and hydrodynamic control of the sample model.

1. INTRODUCTION

The analyzing of concrete dams due to the presence of interaction effects is a very complex problem. Besides the static pressure of water, the dam is subject to dynamic forces of the reservoir when earthquake ground motion affects the system. In an earthquake, the dam body connected to the ground starts to fluctuate, while the water behind the dam is not directly affected by the seismic motion of the ground because of the low shear resistance of the water. So, the hydrodynamic pressure is only created by dam vibrations in the reservoir which spreads away to the reservoir upstream. Because of the interaction, the interface between the dam and the reservoir is a main boundary where the hydrodynamic forces are applied on the structure of the dam. These forces cause a considerable contribution to evaluate the seismic performance and design of the dam.1–3 During an earthquake, there is a significant relationship between the hydrodynamic pressure and the seismic response of a concrete gravity dam. The resulting stresses may cause a to crack to occur with propagation in the dam over relatively severe seismic phenomena. The direct approach to reduce the response of a concrete dam is to increase its structural damping to a desired high value. However, due to the nature of the dam structure and the concrete material properties, no significant action can be done for the structural damping aspect of response reduction.4 Bayraktar et al. investigated the effect of reservoir length on seismic performance of gravity dams to near-and far-fault ground motions using the finite element method.5 They considered the linear and nonlinear behavior for a concrete body. The obtained results illustrated that the length of reservoir affects the seismic responses, considerably. Also, the induced stress on the dam body is more due to near-fault ground motions in comparison with far-fault ground motions. Wang et al. investigated the effect of near fault and far-fault ground motion on seismic performance of the Koyana gravity dam using the concrete damaged plasticity model for nonlinear analysis. The obtained results showed the importance of the near-fault ground excitations on seismic performance of the model.6

Recently, energy dissipation was used in the structures with application of seismic isolation and protection systems.7 These systems can reduce the seismic responses imposed on the structure during an earthquake. The performance of these systems presents a new design philosophy that focuses on increasing the capacity of energy dissipation in the structures. In other words, the earthquake energy, instead of absorbing and causing failures in structural elements, is absorbed in this system. As one of the modern technologies, the seismic isolation and control systems have been more used in the structural engineering such as structures, bridges and dams, the practical effects of which are proven in several strong earthquakes. It has become a new improved method for earthquake engineering.

Due to the nature of the dam structure and the concrete material properties, there is little that can be done on the structural damping aspect of response reduction. The idea of reducing the hydrodynamic loading on the dam appears to be a promising approach for seismic response reduction of concrete dams, though it is still in the early stages in the field of safety of large dams. The separation interface is placed on the upstream face of dams. One of the first ideas for reducing hydrodynamic pressure on the dam by an air curtain at the upstream side of the dam has been examined by Lombardo et al., Sheinin, and Savinov et al.7–9

As’kov et al.10 and Gellis et al.11 examined the idea of hydrodynamic isolation by laboratory model tests as well as measurements of the dam prototypes equipped with an air curtain.

In recent researches, Zhang et al. using the finite element model, studied air cushion impact to reduce the Jinping dam seismic responses with 305 m height.10 They used the Eulerian-Lagrangian formulation for modeling and analysis. Their obtained results illustrated that applying an air cushion in
the dam upstream, reduced the hydrodynamic pressure applied to dam heel by 50 percent and also reduced principle stress by approximately 30 percent. Also, Mirzabozorg et al. investigated the effect of air curtains on reducing the hydrodynamic pressure on the Amirkabir arch dam.\textsuperscript{11} They showed that air cushion separators reduce interaction effects between the dam and the reservoir in large amount. Chen et al. studied the isolation and anti-cracking effects of the air-cushion on seismic behavior of a high arch concrete dam.\textsuperscript{12} The obtained results show that the air-cushion decreases the cracking range of the dam body effectively and the isolation effects of the air-cushion are considerable.

The restricted parameters of the use of the seismic air-cushion to dams for the most part are several dependent structures and tools placed on the upstream face of dams and the remainder is reservoir sedimentation. These parameters will limit the range corresponding to the installation of the seismic air cushion. Therefore, to reduce the effect of the dam and reservoir interaction on seismic responses the using of hydrodynamic dampers on the upper face of the dam is a better option. The use of hydrodynamic dampers, in addition to reducing the hydrodynamic pressure applied on the dam body, is economically less costly than other designs.\textsuperscript{4} The effects of soft material that reduce the seismic response of the dam were investigated by Hall et al.\textsuperscript{15,16} They reported promising results in the reductions of the hydrodynamic force exerted on the dam due to the effect of the isolation layer. Hatami studied the reduction of hydrodynamic pressure using an isolation layer as an absorbing boundary at the contact site of the dam and reservoir using analytical solutions in the frequency domain.\textsuperscript{4} The researches show that there are few problems associated with the use of the isolation layer considering appurtenant structures and performance restrictions.

So, in this paper the possibility of reducing induced hydrodynamic pressure on the dam body by means of the rubber damper as an isolation layer under seismic loading is investigated in detail. To obtain the optimum thickness and height of the isolation layer, the Monte Carlo simulation with Latin Hypercube Sampling (LHS) is used. The thickness and height of the rubber damper are assumed as random input variables. During the earthquake, the highest amount of horizontal movement of the dam crest, maximum hydrodynamic pressure, 1st principle tensile stress of the heel and 3rd principle compressive stress of the dam toe are considered as critical responses in the probabilistic model analysis.

2. RUBBER DAMPER ISOLATOR

The interface of the dam with impounded water is an important boundary where the static and hydrodynamic forces are applied to the dam structure. These forces provide a significant contribution to the seismic response analysis and design of the dam. The maximum hydrodynamic pressure on the dam when subjected to a moderately strong earthquake ground motion may reach the magnitude of the hydrostatic pressure. The isolation layer can have a reducing effect on responses of the dam model because of damping the induced hydrodynamic pressure on the upstream face of the dam due to an earthquake.

The good performance of a rubber damper is indicative of the significant effect of the water – structures’ interaction on seismic responses against the dam. Therefore, in old or damaged structures, where the water-structure interaction is not considered in their design, the use of a rubber damper is very convenient.

The performance of the rubber damper as an isolation layer is not complex in terms of behavior and modeling. The governing equation of the rubber damper behavior is similar to the solid-part equation of the dam-reservoir-foundation system. However, due to the low modulus of elasticity for a rubber damper, stiffness is low and flexibility is high. High flexibility ensures that when the hydrodynamic pressure waves collide with the isolation layer, most of the pressure is absorbed partially and depreciated inside the rubber damper. As a result, little amount of hydrodynamic pressure enters the dam body or reflected to the reservoir.\textsuperscript{3} This mechanism will result in the system withstanding less hydrodynamic force because of dam-reservoir interaction during an earthquake.

3. MONTE CARLO PROBABILISTIC ANALYSIS

Static and dynamic analysis of structures is usually done by assuming a certain amount of parameters will influence the structural seismic responses. Although, assuming the certainty of the model leads to faster and simpler analysis, structural analysis should be conducted in an uncertainty analysis despite the uncertainty in loading, geometric and material properties. Many researchers have been focused on the study of structures by considering the uncertainty characteristics of materials, geometry and so on. Recently, the probabilistic simulation methods are used in different sciences. Understanding the strengths and weaknesses of this efficient tool will create new possibilities for researchers. Among the various methods of probabilistic simulation, the Monte Carlo method is a more suitable tool for the research than other methods. This method has been used for seismic analysis of concrete dams in the recent years.

Since the design parameters of the dams cannot be calculated accurately and there is uncertainty in the calculations, calculating the stability of the dam will be also uncertain. The LHS simulation is one of the most widely used and accurate methods of structural reliability analysis. In this method, different design parameters, which cannot be calculated accurately, can be considered as a probability distribution function. Fairbairn et al. studied the probability of crack expansion in a concrete gravity dam by using the LHS simulation.\textsuperscript{17} In their study, the results obtained from the LHS simulation were compared to the results of testing a laboratory model indicating a good match between the simulation results and the laboratory results. Kostov et al. evaluated the reliability of a dam using a finite element software called NISA which was capable of simulating the LHS method.\textsuperscript{18} The results of this simulation are presented as cumulative distribution graphs. Yannaz and Beşer performed the reliability analysis of the Pursuk dam in Turkey through the LHS simulation.\textsuperscript{19} In their study, they defined seven parameters as variable and provided the results in a table indicating the probability of failure based on dam safety factors. Rohaninejad and Zarghami presented a new method which was a new combination of Monte Carlo simulation and finite element method.\textsuperscript{20} In their study, they evaluated the dam behavior and showed the simulation results as a cumulative distribution function curve. Lupoi and Callari proposed the probabilistic procedure applying Monte Carlo simulation.
which is able to account for uncertainties in material properties and an external condition such as ground motion and reservoir heights with computation of fragility curves. Minor differences were observed in obtained results considering uncertainties in material behavior assuming three levels in the reservoir. An applied probabilistic model in their research was able to estimate the corresponding reduction in the probability of failure. Altarejos et al. studied the reliability of a concrete gravity dam against sliding using Monte Carlo simulation. They considered ten variable parameters in their study and showed the results as a failure probability curve against sliding for each variable.

Feng et al. attempted to analyze the reliability of hydraulic structures by combining MATLAB with Monte Carlo method for the reliability of large hydraulic structures. The results of their study revealed that the methods based on MATLAB-Monte Carlo model are reliable and fast in the reliability of hydraulic structures and have a good function. Mirzabozorg et al. used the Monte Carlo method to create three-dimensional non-uniform ground motions for comparing the effect of three-dimensional uniform and non-uniform ground motions on the response of the Dez arch dam. The results of their study indicated that the responses obtained from considering three-dimensional non-uniform ground motions are different from uniform ground motions and can enhance the structural responses of the system. Alembagheri and Seyedkazemi used Monte Carlo probabilistic analysis and the LHS method to show that the modulus of elasticity and tensile strength of concrete play a more significant role than its final strain against the earthquake. Pasbani Khiavi studied the sensitivity of seismic responses of a concrete gravity dam to the reservoir bottom absorption using Monte Carlo probability analysis and the LHS method. The results of this study indicated the effect of a reservoir bottom on the seismic performance of the dam. Chiti et al. used the subset simulation with Markov Chain and the LHS as an advanced reliability analysis tool to evaluate the probability of dam failure with the lowest number of samples.

Also, Pasbani Khiavi simulated the influence of the concrete stiffness on the seismic performance of concrete gravity dams. The Monte Carlo simulation was used in the analysis. According to the results, the optimized value of the concrete Young modulus can be achieved to access the confident response of the structure with considering economic aspects. Sevieri et al. presented a new hybrid-predictive model using the probabilistic method based on the Bayesian framework for parametric evaluation of finite elements models of concrete gravity dams. The computational effort is reduced by using the proposed model of the dam displacements. The proposed probabilistic model consists of a set of functions that correct the model bias with calibration of the unknown parameters and is able to control the structural performance after an earthquake.

Currently, the reliability analysis methods are mainly used based on the probability theory in which the random parameters are assumed to follow a certain specified distribution. The reliability indexes are varied according to different distribution forms and a certain selected distribution form cannot be applied in all conditions. Moreover, the probabilistic reliability analysis model is highly sensitive to the values of parameters.

The theory of probabilities is a mathematical framework for quantifying the uncertainties in decision making. Almost all of the parameters needed to design a structure, such as mass, damping, material properties, boundary conditions and ground motion are uncertain. These uncertainties should be identified to design a safe structure. The concept of a safe structure refers to the structures that continue to function without damage for many years and the builder is responsible for constructing the structures in such a way that failure does not occur in them.

3.1. Basic Concept

There are several methods for solving the problems related to structural reliability. Simulation methods are a possible method of solving such problems. The concept of simulation is quite clear while the process can be very difficult. The Monte Carlo method is a special technique which can be used for generating some numerical results without any physical test. In this method, the results obtained from previous tests can be used for creating probabilistic distributions from important parameters in problems. Then, this distribution of information can be used for generating some samples of data. The Monte Carlo method along with LHS is often used in three cases:

- In solving complicated problems which are solved with difficulty or are impossible to be solved. For example, it can be used in solving probabilistic problems including complicated nonlinear finite element models with high computational effort;
- it can be used in solving the problems which can be solved approximately by considering simplifying assumptions. In fact, such problems can be studied without performing such simplifications through the Monte Carlo simulations and the results are closer to the real results; and,
- this method can be used for evaluating the results of other solving methods.

3.2. Latin Hypercube Sampling Method

The Monte Carlo sampling and simulation method is a very useful and powerful method. However, sometimes the analysis problem is very complicated and may take a long time to complete the analysis. As a result, the required time for hundreds or thousands of simulations may be impossible. LHS is a technique for reducing the number of required simulations to obtain acceptable results. In this method, a range of possible values of random input variables is partitioned inside some layers and a value of each layer is selected randomly as the representative value. These representative values which represent each random variable are combined together in such a way that each representative value is considered only once in the process of simulation. In this method, all possible values of random variables are presented in the simulation. Consider the limit state function $Y$ with the following random variable $K$:

$$Y = f(X_1 \cdots X_k).$$

The basic steps of the LHS technique are as follows:

- Each $X_i$ is partitioned at certain intervals ($N$ intervals). This partitioning should be in such a way that the probability of the occurrence of any value of $X_i$ at these intervals is equal to $1/N$. 

For each variable $X_i$, each of its $N$ interval is selected randomly as a representative value. In practical applications, if the number of intervals is too high or large intervals are selected, the central point of each interval will be selected instead of random sampling.

After the above steps, the representative value $N$ is obtained for each random variable $K$. In general, there may be $N^N$ possible combinations of these values.

The objective of the Latin hypercube sampling method is to select $N$ combinations, so that each representative value appears only once in $N$ combinations.

In order to obtain the first combination, one of the representative values for each random input variables $K$ is selected randomly. For obtaining the second combination, one of the $N−1$ remaining representative value of each random variable is randomly selected. In order to obtain the third combination, one of the $N−2$ remaining representative values of each random variable is selected randomly. This selection process continues until $N$ combinations are composed of the values of input variables.

Eq. (1) is evaluated for each $N$ combination of the above-mentioned input variables. Thus, the $N$ functions of $Y_i$, ($i = 1, 2, \cdots, N$) will be obtained.

This process creates simulation data; and using data should be determined for estimating statistical parameters for $y$. The most common formulas include:

$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i;$$

$$Y = \frac{1}{N}\sum_{i=1}^{N} (y_i)^m;$$

$$F_p(y) = \frac{\text{number of times } y_i \leq y}{N}. $$

The basis for all LHS simulation steps is generating random numbers being uniformly distributed between 0 and 1. When there is an understanding that $U$ is related to the uniform distribution of random number $U$ between 0 and 1, it is possible to produce $x$ related to the uniform distribution of random number $X$ between both values $a$ and $b$ ($a \leq x \leq b$) using the following equation:

$x = a + (b - a) \cdot U.$

In addition, the generated sample values are derived from the uniform random distribution between two integers $a$ and $b$ were considered through the following equation:

$$i = a + \text{TRUNC}([b - a + 1] \cdot U);$$

where TRUNC refers to the function which eliminates the decimal part of the variable.

In the present study, the correlated variables are analyzed and it is explained how the simulation process creates such a correlation. A conversion method for simulating the correlated normal random variables is indicated below. Although it is valid only for normal random variables, it can be also used approximately for other types of random variables.

### 4. GOVERNING EQUATIONS OF THE SYSTEM

The governing equation of the solid part of the system is the motion equation. In the fluid and solid interface the load of hydrodynamic pressure must be added to dynamic loads. Therefore, the system equation for dam, foundation and isolation layer body is presented as:

$$M\ddot{u} + Cu + Ku = M\ddot{u}_g + F^{Pr}. \quad (7)$$

In Eq. (7), $M$ refers to mass matrix, $C$ represents the damping matrix and $K$ is the structural stiffness matrix. $\ddot{u}$ represents the relative movement vector, $\ddot{u}_g$ refers to the ground acceleration vector. $F^{Pr}$ is the vector of induced hydrodynamic pressure applied at the contact of water with solid parts.

Assuming that the water inside the reservoir is inviscid, compressible with small displacement, the combination of equations of continuity and momentum leads to the wave equation. So, the governing equation of reservoir is defined as:

$$1 \frac{\delta^2 p}{C^2 \delta t^2} - \nabla^2 P = 0; \quad (8)$$

where $c$ represents the velocity of sound in the fluid environment, $P$ refers to the hydrodynamic pressure and $t$ represents the time.

### 5. CASE STUDY

To show the efficiency of the analysis process offered in the current study and the effect of the isolation layer on seismic optimization of concrete gravity dams, the response of Pine flat dam under the both components of longitudinal and vertical of El Centro, San Fernando and North Ridge earthquakes is presented. The longitudinal and vertical components of selected earthquakes have been applied to dam-reservoir system in both horizontal and vertical directions. The Pine Flat dam geometry and finite element model related to dam-reservoir-foundation system with the isolation layer attached to upstream face is presented in Fig. 1. The characteristic of material parameters of the system have been shown in Table 1. For the reservoir, density and bulk modulus corresponding to water are considered as 1000 kg/m$^3$ and 2.1 GPa.

### 6. MODEL ANALYSIS

In this research, the FE-ANSYS software is used for the analysis and seismic optimization of a concrete dam. This software has the ability to consider the irregular geometry domains and interaction effects for seismic analysis. Accordingly, suitable components have been provided for fluid compressibility behavior in FE-ANSYS. According to the geometry and behavior of a concrete gravity dam and reservoir, the dam is intended for two-dimensional plane stress behavior, and the effects of interactions with the foundation has been considered.
in the model. An 8-node SOLID182 element has been selected for the discrete solid part, and 4-node FLUID29 element has been used in the reservoir domain. The output variables, including the dam displacement, induced stress of the dam body, and the hydrodynamic pressure generated in the reservoir, are extracted by analyzing the model. According to the UBC by-laws, horizontal and vertical components of earthquakes extracted from the peer site, shown in Table 2 are scaled to the maximum acceleration of the dam area (0.3 g). The amounts of integration factors of the Newmark technique are considered as $\beta = 0.25$ and $\gamma = 0.5$. Moreover, Sommerfeld indicated that the boundary condition is applied as a far field reservoir boundary. Using the first and second frequency of the system, Rayleigh damping coefficients are extracted as $\alpha_1 = 0.5202$ and $\alpha_2 = 0.0046$. The results of the Monte Carlo probability analysis should be a proper convergence. To achieve this goal, the Monte Carlo simulation setting in the FE-ANSYS software is intended in Table 3.

### 6.1. Model Sensitivity to Discretization

One of the most important modeling steps in finite element software is the discretization of the system. Discretization in the model must be converged so finer discretization does not affect the results considerably. Hence, in this paper, through Monte Carlo probability analysis and with APDL programming in ANSYS software, optimum discretization of the model was achieved with regard to investigation of the effect of discretization size on seismic responses of the concrete gravity dam.

The results which indicate the sensitivity of seismic responses of the dam to the size of the mesh in the model without the rubber damper under El Centro earthquake are extracted from probability analysis in ANSYS software. It is noteworthy to mention that the horizontal axis of the diagrams indicates the size of the mesh, wherein rise presents finer meshes. Figs. 2 show the sensitivity of the seismic responses of the model to the size of the mesh under the El Centro earthquake.

### Table 3. Monte Carlo setting in FE-ANSYS software.

<table>
<thead>
<tr>
<th>Random input</th>
<th>Distribution type</th>
<th>Number of simulations</th>
<th>Number of repetitions</th>
</tr>
</thead>
<tbody>
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<td>Thickness</td>
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<td>3</td>
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### Table 4. Characteristics of uncertainty parameters of the model for the Monte Carlo probabilistic analysis settings.

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<th>Input variable</th>
<th>Number of repetitions</th>
<th>Number of simulation loops</th>
<th>Range of variation</th>
<th>Type of distribution</th>
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</thead>
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<td>40</td>
<td>$0 &lt; X &lt; 120$</td>
<td>uniform</td>
</tr>
</tbody>
</table>

### 7. MODEL ANALYSIS

According to the results, finer meshes affect the seismic responses of the dam until a limited value; beyond that more meshings do not affect the responses noticeably. Therefore, through employing Monte Carlo probability analysis, it is possible to specify the optimum meshing for every earthquake.

#### 7.1. Model Sensitivity to Number of Samples

In the Monte Carlo probabilistic analysis, histogram diagrams of output variables must be converged to the number of simulations. Hence probability analysis settings in ANSYS software are mentioned in Table 4.

After Monte Carlo probabilistic analysis of the dam model under El Centro earthquake, to verify the results, first, histogram diagrams of mean values of output variables with the number of samples are studied. If the number of simulation samples are enough, there is convergence for the mean value of output variables and the curves tend to be flat at their end. Figs. 3 illustrates the histogram curve of output parameters with the number of samples.

The horizontal end of the simulated histogram curves in Figs. 3 indicate the number of simulation samples are sufficient for the probabilistic Monte Carlo analysis.
than 30 cm and a further increase in thickness of the rubber damper has an insignificant effect on hydrodynamic pressure. The San Fernando earthquake represents a suitable performance of the isolation in the reduction of hydrodynamic pressure application on the dam’s upstream face and is a layer with a thickness of 10 cm. The hydrodynamic pressure acting on the dam is effectively reduced for a layer of 20 cm of the rubber damper under the Northridge earthquake.

Fig. 5 demonstrates the proper performance of the rubber damper in order to reduce the horizontal displacement of the dam crest with a thickness of less than 30 cm under the El Centro earthquake. For the San Fernando earthquake, the horizontal displacement of the dam crest is effectively reduced in thickness of less than 10 cm and the rubber damper shows a proper performance. In addition, it was found that the further increase in thickness does not have a significant effect in the
reduction of response. The horizontal displacement of the dam crest is significantly decreased as long as the thickness of the rubber damper is smaller than 20 cm due to the Northridge earthquake.

Fig. 6 illustrates that the rubber damper of a thickness of 20 cm has a proper performance in reducing the 1st principle stress of dam heel under the El Centro earthquake. In this figure, the first principle stress of the dam heel, is decreased significantly while the rubber damper thickness is less than 20 cm. This figure shows that increasing the thickness of the rubber damper by more than 20 cm, does not demonstrate a significant effect on reducing the 1st principle stress of the dam heel for the system due to the San Fernando earthquake. Also, it is obvious from this figure the 1st principle stress of the dam heel is effectively reduced for the rubber damper with a thickness of 10 cm due to the Northridge earthquake. However, further increase in the thickness of the rubber damper does not have a significant effect in reducing response.

Fig. 7 shows that under the El Centro earthquake the third principle stress in the toe of the dam is decreased significantly in a thickness of 20 cm of the rubber damper. It is obvious from the figure that the third principle stress of the dam toe is effectively reduced to a thickness of 20 cm of the rubber damper due to the San Fernando earthquake. The reduction of the third principle stress in the toe of the dam is significant for the rubber damper with a 10 cm thickness under the Northridge earthquake.

Finally, for the proper evaluation of the isolation layer effect with different thicknesses on reducing seismic responses, the numerical values of results are shown in the Tables 5, 6 and 7 for selected earthquakes. According to the tables, the rubber damper with a thickness in the range of 10 cm to 15 cm has the best performance for seismic optimization of the model under the El Centro, the San Fernando and the Northridge earthquakes, respectively.

8.2. Optimization of the Rubber Damper Height

For optimization of the rubber damper thickness using probabilistic analysis and in order to increase the accuracy of the results of the analysis, the sensitivity of the seismic responses of the dam relative to the height of the hydrodynamic damper is extracted from the hydrodynamic damper model. Considering the height of the hydrodynamic damper as a design variable, the results of the probabilistic analysis under the El Centro, San Fernando and Northridge earthquakes are presented in Figs. 8 to 11.

As shown in Fig. 8, the better performance of the isolation layer in the reduction of hydrodynamic pressure is the rubber damper with the height of 80 m, 20 m and 60 m under the El Centro, San Fernando and Northridge earthquakes, respectively.

Fig. 9 also shows the effectiveness of the isolation layer height on reducing the horizontal displacement of the dam crest for three modeled earthquakes is similar and the rubber damper with a 100 m height reduced displacement significantly. Exceeding this amount does not have a considerable effect on reducing the horizontal displacement of the dam crest.
Table 5. Characteristics of uncertainty parameters of the model for the Monte Carlo probabilistic analysis settings.

<table>
<thead>
<tr>
<th>Layer thickness (cm)</th>
<th>S1 Value (MPa)</th>
<th>S1 Reduction (%)</th>
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<th>S3 Reduction (%)</th>
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Table 6. Results from the probabilistic analysis for the model under San Fernando earthquake.

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Figure 8. Sensitivity of the hydrodynamic pressure of the dam heel to the height of the rubber damper.

Figure 10. Sensitivity of the 1st principle stress of the dam heel to the height of the rubber damper.

Figure 9. Sensitivity of the horizontal displacement of dam crest to the height of the rubber damper.

Figure 11. Sensitivity of the 3rd principle stress of the dam heel to the height of the rubber damper.

According to Fig. 10, the hydrodynamic damper reduces the main tensile stresses applied on the dam up to 90 meters effectively for models under selected earthquakes and exceeding this amount has no significant effect in reducing the main tensile stress on the dam.

Fig. 11 represents that the hydrodynamic damper up to a height of approximately 100 m can reduce the induced main compressive stress on the dam considerably for the system under the El Centro and Northridge earthquakes. For the San Fernando earthquake, the best performance for the isolation layer is the rubber damper with a 70 m height.

Finally, for a more appropriate investigation, the effect of a hydrodynamic damper height on reducing the seismic responses of numerical values of responses due to applied earth-
quakes are presented in Tables 8 to 10. According to these tables, choosing the height of a hydrodynamic damper in the range of 80 m to 90 m, leads to an optimal use of the hydrodynamic damper with a safe and economic design of the case model of the concrete gravity dam.

### 9. CONCLUSION

In this paper, to control the interaction effects and dissipation of induced hydrodynamic pressure, the formulation is used in a study of the effects of the rubber damper as an isolation layer on the seismic optimization of the dam model. In particular, the computational treatment of the dam-reservoir–foundation region with an upstream rubber damper by the software and finite element model is described in detail. To assess the optimum thickness and height of the rubber damper, the Monte Carlo simulation and sensitivity analysis are applied.

The obtained results show that, the isolation layer reduces the displacement and stresses of the dam because the absorptive and flexible isolation layer has a damping effect on induced hydrodynamic pressure during earthquake. Also, the thickness and height of the isolation layer have a different effect on the reduction of seismic responses which reduce the amplitude of the hydrodynamic pressure at the dam-reservoir interface.

The comparison of the proper function of the Monte Carlo probability analysis versus the assessment of the rubber damper thickness and height on the seismic behavior shows that the Monte Carlo method is a useful tool for the optimization and safe design by understanding the effects of various parameters on the seismic performance.

Using the results of probabilistic analysis of the model which represents the sensitivity of seismic reaction of the dam to the rubber damper thickness and height, the suitable range of the rubber damper thickness and height for optimum design is presented. Based on the obtained results and considering costs and economic points, it can be concluded that the rubber damper with a thickness of 10 to 20 cm and height of 80 to 90 m has the best performance on reducing seismic responses of the dam. The optimal selection of the isolation layer height and thickness depend on its effectiveness in reduction of response considering economic and executive justification.

Finally, it can be concluded that the presented model is applicable in the safe design of concrete dams and even existing dams due to the scarcity of water resources and the enormous costs of building dams. The concept of a safe structure refers to the structures that continue to function without damage and failure for many years.

### REFERENCES


10. As’kov, V. L., Kalitseva, I. S., Komarov, A.I. and Sheinin, I. S. Physical modeling of phenomena of the interaction


### Table 8. Results of probabilistic analysis for the model under El Centro earthquake.

<table>
<thead>
<tr>
<th>Layer</th>
<th>S1 Height (m)</th>
<th>Value (MPa)</th>
<th>Reduction (%)</th>
<th>S3 Height (m)</th>
<th>Value (MPa)</th>
<th>Reduction (%)</th>
<th>Press Value (MPa)</th>
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### Table 9. Results of probabilistic analysis for the model under San Fernando earthquake.

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<th>Layer</th>
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<th>Reduction (%)</th>
<th>S3 Height (m)</th>
<th>Value (MPa)</th>
<th>Reduction (%)</th>
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Table 10. Results of probabilistic analysis for the model under North Ridge earthquake.

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<th>Layer Height (m)</th>
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<th>Reduction (%)</th>
<th>S3 Value (MPa)</th>
<th>Reduction (%)</th>
<th>Press Value (MPa)</th>
<th>Reduction (%)</th>
<th>Ux Value (Cm)</th>
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Vibration Analysis of Axially Functionally Graded Tapered Euler-Bernoulli Beams Based on Chebyshev Collocation Method

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The bending vibration behavior of a non-uniform axially functionally graded Euler-Bernoulli beam is investigated based on the Chebyshev collocation method. The cross-sectional and material properties of the beam are assumed to vary continuously across the axial direction. The Chebyshev differentiation matrices are used to reduce the ordinary differential equations into a set of algebraic equations to form the eigenvalue problem associated with the free vibration. Some calculated results are compared with numerical results in the published literature to validate the accuracy of the present model. A good agreement is observed. The effects of the taper ratio, volume fraction index, and restraint types on the natural frequency of axially functionally graded beams with non-uniform cross section are examined.

1. INTRODUCTION

Functionally graded (FG) materials are special composites with smoothly varying material properties along any desired spatial direction. These graded properties can be achieved by gradually changing the volume fraction of constituents along a typical direction according to the polynomial, power, and exponential laws. Due to these particular graded properties, the applications of structures with FG materials have received wide attention in civil, automobile and aerospace industries for the past few decades. Because a beam is both a commonly and widely used member in structures and machines, to better understand the dynamic behavior of a beam made of FG materials is necessary from a structural design point of view. In past years, the dynamic behaviors of FG beams with material properties varying along the beam thickness have been thoroughly investigated by many researchers using various analytical and numerical methods. It is well-known that beam structures with varying cross-sectional and material properties along the length direction are commonly used in buildings, bridges and mechanical components due to the fact that they are capable of optimizing the strength and weight of the structure. Hence, it is important to accurately predict and evaluate the dynamic characteristics of such beam structures. In this regard, only the literature related to the FG beams with axially varying properties will be discussed next.

Due to the variable coefficients in the governing equation, the dynamic analysis of axially FG beams with tapered cross-sections becomes more mathematically complex. Therefore, the dynamic problems of FG beams with material and cross-sectional properties varying along the axial direction have been largely studied by numerical methods based on various beam theories. Aydogdu investigated the vibration and buckling analysis of simply supported FG Euler-Bernoulli beams with axial variation of material properties based on a semi-inverse method. Huang and Li applied the integral equation method to study the free vibration of non-uniform axially FG Euler-Bernoulli beams. The effects of material graded parameters on natural frequencies of the beams were investigated. Based on the finite element method, Shahba et al. dealt with the free vibration and stability of axially FG tapered Timoshenko beams with respective classic and non-classical boundary conditions. Hein and Feklistova studied the free vibration of non-uniform axially FG beams using the Euler-Bernoulli beam theory and Haar wavelet approach. The results revealed that the Haar wavelet approach was capable of calculating the frequencies of beams with different non-uniform cross-sections, bending rigidity and mass density. Shahba and Rajasekaran studied the free vibration and stability of axially FG tapered Euler–Bernoulli beams by using both the differential transform element method and differential quadrature element method of lowest-order. The free longitudinal and transverse natural frequencies, and the critical buckling loads, of the beams were determined by the two numerical methods. Li et al. presented the exact frequency equations of the free vibration for axially exponentially FG beams with different end conditions based on an analytical approach. Rajasekaran analysed the bending vibration of rotating axially FG tapered Euler–Bernoulli beams based on the differential transform element method and differential quadrature element method of lowest-order. The effects of material property, taper ratio, rotating speed, hub radius and tip mass on the natural frequencies were investigated. Huang et al. investigated the free vibration of non-uniform and axially FG Timoshenko beams with various boundary conditions by using a unified approach. Sarkar and Ganguli studied the free vibration of axially FG uniform Timoshenko beams with fixed–fixed boundary conditions. Tang et al. derived the exact frequency equations of the free transverse vibration of exponentially FG beams with non-uniform cross-section based on the Timoshenko beam theory. Liu et al. analysed the free bending vibration of axially FG Euler-Bernoulli beams with tapered cross-section using the spline finite point method. The effects of the material and cross-sectional properties varying along the axial direction on the natural frequencies were discussed. Cao and Gao investigated the free vibration of axially FG beams with non-uniform cross-section by using the asymptotic development method.
length, width, and thickness directions, respectively. \( A, I, E \) and \( \rho \) represented the respective cross-sectional area, area moment of inertia, Young modulus, and density; the subscripts \( o \) and \( L \) denoted the axial position of the given properties at \( x = 0 \) and \( x = L \), respectively. The governing equation of motion of such a beam was expressed by a fourth-order partial differential equation as follows:

\[
\frac{\partial^2}{\partial x^2} \left[ S(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right] + m(x) \frac{\partial^2 w(x, t)}{\partial t^2} = 0, \quad 0 \leq x \leq L. \tag{1}
\]

Here \( w \) was the transverse displacement, \( L \) was the beam length, and \( x \) was the axial coordinate. \( S(x) = E(x)I(x) \) was the bending rigidity where \( E(x) \) and \( I(x) \) were the Young modulus and area moment of inertia at position \( x \), respectively. \( m(x) = \rho(x)A(x) \) was the beam mass per unit length in which \( \rho(x) \) and \( A(x) \) were the respective density and cross-sectional area at location \( x \). Assume that \( w(x, t) = W(x)e^{i\omega t} \) and substitute it into Eq. (1) to yield:

\[
\frac{d^2}{dx^2} \left[ S(x) \frac{d^2 W}{dx^2} \right] - m(x)\omega^2 W = 0, \quad 0 \leq x \leq L; \tag{2}
\]

where \( \omega \) was the natural frequency. Changing the spatial variables by \( \xi = 2x/L - 1 \) and letting \( \Omega^2 = \omega^2L^4/16 \) allowed Eq. (2) to be rewritten as:

\[
\frac{d^4}{d\xi^4} \left[ S(\xi) \frac{d^4 W}{d\xi^4} \right] - m(\xi)\Omega^2 W = 0, \quad -1 \leq \xi \leq 1; \tag{3}
\]

or

\[
S(\xi) \frac{d^4 W}{d\xi^4} + 2S'(\xi) \frac{d^3 W}{d\xi^3} + 2S''(\xi) \frac{d^2 W}{d\xi^2} = m(\xi)\Omega^2 W, \quad -1 \leq \xi \leq 1. \tag{4}
\]

The associated classic boundary conditions were represented as follows:

\[
W = 0, \quad \frac{dW}{d\xi} = 0 \quad \text{(Clamped end)}; \tag{5}
\]

\[
W = 0, \quad \frac{d^2W}{d\xi^2} = 0 \quad \text{(Pinned end)};
\]

\[
\frac{d^2W}{d\xi^2} = 0, \quad \frac{d}{d\xi} \left[ S(\xi) \frac{d^2 w}{d\xi^2} \right] = 0 \quad \text{(Free end)}. \tag{5}
\]

The free bending vibration problem of non-uniform axially FG tapered beams in Eqs. (4) and (5) were solved using the Chebyshev collocation method. The following Gauss-Chebyshev-Lobatto collocation points within the interval \([-1, 1]\) were used:

\[
\xi_j = \cos \frac{\pi j}{N}, \quad j = 0, 1, 2, \ldots, N. \tag{6}
\]

Thus, the displacement function \( W(\xi) \) was expanded by the \( N \)-th order Chebyshev polynomials as follows:

\[
W(\xi) \approx \sum_{j=0}^{N} \gamma_j(\xi) W(\xi_j), \quad j = 0, 1, 2, \ldots, N. \tag{7}
\]

As mentioned above, various analytical and numerical methods had been used to effectively investigate the influences of material and geometric parameters on the dynamic characteristics of axially FG beams. It is well-known that the Chebyshev collocation method has been applied to different mathematical and engineering models because of its high rate of convergence and predictable accuracy.\(^\text{[5–10, 12, 25–34]}\) However, the application of this method to the bending vibration of axially FG tapered beams has seldom been reported. Hence, the present paper attempts to study the free transverse vibration of the axially FG Euler-Bernoulli beam with a tapered cross-section based on the Chebyshev collocation method. The Chebyshev differentiation matrices are applied to transform the governing differential equations into a set of algebraic eigenvalue equations. The lateral natural frequencies of tapered axially FG beams with various boundary conditions are then obtained by solving the generalized eigenvalue equation. The material properties axially graded according to the exponential, polynomial and power functions are considered. The rectangular cross section is assumed to be tapered linearly in the width and height directions along the beam length. The effects of the taper ratios, axially graded properties, and boundary conditions on the free vibration behaviors of the axially FG tapered beams are investigated. Several numerical results are evaluated and compared with those in the published literature to validate the accuracy of the present model. The results reveal that the proposed method can be used to study the free vibration of tapered Euler-Bernoulli beams with different axially graded material properties described by typical functions under various boundary conditions with good accuracy.

2. PROBLEM FORMULATION

The present paper investigates the bending vibration behavior of non-uniform axially FG beams under various boundary conditions. The material and cross-section properties were assumed to be varied along the longitudinal direction of the beam. Figure 1 illustrates the geometric configuration and coordinate system of the beam with a tapered section along the height and width directions. The origin \( o \) of the coordinate \( xyz \) was the centre of the left-end plane of the beam. The \( x \)-axis, \( y \)-axis and \( z \)-axis originating from the origin were found in the
with
\[
\gamma_j(\xi) = \frac{(-1)^{j+1}(1 - \xi^2)T_j^N(\xi)}{c_j N^2(\xi - \xi_j)};
\]
\[
T_N(\xi_j) = \cos\left(N\cos^{-1}(\xi_j)\right);
\]
\[
\gamma_j(\xi_k) = \delta_{jk};
\]
\[
c_j = \begin{cases} 2 & j = 0, \quad N = 0 \\ 1 & j = 1, 2, \ldots, \quad N - 1 \end{cases}.
\] (8)

Then, the first derivative of the displacement function \(W(\xi)\) was obtained by the following matrix vector multiplication:
\[
W'(\xi_j) = \sum_{j=0}^{N} (D_N)_{ij} W(\xi_j), \quad j = 0, 1, 2, \ldots, N.
\] (9)

Here \((D_N)_{ij}\) was the \(i, j\) element of an \((N+1) \times (N+1)\) Chebyshev differentiation matrix \(D_N\). The entries of this matrix are:
\[
(D_N)_{00} = \frac{2N^2 + 1}{6}; \quad (D_N)_{NN} = -\frac{2N^2 + 1}{6};
\]
\[
(D_N)_{jj} = -\frac{\xi_j}{2(1 - \xi_j^2)}; \quad j = 1, 2, \ldots, N - 1;
\]
\[
(D_N)_{ij} = \frac{c_i(-1)^{j+1}}{c_j(\xi_i - \xi_j)}; \quad i \neq j, \quad j = 1, 2, \ldots, N - 1.
\] (10)

For simplicity, the first derivative of the Chebyshev differentiation matrix was denoted by \(D_1\). The \(k\)th derivative was obtained by \(D_k = (D_1)^k\).

Based on the Chebyshev collocation method as described, the ordinary differential equation in Eq. (4) was rewritten in terms of Chebyshev differentiation matrices as:
\[
(K_1D_4 + 2K_2D_3 + K_3D_2)\overline{W} = \Omega^2M\overline{W};
\] (11)

Likewise, the boundary equations in Eq. (5) at the supporting ends of the beam were also expressed by Chebyshev differentiation matrices as given in Table 1. When the homogeneous boundary conditions were imposed on the governing Eq. (11), some mathematical operations were performed. First, the first and second equations of the system were replaced by the two boundary conditions at the right end. The \(N\)th and \((N+1)\)th equations were replaced by those at the left end. Then, the \(N\)th and \((N+1)\)th equations of the new system were moved up to become the third and fourth equations. Finally, by shifting the displacements \(W_N\) and \(W_{N+1}\) to the third and fourth rows of the displacement vector \(\overline{W}\), the generalized eigenvalue problem was obtained as:
\[
\begin{bmatrix}
K_{BB} & K_{BI} \\
K_{IB} & K_{II}
\end{bmatrix}
\begin{bmatrix}
\overline{W}_B \\
\overline{W}_I
\end{bmatrix} = \Omega^2
\begin{bmatrix}
O & M_{II} \\
M_{IB} & M_{II}
\end{bmatrix}
\begin{bmatrix}
\overline{W}_B \\
\overline{W}_I
\end{bmatrix};
\] (13)

\[
\overline{W}_B = \{W_1 W_2 W_N W_{N+1}\}^T;
\]
\[
\overline{W}_I = \{W_3 W_4 \ldots W_{N-1}\}^T.
\] (14)

The subscripts B and I denoted the boundary and internal collocation points associated with the boundary condition and the governing equation, respectively. The sizes of the stiffness matrices \(K_{BB}, K_{BI}, K_{IB}\) and \(K_{II}\) were \(4 \times 4, 4 \times (N-3), (N-3) \times 4\) and \((N-3) \times (N-3)\), respectively. The sizes of the inertia matrices \(M_{IB}\) and \(M_{II}\) were the same as the corresponding stiffness matrices \(K_{IB}\) and \(K_{II}\).

To solve the general eigenvalue equation in Eq. (13), it was expanded to yield:
\[
K_{BB}\overline{W}_B + K_{BI}\overline{W}_I = 0;
\] (15)
\[
K_{IB}\overline{W}_B + K_{II}\overline{W}_I = \Omega^2(M_{IB}\overline{W}_B + M_{II}\overline{W}_I);
\] (16)

Then, after introducing Eq. (15) into Eq. (16), the algebraic eigenvalue equation was reduced to the following form:
\[
AW_I = \Omega^2BW_I;
\] (17)
\[
A = -K_{IB}(K_{BB})^{-1}K_{BI} + K_{II};
\]
\[
B = -M_{IB}(K_{BB})^{-1}K_{BI} + M_{II}.
\] (18)

Thus, natural frequencies of the free bending vibration of non-uniform axially FG Euler-Bernoulli beams with various classical boundary conditions can be obtained as the solution:
\[
det(A - \Omega^2B) = 0.
\] (19)

3. RESULTS AND DISCUSSIONS

To assure the successful application of the Chebyshev collocation method in the vibration analysis of axially FG Euler-Bernoulli beams with non-uniform cross-sections, the accuracy studies are carried out through various numerical examples. First example to be concerned is a uniform axially FG beam with the bending rigidity \(S(x)\) and mass per unit length \(m(x)\) according to the following distribution:
\[
S(x) = \left[E_o + (E_L - E_o)e^{a \xi/L} - \frac{1}{e^a - 1}\right]I, \quad \alpha \neq 0;
\]
\[
m(x) = \left[\rho_o + (\rho_L - \rho_o)e^{a \xi/L} - \frac{1}{e^a - 1}\right]A, \quad \alpha \neq 0;
\] (20)
where $\alpha$ is the material graded index which describes the distribution of material properties. For $\alpha > 0$, a larger value of $\alpha$ (e.g., 10) represents a more sudden increase in the properties $E(x)/E_o$ and $\rho(x)/\rho_{o}$ near the right surface. In contrast, the properties vary abruptly near the left surface for a smaller value of $\alpha$ (e.g., -10) as $\alpha < 0$. The beam is made of aluminum and zirconia, whose properties vary axially according to the exponential function in Eq. (20). Meanwhile, the beam is aluminum-rich at $x = 0$ and zirconia-rich at $x = L$. The dimensionless fundamental natural frequencies $\lambda = \omega L^2 (\rho_o A_o/E_o I_o)^{1/2}$ of the beams with various values of $\alpha$ and boundary conditions are presented in Table 2. The present results agree well with those given by Huang and Li, Hein and Feklistova and Liu et al.\textsuperscript{14,16,23} The results reveal that the restraint and material graded index have a significant impact on the frequencies. The fundamental frequency decreases first and increases with the increasing $\alpha$ for the clamped-free (CF) beam, reduces with the increasing $\alpha$ for the pinned-ipped (PP) and clamped-pinned (CP) beam, and varies irregularly with $\alpha$ for the clamped-clamped (CC) beam.

Secondly, a non-uniform axially FG beam is examined whose bending rigidity $S(x)$ and mass per unit length $m(x)$ are represented as:\textsuperscript{17}

$$S(x) = E(x)I(x) = \left[ E_o \left( 1 + \frac{x}{L} \right) \right] \left[ 1 - C_h \frac{x}{L} \right] \left[ 1 - C_h \frac{x}{L} \right]^3 I_o;$$

$$m(x) = \rho(x)A(x) = \left[ \rho_o \left( 1 + \frac{x}{L} + \left( \frac{x}{L} \right)^2 \right) \right] \left[ 1 - C_h \frac{x}{L} \right] \left( 1 - C_h \frac{x}{L} \right) A_o.$$  \hspace{1cm} (21)

Here $C_b$ and $C_h$ denote the width and height taper ratios, respectively, whose values range from 0 to 1. The non-uniform beam becomes prismatic as $C_b = C_h = 0$, and it tapers to a point at $x = L$ as $C_b = C_h = 1$. The tapered FG beam has the properties of $L = 1$ m, $A_o = 4 \times 10^{-4}$ m$^2$, $I_o = 1.33 \times 10^{-8}$ m$^4$, $E_o = 70$ GPa and $\rho = 2702$ kg m$^{-3}$. Its material properties vary axially according to the polynomial function in Eq. (21). The dimensionless frequency parameter $\lambda = \omega L^2 (\rho_o A_o/E_o I_o)^{1/2}$ is used in the calculations of natural frequencies. Tables 3–5 present the variations of the dimensionless fundamental frequencies against different taper ratios for axially FG beams with CF, PP and CC boundary conditions, respectively. In comparison with the results obtained by Shahba and Rajasekaran and Liu et al.,\textsuperscript{17,23} an excellent agreement is achieved.\textsuperscript{17,23} Depending on the taper ratios and boundary conditions, the increase in height and width taper ratios may decrease or increase the fundamental natural frequencies. For the beams with the same width ratio $C_b$, the natural frequencies reduce with the increasing height taper ratio $C_h$ except for those of CF beams. The frequencies of CF beams enlarge with the increasing $C_h$. For the beams with the same height taper ratio $C_h$, the frequencies of CF beams increase with the increasing width taper ratio $C_b$ but those of PP and CC beams vary differently with $C_h$ depending on the value of $C_b$. It is important to note that the height taper ratio has a more profound impact on the natural frequencies of all beams than width taper ratio while it shows an opposite trend for CF beams.

From the previous comparison study, it indicates that the proposed method can be applied to evaluate the free vibration frequencies of various axially FG tapered beams with high accuracy. In the next, the free vibration of non-uniform beams with axially graded material properties according to a power-law function is studied to show the adaptability of the presented method. Its bending rigidity $S(x)$ and mass per unit length $m(x)$ are given as follows:\textsuperscript{17,19}

$$S(x) = \left[ E_o + (E_L - E_o) \left( \frac{x}{L} \right)^p \right] \left[ 1 - C_h \frac{x}{L} \right] \left[ 1 - C_h \frac{x}{L} \right]^3 I_o;$$

$$m(x) = \left[ \rho_o + (\rho_L - \rho_o) \left( \frac{x}{L} \right)^p \right] \left[ 1 - C_h \frac{x}{L} \right] \left( 1 - C_h \frac{x}{L} \right) A_o.$$  \hspace{1cm} (22)

Here the non-negative exponent $p$ is the volume fraction index. For a larger value of $p$, the properties $E(x)/E_o$ and $\rho(x)/\rho_{o}$ change more suddenly near the right surface and the material at the left surface is the dominant constituent. For a smaller value of $p$, the variation of the properties shows an opposite tendency. The recommended value of $p$ ranges from 1/3 to 3 to insure that the FG material has a proper balance between the percentages of the constituents.\textsuperscript{15} The axially FG beam is composed of zirconia and aluminum with ceramic-rich left side and metal-rich right side. The properties of the beam are

| Table 1. Boundary condition equations in terms of Chebyshev differentiation matrices. |
|---------------------------------|----------------|----------------|----------------|
| Boundary Conditions             | Left end ($\xi = -1$) | Right end ($\xi = +1$) |
| clamped                         | $D_1(N + 1, 1)W = 0$ | $D_1(1, 1)W = 0$ |
| pinned                          | $D_1(N + 1, 1)W = 0$ | $D_1(1, 1)W = 0$ |
| free                            | $D_1(N + 1, 1)W = 0$ | $D_1(1, 1)W = 0$ |

$$S(\xi) = D_1(N + 1, 1)W + S(\xi)D_1(N + 1, 1)W = 0$$

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as follows: $E_o = 200$ Gpa, $\rho_o = 5700$ kgm$^{-3}$, $E_L = 70$ Gpa, $\rho_L = 2702$ kgm$^{-3}$, and $r^2 = I_o/A_o L^2 = 0.0001$. $r$ is the inverse slenderness ratio. The effects of the taper ratios, material graded indices and restraint types on the vibration frequencies of such axially FG beams are demonstrated. The frequency parameter $\lambda = \omega L^2 (\rho_o A_o/E_o L_o)_{1/2}$ is used to evaluate the dimensionless natural frequencies.

Tables 6 and 7 present the effects of various values of $C_h$ and $C_b$ on the first three dimensionless frequencies for the CF and PP axially FG beam with $p = 2$, respectively. The variations of the first three natural frequencies with respect to the taper ratios $C_h$ and $C_b$ for the CP and CC axially FG beams are depicted in Figs. 2 and 3 to show the varying trend of the natural frequencies. As seen in Table 6, the first frequency of the CF beam dramatically increases with the increase of taper ratios $C_h$ and $C_b$. However, its second and third frequencies reduce with $C_h$. It is also noted that the first frequency is affected much more by the increasing width taper ratio $C_b$ than the height taper ratio $C_h$. In contrast, the height taper ratio has a significant impact on the
reduction of the first three frequencies of the PP beams. The first frequency is most significantly affected by the increasing $C_h$, followed by the second and third frequencies. When the width taper ratio $C_b$ increases, the first frequency slightly reduces but the second and third frequencies increase. The only exception is the second frequency of the beam with $C_h = 0.1$, which is decreased as $C_b$ is increased from 0.8 to 0.9. As observed, the height taper ratio will enhance the effect of width taper ratio on the natural frequencies, especially for the first mode. It can be found in Fig. 2 that all three frequencies of CP beams reduce with the increase in $C_l$ and slightly increase with $C_b$. Unlike the PP beams, the increasing $C_b$ has a more profound effect on the reduction rate of higher mode frequencies. Like the PP and CP beams, Fig. 3 reveals that the first three frequencies of CC beams decrease as the height taper ratio $C_h$ increases. Like the CP beams, the reduction rate of higher mode frequency with respect to the increasing height taper ratio is higher than that of lower ones. With the increase in $C_b$, the first three frequencies of CC beams slightly increase first and then reduce. However, the effect of the increasing $C_b$ on the reduction of frequencies is gradually diminished while the CC beam has a higher height taper ratio $C_h$.

To summarize the results reported previously, several conclusions on the effects of taper ratios for axially FG beams with various boundary conditions can be made as follows. In general, the height taper ratio remarkably affects the natural frequencies of all beams with the same width ratio.
With the increasing axially FG tapered beams of various boundary conditions. Therefore, it is difficult to predict the natural frequencies for axially FG tapered beams of various boundary conditions with respect to the taper ratios.

Table 8 presents the effects of material gradation on the first three frequencies of axially FG beams with \( C_b = 0.3 \) and \( C_b = 0 \) under different boundary conditions. As can be seen, the variations of natural frequencies of axially FG tapered beams with respect to \( p \) depend on the boundary conditions. With the increasing \( p \), the first and second modes of CF beam increase first, and then decrease while the third mode continues to increase. All modes of PP beam enlarge with \( p \). For CP and CC beams, the fundamental frequency increases and decreases alternatively with \( p \), while other two modes show an increasing trend. As cited by Shahba et al., it is important to note that the effects of \( p \) on the variation of the natural frequencies of tapered axial FG beams are hard to be predicted because both the stiffness and mass of the beam are enhanced with the increase in \( p \).

### 4. CONCLUSIONS

The bending vibration of various axially FG Euler-Bernoulli beams with tapered cross section is section based on the Chebyshev collocation method. The effects of the material and cross-sectional properties varying along the beam length direction on the vibration behaviors are investigated. Natural frequencies for the uniform axially FG beams with the exponential function gradient and the tapered axially FG beams with polynomial function gradient are evaluated and compared with the published ones to confirm the effectiveness of the present method. The results indicate that the present study can analyze the free vibration of Euler-Bernoulli beams with different axially graded material properties and varying cross-sectional properties under various boundary conditions. Finally, the axially FG tapered beam with power law gradient are examined to demonstrate the adaptability of the present method to different graded material properties. Hence, it is believed that the present method can be extended to study the dynamic problem of elastically supported bi-directional FG beams resting on elastic foundations in the future work.

### ACKNOWLEDGEMENT

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### REFERENCES


Table 8. Dimensionless frequencies $\lambda_i$ of axially FG Z/A beams with different material graded indices and boundary conditions ($C_A = 0.3$, $C_B = 0$).

<table>
<thead>
<tr>
<th>$B/C$</th>
<th>mode</th>
<th>$p = 0.2$</th>
<th>$p = 0.5$</th>
<th>$p = 1$</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 5$</th>
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<tr>
<td>CF</td>
<td>$\lambda_1$</td>
<td>3.8182</td>
<td>4.2269</td>
<td>4.4481</td>
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<td></td>
<td>$\lambda_3$</td>
<td>50.9055</td>
<td>50.1804</td>
<td>50.4105</td>
<td>50.1415</td>
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<td>PP</td>
<td>$\lambda_1$</td>
<td>7.3610</td>
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<td>$\lambda_3$</td>
<td>67.2914</td>
<td>69.4157</td>
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<td>$\lambda_2$</td>
<td>39.4242</td>
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<td>$\lambda_1$</td>
<td>17.5197</td>
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<td>47.6082</td>
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Inertial Actuator with Virtual Mass for Active Vibration Control

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Inertial actuators (IAs) are often used as control units in active noise and vibration control systems. It is well-known that the IA’s natural frequency should be far below that of the structure under control to ensure good stability margins. However, under normal circumstances, an IA with low natural frequency either increases the additional weight or causes unwanted static displacement of the IA’s proof-mass. In this study, an IA with virtual mass is presented to reduce the IA’s natural frequency without changing its physical design. The virtual mass of the IA is realized by using the proof-mass acceleration feedback as a local loop within the IA. Thus, the IA’s natural frequency can be shifted to low frequency for active control application. The proposed IA with virtual mass is then applied to actively control a clamped beam’s vibration based on the velocity feedback control system. The experimental results show that the stability of the control system and the control performance can be improved significantly as the IA’s natural frequency is reduced with virtual mass.

NOMENCLATURE

- \( a_{a}(\omega) \) Complex amplitude of the proof-mass acceleration of the IA
- \( Bl \) Electromagnetic transduction coefficient
- \( C_{a} \) Suspension damping coefficient of the IA
- \( f_{n} \) Natural frequency of the IA
- \( g_{p} \) Positive or negative acceleration feedback
- \( G_{c} \) Open-loop FRF of the control path
- \( h \) Fixed feedback control gain
- \( i(\omega) \) Complex amplitude of the current flowing through the coil
- \( I(t) \) Current flowing through the coil in time-domain
- \( j \) Imaginary unit
- \( K_{a} \) Suspension stiffness of the IA
- \( L_{c} \) Inductance of the coil
- \( M_{a} \) proof mass of the IA
- \( M_{v} \) Virtual mass
- \( R_{c} \) Resistance of the coil
- \( T_{a} \) Blocked force per unit input voltage
- \( v_{a}(\omega) \) Complex amplitude of the proof-mass velocity of the IA
- \( v_{d}(\omega) \) Error sensor location velocity due to primary force
- \( v_{s}(\omega) \) Complex amplitude of base velocity of the IA
- \( V_{s}(\omega) \) Complex amplitude of voltage applied to the coil
- \( V_{in}(\omega) \) Complex amplitude of input voltage of the velocity feedback loop
- \( x_{a}(\omega) \) Complex amplitude of proof-mass displacement of the IA
- \( x_{b}(\omega) \) Complex amplitude of base displacement of the IA
- \( X_{d}(t) \) Proof-mass displacement of the IA in time-domain
- \( X_{b}(t) \) Base displacement of the IA in time-domain
- \( Z_{a} \) Undriven mechanical impedance of the IA
- \( \omega \) Circular frequency

1. INTRODUCTION

Inertial actuators (IAs) are efficient control units in active noise and vibration control systems for reducing sound and vibration in the low frequency range.\(^{1-9}\) The main advantage of the use of IAs is that there is no requirement of any other structure to react off.\(^{1,2,5}\) At frequencies above the natural frequency of the IA, it can effectively generate a constant actuation force and in-phase with the input driving voltage.\(^{9}\) Consequently, when an accelerometer with integrated output is colocated at primary structure, the IA can be used to perform velocity feedback control, which reduces the structure vibration by means of active damping.\(^{10,12}\)

At frequencies lower than the IA’s natural frequency, the force produced by the IA has 180° phase shift, thus a negative damping effect occurs, which may lead to instabilities in the feedback control system.\(^{4,9}\) It means that the IA’s natural frequency should be far below that of the structure under control in order to ensure good stability margins and improve
its control performance. However, the IA with low natural frequency normally increases the unwanted weight of the IA proof-mass or causes unwanted static displacement of the proof-mass with the lower stiffness of the IA spring.

For velocity feedback control system by using IA with collocated accelerometer sensor, there are two approaches to improve the control performance limitation of the IAs. The first approach is to develop the new type IAs which have much lower natural frequency. For examples, Olaru et al. used a differential magnetic spring to replace the classical elastic suspensions in IAs. Braghin et al. introduced a new design method for the supporting structure of the magnetostriuctive spring, thus the IA can extend its lower frequency working ranges by reduction of its natural frequency. Kras and Gardonio imposed a flywheel element into the traditional IA, thus the IA’s natural frequency moved to lower frequency. This new type flywheel inertial actuator has been experimentally verified. Zilletti presented a theoretical study on an IA with an inerter element. It is found that the IA’s natural frequency can be shifted down because of the inerter. The second approach is to modify the velocity feedback control loop to make the control system perfectly collocated. For examples, Benassi and Elliott imposed additional internal displacement feedback loop to overcome the static deflection problem of the IA with very low natural frequency. Diaz et al. extended this approach for the control of human-induced vibration of pedestrian structures. Rohlfing et al. presented a new modified velocity feedback loop for IAs by adding an additional compensator filter. Diaz and Reynolds imposed a feed-through term into the velocity feedback controller, thus a robust control system has been designed by using the IA as control unit.

Recently, an IA, with a proof-mass acceleration feedback used as the tuneable vibration absorber, has been proposed and experimentally verified. The IA’s natural frequency can be moved to high or low frequency under positive or negative proof-mass acceleration gains, which can produce virtual mass effect. In this study, the IA with virtual mass is extended for active control application. Different to preview studies, the proposed control system includes an IA proof-mass acceleration feedback loop and a velocity feedback loop, as presented in Fig. 1. The use of a proof-mass acceleration feedback loop has the effect of producing the virtual mass effect, thus the IA’s natural frequency can be reduced. This allows the velocity feedback loop with collocated integrated accelerometer output on the vibrating structure has higher feedback control gain and hence higher active damping.

The remainder of this study is organized as follows. In Section 2, the theoretical mathematical model for the IA with virtual mass by using acceleration feedback is presented. In Section 3, the tuning capability, and the blocked force response of the IA by using proof-mass acceleration feedback are experimentally assessed. Section 4 presents the velocity feedback control experiments by using proposed IA in a clamped-clamped beam. Finally, some useful conclusions are provided in Section 5.

2. THE IA WITH VIRTUAL MASS BY USING PROOF-MASS ACCELERATION FEEDBACK

A generic schematic for an IA, with virtual mass by using proof-mass acceleration feedback considered in this study, is presented in Fig. 2. An accelerometer, which was positioned at the IA proof-mass, was used as the error sensor. This means that the feedback signal is the proof-mass acceleration, which was used to shift the IA’s natural frequency.

The IA shown in Fig. 2 consisted of a proof mass \( M_a \), mounted on a damping \( C_a \) and a suspension of stiffness \( K_a \) in parallel. It was assumed that the actuator base mass \( M_b \) was very small and thus could be neglected. A linear lumped parameter model was used in this study assuming the time harmonic vibration motion of the form \( \exp(j\omega t) \), where \( \omega \) was the circular frequency and \( j \) was imaginary unit. Two differential equations were used to describe the IA’s coupled electro-mechanical behaviour. The governing equation of motion of the proof mass \( M_a \), and the equation of the electrical circuit of the IA presented in Fig. 2 was expressed as

\[
M_a \ddot{X}_a(t) + C_a \left[ \dot{X}_a(t) - \dot{X}_s(t) \right] + K_a \left[ X_a(t) - X_s(t) \right] = Bl \cdot I(t) \tag{1}
\]

\[
L_e \ddot{I}(t) + R_e I(t) = V_e \exp(j\omega t) - Bl \cdot \left[ \dot{X}_a(t) - \dot{X}_s(t) \right] \tag{2}
\]

where \( Bl \) was the electromagnetic transduction coefficient of
where $v$ is the complex amplitude of the proof-mass velocity and acceleration. $V_i$ was the amplitude of the voltage applied to the coil.

For harmonic vibration of the form $\exp(j\omega t)$, we set $X_a(t) = x_a(\omega)\exp(j\omega t)$;

$$I(t) = i(\omega)\exp(j\omega t);$$

(3a)

$$X_a(t) = x_a(\omega)\exp(j\omega t);$$

(3b)

$$X_v(t) = x_v(\omega)\exp(j\omega t);$$

(3c)

where $i(\omega)$, $x_a(\omega)$ and $x_v(\omega)$ were the complex amplitude of the electric current, the proof-mass displacement and the base displacement, respectively.

From Eq. (3a), the following equations can be obtained

$$\dot{I}(t) = j\omega \cdot i(\omega)\exp(j\omega t) = j\omega \cdot I(t);$$

(4a)

$$\dot{X}_a(t) = j\omega \cdot x_a(\omega)\exp(j\omega t) = v_a(\omega)\exp(j\omega t);$$

(4b)

$$\dot{X}_v(t) = -j\omega \cdot x_v(\omega)\exp(j\omega t) = a_v(\omega)\exp(j\omega t);$$

(4c)

$$\dot{X}_s(t) = j\omega \cdot x_s(\omega)\exp(j\omega t) = v_s(\omega)\exp(j\omega t);$$

(4d)

where $v_a(\omega) = j\omega \cdot x_a(\omega)$ and $a_v(\omega) = -j\omega^2 \cdot x_a(\omega)$ were the complex amplitude of the proof-mass velocity and acceleration, respectively. $v_s(\omega) = j\omega \cdot x_s(\omega)$ was the complex amplitude of the base velocity.

For brevity, the frequency dependence of the complex amplitudes will be omitted in the remaining part of the paper. By substituting Eqs. (3) and (4), into Eqs. (1) and (2), we got

$$M_a a + C_a (v_a - V_i) + K_a (x_a - x_b) = Bl \cdot i;$$

(5)

$$j\omega L i + R_e i = V_i - Bl \cdot (v_a - v_s).$$

(6)

From Fig.2, it can be found that $V_i$ was the sum of the proof-mass acceleration feedback voltage and an additional input voltage $V_{in}$, such as

$$V_i = V_{in} + a_v \cdot g_p;$$

(7)

where $g_p$ was the positive or negative acceleration feedback.

Substituting Eq. (7) into Eq. (6), the current in Eq. (6) was rewritten as

$$i = V_{in} + a_v \cdot g_p - Bl \cdot (v_a - v_s).$$

(8)

Substituting Eq. (8) into Eq. (5), we got

$$C_a v_a + \left( K_a + \frac{j\omega |B|^2}{j\omega L_e + R_e} \right) x_a + \frac{Bl}{j\omega L_e + R_e} V_{in} = \left( M_a - \frac{Bl \cdot g_p}{j\omega L_e + R_e} \right) a_v + \left( C_a + \frac{|B|^2}{j\omega L_e + R_e} \right) v_a + K_a x_a = (M_a + M_e) a_v + \left( C_a + \frac{|B|^2}{j\omega L_e + R_e} \right) v_a + K_a x_a;$$

(9)

$$a_v = -\omega^2 \left( C_a + \frac{K_a}{j\omega} + \frac{|B|^2}{j\omega L_e + R_e} \right) v_a + \frac{Bl}{j\omega L_e + R_e} V_{in} - \omega^2 (M_a + M_e) + j\omega \left( C_a + \frac{|B|^2}{j\omega L_e + R_e} \right) K_a.$$
natural frequency. Before applying the proposed IA as control unit in active vibration control system, it was necessary to examine the tuning ability of the IA’s natural frequency. A low-cost home-made IA was used in this study, as shown in Fig. 3. The physical parameters of this IA are presented in Table 1.

The blocked force response of the IA was experimentally measured when the actuator base was mounted on a rigid foundation, it means that the base velocity $v_s = 0$. From Eq. (11), it can be found that blocked force response was $T_a = \frac{f_a}{V_{in}} = \frac{M_a a_a}{V_{in}}$ for $v_s = 0$. An experimental setup presented in Fig. 4 was used to measure the blocked force response $T_a$ of the IA. A CA-YD-186 accelerometer (with weight of 28g), which is attached on the IA proof-mass, was used to sense the proof-mass acceleration. A COINV dynamic signal analyzer was used to create the excitation signal and acquire the frequency response functions (FRFs) between input voltage and proof-mass accelerations. A simple switch circuit, by using OP07 Voltage Operational Amplifier (by Analog Devices Inc.), was designed to feedback the negative or positive acceleration signal. Then a power amplifier with two input channels was used to input the excitation signal and acceleration feedback signal. It means that the sum of the acceleration signal and the excitation signal were used to drive the IA. The gain of each channel of the power amplifier was independent tunable, as presented in Fig. 4. By changing the gain of the acceleration feedback channel, the blocked force responses $T_a$, which equal to the measured acceleration signals multiplied the value of proof mass, under different acceleration feedback gains was determined. All measurements were performed at small input voltage level so that the mechanical behavior of the IA is effectively linear. Figures 5 and 6 show the measured and calculated blocked force response of the IA under different feedback gains of the acceleration signal, respectively. As presented in Fig. 5, because of the accelerometer mass, the IA’s natural frequency without feedback control ($g_p = 0$) was about 99.5Hz. When a negative acceleration feedback gain $g_p$ was applied, the IA’s natural frequency was able to move to a much lower frequency.
and the peak amplitudes of blocked force responses were decreased as negative gain increased. Alternatively, the IA's natural frequency was moved to high frequency with positive gains. From Figs. 5 and 6, it can also be found that the phases of the blocked force responses shifted to $180^\circ$ and the amplitudes fell off below the natural frequency, as expected. Above the IA's natural frequency, the amplitudes and phases of the blocked force responses were reasonably flat up.

The measured and numerical calculated open-loop frequency response in Nyquist format, from IA voltage input to signal output from the accelerometer on proof-mass, is also shown in Fig. 7. From Fig. 7, it can be found that the ideal IA was unconditionally stable with negative acceleration feedback gain because of the instability point (-1, 0) is excluded from the loop. However, the measurement Nyquist plot indicates the feedback loop is not unconditionally stable due to the dynamic of the power amplifier. The natural frequency of the IA used in this study was reduced from 99.5Hz to 40Hz when the negative acceleration feedback gain $g_p$ was tuned from 0 to -3.5. The calculated and experimental results presented in Figs. 5–7 confirm that the IA's natural frequency can be reduced significantly by using negative proof-mass acceleration feedback loop as virtual mass.

4. EXPERIMENTAL INVESTIGATION OF VELOCITY FEEDBACK CONTROL

4.1. Experimental setup

In this section, with an example of an aluminium clamped-clamped beam, the control performances of the proposed IA with virtual mass as control actuator were investigated for a velocity feedback control system. The basic idea is shown in Fig.1 and the experimental setup is shown in Fig. 8. The IA with virtual mass mounted near one clamped end was used to control the beam vibration. The size and mass of beam were $580 \times 50 \times 7$ mm and 550g, respectively. The beam was excited by a sweep-sine signal with frequency ranges from 0 – 800Hz by using another IA near another clamped end. The mass of the control IA including accelerometer on the proof-mass was 100g, which equals to 18.18% weight of the beam. Another accelerometer was mounted at centre of control IA location (on the bottom of the beam), its integrated output was used as sensor signal in velocity feedback loop, as shown in Fig. 8.

4.2. Stability and control performance

According to Refs. $^5,^6,^{18},^{24}$ the velocity feedback control system was modelled in terms of the classic disturbance rejection feedback block diagram, as presented in Fig. 9. Since the velocity feedback controller used in this study was not modal-based, it means that the frequency response of the control loop was sufficient to analyze the stability and the state-space model is not needed. From Fig. 9, it can be found that the velocity at
error sensor location can be expressed as

\[ v_s = V_{in} G_c + v_d = -h v_s G_c + v_d; \]  

(14)

where \( V_{in} = -h v_s \) was the control voltage and \( h \) was the fixed feedback control gain. \( G_c = \frac{v_s}{V_{in}} \) was the open-loop FRF between the error sensor velocity \( v_s \) and the control input voltage \( V_{in} \). \( v_d \) was the velocity at error sensor location due to primary excitation source.

From Eq. (14), it can be found that the responses at the error sensor location with feedback control loop was expressed as

\[ v_s = \frac{v_d}{1 + h G_c}. \]  

(15)

The objective of the controller in this study was to minimize the velocity at sensor location under certain stability margin. Notice that the practical sensor–actuator pair is not perfectly dual and collocated, it means that the velocity feedback control loop in this study is not unconditionally stable. Furthermore, the open-loop FRF \( G_c \) can be modified under different virtual masses (or negative acceleration feedback gains). The open-loop FRFs \( G_c \), between the voltage input to the control IA and the integrated beam-accelerometer outputs under different virtual masses were first discussed. The measured Bode and Nyquist plots of the open-loop FRFs are shown in Figs. 10 and 11, respectively.

The effect of adding the virtual mass on the IA’s natural frequency and the stability of the system can be found in both Bode and Nyquist plots. The first natural frequency of the beam (100Hz) was arranged to be closed to the natural frequency of the passive IA (99.5Hz, without virtual mass). Notice that the passive IA acted as a tuned vibration absorber at this situation, the mode splitting phenomenon can be found in Fig. 10. For open-loop FRF without virtual mass case, the peak at 77.5Hz was associated with the lowest mode of the coupled IA-beam structure and has a 180° phase shift in this case. It means that the control gain was very limited because a significant circle on the left-hand side of the Nyquist plot appears, as presented by the solid line in Fig. 11. However, when the proof-mass acceleration feedback loop was closed with feedback gains of 0.5 and 3, the control IA’s natural frequency was reduced to 80Hz and 47.5Hz, respectively. Then the peak associated with the lowest mode was changed to 70Hz and 47.5Hz, respectively. The circles on the left hand side of the Nyquist plot were reduced significantly when the control IA’s natural frequency was reduced. Much higher velocity feedback gains were used. It means that the control system successfully attenuated the structural vibration when the virtual mass (proof-mass acceleration feedback loop) was introduced in the control IA.

The velocity responses measured at the control location of the beam, normalized to the input voltage of the primary excitation IA are presented in Fig. 12. From Fig. 12, it can be
found that the passive IA acted as a tuned vibration absorber on the beam. The first mode of the beam was split into two new resonances at about 77.5 Hz and 120 Hz after mounting the passive IA without any feedback action, as presented by the dotted line in Fig. 12. Furthermore, the amplitudes of the 2nd and 3rd modes were also slightly reduced.

Then the control IA’s natural frequency was shifted down to 80 Hz by using proof-mass acceleration feedback with gain $g_p = 0.5$. In this case, when the velocity feedback control loop was closed with the maximum stable gain, a peak with large amplitude at 70 Hz appeared, as presented by the dot-dashed line in Fig. 12. The reduction of vibration was very limited, where it can even become worse than that of passive case around 70 Hz.

Finally, the control IA’s natural frequency was moved to 47.5 Hz by using proof-mass acceleration feedback with gain $g_p = 3$. In this case, when the velocity feedback gain approached the stability limit, about 8 dB vibration reduction was achieved for the first two modes at the error sensor location, the amplitude of the 3rd mode was also reduced 3.5 dB, as presented by the dashed line in Fig. 12. However, at 47.5 Hz, there was also some control enhancement of the vibration, due to the positive feedback in this frequency region caused by the phase response of the control IA.

5. CONCLUSIONS

In this paper, an IA with proof-mass acceleration feedback is used as a control unit for velocity feedback control approach. It is shown that the proof-mass acceleration feedback loop can be seen as a virtual mass, the effect of the virtual mass is to reduce the IA’s natural frequency. The virtual mass of the IA proposed in this study uses simple acceleration control technology, and it is easy to implement because there is no need of the changes in IA physical design.

With an example of a clamped-clamped beam, the velocity feedback control experiments are performed to check the control performance of the proposed IA with virtual mass. When the IA’s natural frequency is shifted from 80 Hz to 47.5 Hz, the stability of the velocity feedback loop and control performance can be improved significantly. It means that the IA with appropriate proof-mass acceleration feedback gain can provide better control performance, because the IA’s natural frequency is shifted to much lower frequency.

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REFERENCES


Experimental Investigation of Piezoelectric Micropumps with Single, Series or Parallel Pump Chambers

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Three types of piezoelectric micropumps following different configurations: single, series, and parallel connection, are developed and investigated. All the micropumps are fabricated by wet etching technology and sealed by high temperature glass bonding. They share the same dimension characteristic of diffuser/nozzle microchannels. Verifying the impact of adding series or parallel connected pump chambers on single chambers, as well as verifying the performance of the flow rate, pressure and piezoelectric transducer vibration of three micropumps have been examined. Through the comparisons between three kinds of micropumps, the results show that the flow rate of the micropumps with parallel connected pump chambers have a higher flow rate than that of micropumps with single and serial connected pump chambers under the same driving conditions. In addition, both the flow rate and pressure with the serial micropump are the lowest. The pressure of the micropump with single pump chamber is larger than other kinds of micropumps at certain driving frequencies. Consequently, increasing the pump chambers cannot always increase the performance of the micropump. This coincides with the theory analysis. Finally, the vibration performance of piezoelectric transducers with three micropumps have been carried out. The parallel transducer has a higher vibration displacement than the other two kinds of micropumps. These results have great potentials for integration into labs with a chip or microfluidic driven systems.

1. INTRODUCTION

The microfluidic systems play an important role in the industry along with the fast-growing economy, especially in the fields of chemistry, life science, biology and aerospace. For instance, these systems have been used to synthesize, separate and analyze cells, medicines and DNA, thus benefiting the development of new medicine and therapy. To successfully implement the microfluidic systems in these applications, the connection between microscale and macro environments is critical. In most cases, fluids are pumped through the system, and one of the most commonly used tools is the micropump. From the 1990s, A. Olsson put forward the first micropump with a diffuser/nozzle microchannel, followed by numerous micropumps being developed and improved for several decades, including a drug delivery micropump, an EHD driving micropump, a thermo-pneumatic micropump and an electroosmotic pump, etc.

Although the micropumps with a single pump chamber have been widely studied recently, micropumps with series and parallel connections of pump chambers have been proposed due to their favorable pumping performance. For example, Li Guo and Azarbadegan fabricated a micropump with parallel connected pump chambers when the measured flow rate is 151.7 µl/min. Hsu and Fangsheng Huang have tested the performance of a micropump with a series connected pump chamber. The results demonstrated that the series and parallel micropump possessed better performance compared to that of micropumps with a single pump chamber. However, it seemed that these studies did not compare the differences of working principle between single, series and parallel connected micropumps, which are the crucial points resulting in these performance differences. In this paper, these differences are discussed, and more detailed experimental data is given between three types of micropumps with the same diffuser/nozzle microchannels.

2. THEORY ANALYSIS

A schematic configuration of the piezoelectric micropumps with a single chamber, series and parallel connected pump chambers are shown in Fig. 1(a), (b) and (c), respectively. The performance of the micropump is based on a unique trait of the “diffuser-nozzle” microchannels, which has been shown to have lower flow resistance for diffuser direction flow than nozzle direction flow. Consequently, a reciprocating piezoelectric transducer motion results in a net flow from the left to right for the three kinds of micropumps.

The pressure $P$ and flow rate $Q$ of every inlet and outlet are shown in Fig. 1. The conductivity coefficient $C$ for every flow direction have been listed in Fig. 1.

The flow rate in the nozzle direction flow is considered to be proportional to the pressure difference between the inlet and the outlet of the nozzles:

$$ Q = C(\Delta P). $$
The flow rate in the left nozzle for all three kinds of connected manners is:

\[ Q_1 = C_1(P_{12} - P_1); \]  \hspace{1cm} (2)

while the flow rate in the right one is:

\[ Q_2 = C_2(P_{12} - P_2). \]  \hspace{1cm} (3)

Since the fluid is considered to be impressible, therefore the sum of the flow rate should be equal to the change rate of the volume of the pump chamber, \( V_{12} \), namely:

\[ Q_1 + Q_2 = V_{12}. \]  \hspace{1cm} (4)

The change rate of volume of the pump chamber volume is usually assumed to be equal: \(^{22}\)

\[ V_{12} = v_0 w \sin \left( \frac{2\pi t}{T} + \phi \right); \]  \hspace{1cm} (5)

where \( w = 2\pi / T \) is the angular frequency, and \( \phi \) is a phase shift, in this paper the phase shift is assumed to be equal to 0 (\( \phi = 0 \)) for every pump chamber, then \( V_{34} = V_{12} \).

In addition, because of the simultaneous influence of the diffuser and nozzle direction flow on the central microchannel within the micropump with the series connected pump chambers, and the fact that at zero pressure head \( P_3 - P_1 < P_1 - P_2 \) both in “supply” and “pump” mode.

So:

\[ Q_4 < Q_2 < Q_{34}. \]  \hspace{1cm} (14)

That means the highest flow rate comes from the parallel micropump, followed by a single micropump, whereas the flow rate of the series micropump is the lowest.

The pressures between single and series micropumps have been compared, in “supply” mode when the piezoelectric transducer vibrates forward upside, leading to a vacuumed pump chamber. As a result, \( P_3 < P_1 \), thus \( P_{12} > P_{34} \). On the contrary, for “pump” mode, the result is \( P_{12} < P_{34} \). Hence in the whole working process, resulted in:

\[ P_{12} < P_{34} < P_{12} + P_{34} \quad \text{or} \quad P_{12} > P_{34} < P_{12} + P_{34}; \]  \hspace{1cm} (15)

where, \( P_{12}, P_{34}, \) and \( P_{12} + P_{34} \) represent the measured pressure for single, series and parallel micropumps, respectively.

3. DESIGN AND FABRICATION

Three types of micropumps have been designed to investigate their pumping performance with varying operation conditions. Figure 2(a), (b) and (c) show the schematic diagram of the micropumps with single, series and parallel connected pump chambers, respectively. The geometry characteristics of the pump chambers (the diameters of P1 and P2 are 10 mm), diffuser/nozzle microchannel (the length is 1.906 mm) and the inlet/outlet chambers (the diameter is 4 mm) are kept identical for all micropumps. \(^{22}\) The depth is 0.3 mm for the microchannels and inlet/outlet holes, whereas the depth is 1.7 mm for the

Figure 1. Schematic of micropump with (a) single, (b) series and (c) parallel pump chambers.
4. EXPERIMENT

4.1. Micropump Performance Measurement

The performance test of three micropumps include the flow rate and pressure measurement with sine, triangle and square driving signals combinations. The schematic of testing is shown in Fig. 5. To simultaneously actuate two PZT driving transducers P1 and P2 for the series and parallel micropumps, two set of driving and testing equipment including the signal generators, oscilloscope, analytical balances and voltage amplifiers are applied. The fluid medium is deionized water (DI water). The flow rates of the three micropumps were measured by the same procedure in Fig. 5(b). First, the inlet and outlet pipe were placed on the same plane, then the micropumps were actuated for $\Delta t = 2$ min without any difference in height between the inlet and outlet reservoirs (zero pressure head). Finally, the mass of liquid in the outlet reservoir was determined using an analytical balance, and the flow rate was calculated as:

$$ \phi = \frac{W \times 10^6}{\rho \Delta t} \text{ (}\mu\text{l/min)}; \quad (16) $$

where $\phi$ is the flow rate, $W$ is the weight of liquid in the outlet reservoir (in g), and $\rho$ is the liquid density.

The pressures of three micropumps were measured by placing the outlet pipe in a vertical position and measuring the li-
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Figure 6. Phase (a), duty cycle (b), and symmetry (c) of the sine, square, and triangle driving signals.

uid height, $\Delta Z$, in Fig. 5(a), from which the pressure was calculated as:

$$\Delta P = \Delta Z \rho g; \quad (17)$$

where $\Delta P$ is the pressure head at zero flow rate, and $g = 9.80 \text{ m/s}^2$ is the acceleration due to gravity.

The phase for sine signal driving is set to be $0^\circ$; the duty cycle of the square signal and symmetry of the triangle signal are 50% as shown in Fig. 6 from DC power supply (DG1022, RIGOL, China). Moreover, the driving signals imposed on P1 and P2 are synchronized. The driving voltage is swept from 90 Vpp to 130 Vpp, in 20 Vpp steps (here, Vpp means peak-to-peak value of the driving voltage). The driving frequency is swept from 0 Hz to 600 Hz.

4.2. Vibration Measurement of Piezoelectric Transducer

The vibrational performance of the piezoelectric transducers incorporated into three micropumps is tested under forced vibration conditions with PSV400 scanning vibrometer.33–35 Fig. 7 shows the schematic of vibration displacement. The driving signals on P1 and P2 are similar to the working conditions. The material for the piezoelectric transducer is PZT-5H. The detailed parameter information is shown in table 1.

5. RESULTS AND DISCUSSION

The performances of three micropumps with different driving voltages, frequencies and signals are performed on P1 and P2. For a better visualization, the results have been interpreted in terms of three aspects: flow rate, pressure and vibration performance. All datum was conducted and processed by Matlab software.

5.1. Flow Rate Comparison

The flow rates of micropump chambers are depicted in Fig. 8 as follows: single, solid and black line, series, dotted and red line and parallel dash and blue line. The driving signals imposed on P1 and P2 are a sine signal as shown in Fig. 8(a), a triangle signal as shown in Fig. 8(b) and a square signal as shown in Fig. 8(c). Three driving voltages (90 Vpp, 110 Vpp and 130 Vpp) are applied as shown in Fig. 8(a), (b) and (c). The higher flow rates can be obtained through the increase of the driving voltages regardless of the type of the micropump and the input signal as shown in Fig. 8. This can be attributed to the larger displacement resulting from high driving voltage.

But when it comes to the frequency dependence, the flow rate increased at first, and then decreased along with the frequency that increased under three kinds of voltages and signals. There is a peak value (maximum flow rate value) on every flow rate curve. That is because the resonant frequency is around 50–400 Hz, at which the volume efficiency reaches the biggest. Furthermore, the flow rate generated by a parallel micropump is higher than that of single and series micropumps under the same driving conditions. The flow rate generated by a series micropump is the least among the three signals driving. These results are in accordance with the theory analysis in Section 2. Consequently, adding parallel connected pump chambers can increase the flow rate, while adding series connected chambers reduces the flow rate.

5.2. Pressure Comparison

Figure 9 shows the comparison between the pressures generated by different micropumps under sine, triangle and square signals driving signals. Similarly, the driving voltages are set to be 90 Vpp, 110 Vpp and 130 Vpp. A similar change in the trend of pressures for the three types of micropumps was found for flow rates. Specifically, the pressure increases at first, and then decreases as the frequency increases. Moreover, the pressure generated by the parallel micropump is higher than that of the single in low frequency, whereas when the frequency is 200 Hz higher, the pressure of the single micropump is found to be larger than that of the parallel micropump, which does not agree with the trend for flow rates. Here, the reason can be the fact that vibration performance is weaker at a lower frequency.
so the pressure of parallel micropump is almost twice of that of the single micropump according to Eq. (15). When the frequency increases and exceeds the resonant frequency of the PZT transducer resonant frequency, the vibration performance will be anabatic, also inlet/outlet holes for measured pressure is far away from the pump body as shown in Fig. 3(c), therefore the fluid flow becomes out of order for the parallel micropump. As a result, the pressure decreases. Note: the pressure generated by the series micropump is still the least among three micropumps.

5.3. Vibration Deformation Comparison

The vibrational performances are also measured for single, series and parallel micropumps. The same driving signals are imposed on P1 and P2 as sine-sine, triangle-triangle and square-square. The driving voltages are 110 Vpp for P1 and P2. The phase, duty cycle and symmetry are $0^\circ$ and 50% for sine, square and triangle single.

As shown in Fig. 10, the vibrational displacements of the P1 and P2 piezoelectric transducers on parallel micropumps are the highest, followed by that from the single micropump lying in between P1 and P2. The PZT displacement for the series micropump is the lowest among the three micropumps. The vibration displacement of PZT responds to micropump efficiency according to the volume pump theory directly. Due to the fact that larger vibration displacement induces larger volume efficiency, this explains why the flow rate and pressure of the serial micropump are the lowest among the three kinds of micropumps. This emphasizes that increasing the capacity in the pump chamber sometimes cannot increase the micropump performance.\textsuperscript{24,27} Comparing Figures 8, 9 and 10, it is seen that there are some differences regarding the resonance frequency due to the flow rate and head pressure having been measured in the outlet that is reflected in the total performance of the micropump as shown in Figs. 8 and 9. However, the displacement of every PZT transducer with three kinds of micropumps have been measured as shown in Fig. 10. The displacements reached the maximum values when the frequency was below 200 Hz. The optimal performance with a low driving frequency is in accordance with the flow rate and pressure head.

6. SUMMARY AND CONCLUSIONS

In this paper the micropumps with single, series and parallel connected pump chambers have been designed and fabricated. The flow rate and pressure of three micropumps have been measured and analyzed. In order to figure out the reason for changing pressure and flow rate between three kinds of micropumps, the vibration displacement of the PZT transducer with three kinds of micropumps have been tested and discussed. The results are as follows:

(1) The change trend of the flow rate and pressure for the single, series and parallel micropumps are similar. Specifically, it increased at first, and then decreased when the frequency increased. There was a maximum flow rate and a pressure value on each curve.

(2) The flow rate and pressure generated by the parallel micropump are the largest among three micropumps (except for 200 Hz higher frequency) under the same driving conditions, with those of series micropump being the least. The experi-
Experimental results coincided with the theory analysis.

(3) The vibration displacement of parallel and single micropumps were larger than that of the series micropump for both P1 and P2. The displacement of the single micropump was between the P1 and P2. These results explained changing trends of the flow rate and pressure with three kinds micropumps again.

Overall, these three kinds of micropumps have their own variety of applications. For example, in the situations where the microinjected is needed in the microfluidic systems, or the use prohibits the application of electric fields in the microchannel where the fluid flow and pressure based flow are particularly desirable.

Figure 8. Flow rate of single, series and parallel micropumps with three kinds of signal driving.

Figure 9. Pressure of single, series and parallel micropumps with three kinds of signal driving.

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**Figure 10.** Vibration displacement of single, series and parallel micropumps with sine (a), triangle (b) and square (c) driving signals.

**REFERENCES**


A New Intelligent Weak Fault Recognition Framework for Rotating Machinery

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The presence of strong background noises makes it a challenging task to detect weak fault characteristics in vibration signals collected from rotating machinery. Thus, a two-stage intelligent weak fault recognition framework, which includes signal enhancement and intelligent recognition, is proposed in this work. The signal enhancement is accomplished via an optimized relevant variational mode decomposition (ORVMD) algorithm. Specifically, the optimal parameters is derived by combining a particle swarm optimization (PSO) algorithm and the novel defined relevant energy (Re) index. This optimized VMD algorithm can extract the principal components from the raw signals. Then, the enhanced vibration signals via the proposed ORVMD are converted into spectral signals and fed into an improved stacked auto-encoder (ISAE) algorithm for fault recognition. Experimental results demonstrate the effectiveness of the proposed algorithms and fault diagnosis framework in rotating machinery fault recognition and detection.

1. INTRODUCTION

Large-scale rotating machinery, such as steam turbines and wind turbines, are widely used in current industrial fields. They play a vital role in economic production and industrial manufacturing. However, rotating machinery is prone to failure, which could damage property and human life. Early detection or diagnosis of the potential weak fault in these systems is desirable to cut down the economic loss of industrial production and increase the benefits.

The vibration signals captured from rotating machinery are mostly non-stationary and consist of multiple-component aliased vibration signals. These signals are usually affected by the attenuation of the vibration in the transmission path and background noise. These signal characteristics seriously impede the fault information extraction from the raw industrial vibration signals. Compared with the severe or late-in-life faults of rotating machinery, it is a more challenging topic to extract and recognize weak fault feature information in advance.

Moreover, the multiple-component vibration signals are separated into many single component signals from each vibration source. If the fault features can be effectively extracted from the noises, fault diagnosis techniques become more effective.

The weak fault diagnosis method based on time-frequency analysis is widely used in the field of fault diagnosis and detection of rotating machinery. It is mainly divided into two modes: one is recursive signal decomposition, and the other is non-recursive signal decomposition. For the recursive signal decomposition, Huang et al. proposed an adaptive signal processing method called empirical mode decomposition (EMD). In the EMD, a given multiple-component signal is decomposed into a series of components under different frequency bands. However, there still exist some problems, such as mode mixing, which will result in obstacles on signal decomposition and reconstruction. Inspired from EMD, Smith et al. proposed a new time-frequency signal decomposition method, namely local mean decomposition (LMD), which can adaptively decompose a given nonstationary signal into a linear combination of multiple product function (PF) components. Su et al. developed an early fault diagnosis method based on EMD and spectral kurtosis. Bin et al. constructed a feature extraction approach built on the combination of wavelet packet decomposition (WPD) and EMD. Li et al. proposed an LMD algorithm based on differential rational splines (DRS-LMD) for early fault diagnosis in rotating machinery. At the same time, Kullback-Leibler (K-L) divergence was applied to select the sensitive product function (PF) component signals. Due to the interference of background noise and amplitude modulation effects, the weak faults in planetary gear systems are difficult to detect. To detect these weak faults in planetary gearboxes, a comprehensive diagnostic method called improved maximum correlation kurtosis deconvolution (IMCKD) was proposed by Zhang. The weak root crack in the single-stage planetary sun gear was used to prove its effectiveness. Cheng et al. proposed an improved symplectic ge-
ometric mode decomposition (ISGMD) that contains the vast majority of fault feature information for the early processing of gear fault signals. Simulation and experimental results suggest that ISMGD is effective for early fault signal decomposition. To reveal the dynamic characteristics of the rubbing fault signal of a rotor system, Yu et al. proposed a second-order multi-synchronous compression algorithm, which embedded the second-order two-dimensional instantaneous frequency estimation into multi-synchronous compression.

The signal decomposition algorithms are all recursive mode. For the recursive signal decomposition, Dragomiretskiy developed an adaptive signal decomposition algorithm known as variational mode decomposition (VMD). In this method the component decomposition is iterative and an optimal search is performed to determine the center frequency and bandwidth of each component. Compared with other existing recursive “filtering” mode methods, the VMD algorithm has a solid mathematical and theoretical basis. Hence, the given signals are decomposed more accurately via VMD.

Currently, VMD has been widely applied in fault diagnosis, and has demonstrated its capability in signal decomposition and feature extraction. Mohanty et al. utilized the advantage of VMD to decompose vibration signals of the rolling bearing into a series of IMF components signals from which the different health conditions of rolling bearing signals were recognized. And Yang showed the noise signal of hydro power units was decomposed via VMD, then the decomposed signal was naturally reconstructed. Wang et al. used VMD to successfully recognize the different friction and collision faults of a rotor system.

Although VMD has been successfully applied to the intelligent fault diagnosis, the parameter optimization and component selection issues remain the most influencing factors for the decomposition performance. In other words, VMD has two important parameters, namely, the number of components \( K \) and the penalty factor \( \alpha \). If these two parameters are not optimized, some errors of signal decomposition will occur and cut down diagnostic accuracy. In addition, it is of great concern to select the sensitive and relevant principal components of VMD from signal decomposition.

To overcome the deficiencies of VMD, a novel optimization algorithm, called particle swarm optimization (PSO), is employed to optimize the two parameters of VMD in this work. A new well-defined relevant energy (Re) index is also applied to evaluate the reconstructed signals. A designed signal enhancement method (VMD with parameter optimization and components selection index Re) called optimized variational mode decomposition (ORVMD) is presented in this study.

Furthermore, the intelligent recognition algorithm is urgently needed to implement the weak fault detection in an automatic approach. As one of the most advanced data processing and pattern recognition methodologies, deep learning (DL) has been broadly applied in many fields such as machine vision, speech processing and intelligent fault diagnosis. Among them, auto-encoder (AE) is one of the more widely applied deep learning architectures that can effectively extract the underlying data into high-level and meaningful features. For instance, Sun et al. proposed a novel induction motor fault diagnosis based on a sparse auto-encoder (SAE). In the work of Junbo et al., a novel intelligent fault diagnosis of rolling bearings built on wavelet transform and AE was developed to enhance the availability of fault diagnosis.

It has been demonstrated that the traditional SAE algorithm is very sensitive to industrial noise. Currently, the sparsity and generalization are beneficial to signal processing and fault diagnosis. As is well-known that \( l_1 \) norm can make the data sparser, and \( l_2 \) norm is employed to prevent over-fitting of data. Therefore, if the advantages of the \( l_1 \) and \( l_2 \) norms are combined, the sparse and generalization performance can be simultaneously improved. To comprehensively utilize the above-mentioned two types of regularization techniques, a norm combination (termed \( l_1/l_2 \)) is developed to strengthen the generalization of SAE. The so-called improved stacked auto-encoder (ISAE) with the generalized \( l_1/l_2 \) norm is constructed in this work.

Specifically, the advantages of the proposed ORVMD algorithm in weak signal processing and the ISAE algorithm in intelligent fault recognition are integrated in this work. A novel intelligent weak fault diagnosis framework based on ORVMD and ISAE algorithms for fault diagnosis of rotating machinery is proposed. The main contributions of this work are outlined as:

1) An optimized VMD algorithm (called ORVMD) is proposed to overcome the deficiencies of the original VMD, which can remove redundant and irrelevant fault components and implement weak signal enhancement.

2) To utilize two types of regularization norms, a new improved stacked Auto-encoder (ISAE) with generalized \( l_1/l_2 \) norm is constructed.

3) A new intelligent weak fault recognition framework of rotating machinery built on ORVMD and ISAE is proposed. And this framework is validated with two case studies of vibration signals from roller bearings.

The remainder of this paper is organized as below. A brief review of the background theories is provided in Section 2. In Section 3, two proposed algorithms (ORVMD and ISAE) are presented. In Section 4, the intelligent weak fault diagnosis framework of rotating machinery via ORVMD and ISAE is proposed. In Section 5, the proposed algorithm and fault diagnosis framework are validated with the two cases of experimental signals from roller bearings. Conclusions are given in Section 6.

2. BACKGROUND

2.1. Brief Review of the General VMD

To obtain an intrinsic mode function (IMF), VMD gets rid of the circular stripping signal used by EMD, and the signal decomposition of VMD is converted into the variational framework. The adaptive decomposition is implemented through an optimal solution search constraint variational model, and then the center frequency and bandwidth are updated. If the input signal \( f(t) \) is composed of a series of components with different bandwidths, then the sum of the components is equal to the constraint of the original input signal \( x(t) \). Ultimately, the aggregation bandwidth sum of the components is sought to
be minimum. Its corresponding constraint variational model is rewritten as
\[
\min_{\{u_k\},\{w_k\}} \left\{ \sum_k \left\| \partial_t \left[ (\delta(t) + \frac{j}{\pi t}) * u_k(t) \right] e^{-jw_k t} \right\|^2_2 \right\}; \\
\text{s.t. } \sum_k u_k = f;
\]
where \(\{u_k\}\) represents IMF components, \(\{w_k\}\) is the frequency center of each component. \(\delta(t)\) is the Dirichlet function.

To resolve the above-mentioned constrained variational phenomenon, the Augmented Lagrange (AL) function is introduced as
\[
L(w_k, u_k) = \alpha \sum_k \left\| \partial_t \left[ (\delta(t) + \frac{j}{\pi t}) * u_k(t) \right] e^{-jw_k t} \right\|^2_2 + \\
\left\| f(t) - \sum_k u_k(t) \right\|^2_2 + \left\langle \lambda(t), f(t) - \sum_k u_k(t) \right\rangle;
\]
where \(\alpha\) is the penalty parameter, and \(\lambda\) is the Lagrange multiplier. Subsequently, the so-called Alternate Direction Method of Multipliers (ADMM) is employed to obtain the “saddle point”. By performing a loop, \(u_k\) and \(w_k\) are timely updated. For the sake of finding the optimal solution of the constrained variational model in Eq. (2), the original signals are decomposed into \(K\) narrow-band components. Thus, the value of \(u_k\) is described as
\[
\begin{align*}
\hat{u}^{n+1}_k &= \arg \min \left\{ \alpha \sum_k \left\| \partial_t \left[ (\delta(t) + \frac{j}{\pi t}) * u_k(t) \right] e^{-jw_k t} \right\|^2_2 + \\
&\left\| f(t) - \sum_k u_k(t) \right\|^2_2 + \left\langle \lambda(t), f(t) - \sum_k u_k(t) \right\rangle; \right. \end{align*}
\]
where Eq. (3) is transformed into the frequency domain, the updated frequency domain of each mode is obtained as
\[
\hat{a}^{n+1}_k(w) = \frac{\hat{f}(w) - \sum_{i\neq k} \hat{a}(w) + \frac{\lambda(w)}{2}}{1 + 2\alpha(w - w_k)^2}.
\]

According to the same calculation process, the central frequency is converted into the frequency domain. Its central frequency is updated by
\[
w_{\gamma+1}^{n+1} = \frac{\int_{-\infty}^{\infty} w |\hat{u}_k(w)|^2 dw}{\int_0^{\infty} |\hat{u}_k(w)|^2 dw};
\]
where \(u_{\gamma+1}^{n+1}\) is equivalent to the Wiener Filter for the current residual amount \(f(w) - \sum_{i\neq k} \hat{a}(w)\) where \(w_k\) is the power spectrum center of the current mode function. \(\{u_k\}\) is calculated using an inverse Fourier transform.

2.2. Brief Review of the General SAE

Traditional signal analysis has difficulty accurately diagnosing a fault when the fault impulse signal is not obvious. Thus, an intelligent diagnosis based on DL is needed to implement fault recognition. SAE is composed of several AEs. Where the mapping function from inputting layer to encoding layer is usually a nonlinear function, which is described as
\[
h_i = f(x_i) = \frac{1}{1 + \exp(-(w\xi + b))};
\]
where \(w\) is the weight between layers and \(b\) is the deviation. Similarly, the nonlinear function from the encoding layer to the decoding layer can be written as
\[
x'_i = g(h_i) = \frac{1}{1 + \exp(-(w'\xi + b'))};
\]
where \(w'\) is weight, and \(b'\) is the deviation. The objective optimization is described as
\[
J = L(x'_i, x) = \min \frac{1}{N} \sum_{i=1}^{N} \left\| x'_i - x_i \right\|.
\]

3. THE PROPOSED ALGORITHMS: ORVMD AND ISAE

3.1. Improved Signal Enhancement Method: ORVMD

Compared with the general VMD algorithm, the core of the proposed ORVMD can take advantage of parameter optimization and the new well-defined index to select sensitive principal components. The specific step of the ORVMD method are presented in the remainder of this section.

3.1.1. Parameter Optimization and Signal Decomposition

The penalty coefficient \(a\) and the number of decomposition \(k\) in the VMD algorithm have a great influence on signal decomposition and reconstruction. Owing that the signal decomposition and reconstruction are changeable, the parameter combination \([k, a]\) is difficult to be determined in advance. Particle swarm optimization (PSO)27,41 has a desired global optimization ability. To avoid the interference of human subjective factors, the PSO algorithm is used to enhance the parameter optimization capability of the VMD method in this work.

For \(D\) dimensional space, the population of \(N\) particles is \(XX = (XX_1, XX_2, \ldots, XX_N)\), the position of the \(i\)-th particle in the \(d\)-dimensional search space is \(XX_i = (x_{i1}, x_{i2}, \ldots, x_{iD})\). The velocity of the \(i\)-th particle is \(V_i = (v_{i1}, v_{i2}, \ldots, v_{iD})\), the individual local extreme value is \(P_i = (p_{i1}, p_{i2}, \ldots, p_{iD})\), the global extreme value is \(G = (g_1, g_2, \ldots, g_D)\), each particle can update its speed and position through local extremum and population global extremum, as described by
\[
\begin{align*}
v_{id}^{k+1} &= w v_{id}^k + c_1 \xi (g_{id}^k - x_{id}^k); \\
x_{id}^{k+1} &= x_{id}^k + v_{id}^{k+1};
\end{align*}
\]
where \(w\) is the inertia weight, \(k\) is the current iteration number. In the publication by Yan et al.,13 the envelope signals obtained after signal demodulation are processed into a probability distribution sequence, and the envelope entropy value
(EEV) is calculated according to the sparse characteristics of the original signals. After the fault signals are decomposed by the parameter optimized VMD, the sparsity of the component signals is weak, its envelope entropy value (EEV) is larger. Conversely, the envelope entropy (EEV) is smaller. As a result, the local minimum envelope entropy is deemed as the fitness value of the optimized VMD in this study.

3.1.2. Component Selection and Signal Reconstruction

To remove redundancy components, the defined component selection index combines the concept of relevance coefficient and energy ratio. The relevant coefficient \( R(x_i, x_j) \) is applied to describe the related degree between the statistical vector \( x_i \) and \( x_j \). \(^{42}\) Therefore, relevant coefficient is defined as

\[
R(x_i, x_j) = \frac{EX(x_i) - EX(x_i) - EX(x_j)}{\sqrt{D(x_i)}\sqrt{D(x_j)}},
\]

where \( EX(x_i) \) and \( EX(x_j) \) are mathematical expectation, \( D(x_i) \) and \( D(x_j) \) are variance. To be specific, the proposed \( Re \) index is defined as

\[
Re(\text{imf}(i)) = \frac{E(\text{imf}(i)) \times R(x, \text{imf}(i))}{E_{\text{total}}};
\]

where \( E(\text{imf}(i)) \) is the energy of component signals, \( E_{\text{total}} \) is the energy of the raw signals. Consequently, as the value of \( Re \) is increased, the correlation with the fault signal component is increased. Compared with the single relevance coefficient or energy ratio, the proposed \( Re \) index can efficiently reflect fault information. When the cumulative component index reaches more than 80%, signal reconstruction can be carried out.

In summary, the steps of the ORVMD algorithm are:

- Initialize the parameters of the PSO and the fitness function. The parameter combination \( [a, k] \) is regarded as a particle position, a certain parameter combination as the initial particle position can be randomly generated, its moving speed of each particle was initialized.

- The original signals are decomposed via VMD at different particle positions to calculate the fitness value of each particle position.

- Loop iteration, go to step 3 until the iteration number reaches the maximum setting value.

- After optimized combination parameters, the \( Re \) value of IMF components was calculated and sorted it from large too small. When the cumulative \( Re \) value reached to 80%, the filtered signal components are reconstructed to obtain the enhanced signals.

The proposed ORVMD algorithm is illustrated in Fig. 1.

3.2. Intelligent Fault Recognition Algorithm: Improved Stacked Auto-Encoder (ISAE)

It is illustrated that the SAE algorithm is not robust.\(^ {34-36} \) Fortunately, the sparsity and generalization are the increasingly important research topic in recent years. The generalized \( l_p \) norm and its combined norm are a large family of sparse measures.\(^ {34-37} \) The \( l_p \) norm can enhance the sparseness and generalized performance of data. The \( l_1 \) is a well-known norm, which can make the data sparser, and the \( l_2 \) norm can prevent over-fitting of data. To utilize the above-mentioned regularization techniques, the norm combination \((l_1/l_2)\), i.e. combining with \( l_1 \) and \( l_2 \) norms is developed to up-grade the generalization of the general SAE algorithm. This is termed as an improved stacked auto-encoder (ISAE). Therefore, the \( l_1 \) norm is expressed as the sum of absolute values of each element of \( w \) vector, which can be written as

\[
l_1 = \| w \|_1 = \sum_{i=1}^{n} |w_i|,
\]

where \( l_1 \) norm is usually sparse to choose eigenvectors with a small number. The \( l_2 \) norm is the 1/2 power of the sum of squares for each element of \( w \) vector. Accordingly, \( l_2 \) norm is called as Euclidean norm (Euclidean distance)

\[
l_2 = \| w \|_2 = \sqrt{\sum_{i=1}^{n} w_i^2}.
\]

\( l_p \) norm can prevent over-fitting of data. To utilize the above-mentioned regularization techniques, the norm combination \((l_1/l_2)\), i.e. combining with \( l_1 \) and \( l_2 \) norms is developed to up-grade the generalization of the general SAE algorithm. This is termed as an improved stacked auto-encoder (ISAE). Therefore, the \( l_1 \) norm is expressed as the sum of absolute values of each element of \( w \) vector, which can be written as

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\[
l_2 = \| w \|_2 = \sqrt{\sum_{i=1}^{n} w_i^2}.
\]
According to Vincent et al.,\textsuperscript{34} the $l_1/l_2$ norm can be obtained as
\[ \frac{l_1}{l_2} = \frac{\left\| w \right\|_1}{\left\| w \right\|_2} = \frac{\sum_{i=1}^{n} |w_i|}{\left( \sum_{i=1}^{n} w_i^2 \right)^{1/2}}. \]  
(14)

Moreover, according to the proposition by Jia et al.,\textsuperscript{37} $l_1/l_2$ norm can be normalized into a general framework, which is given as
\[ M_{l_1/l_2} = \log \left( \frac{2}{3} \right) \cdot \frac{\sum_{i=1}^{n} |w_i|}{\left( \sum_{i=1}^{n} w_i^2 \right)^{1/2}}. \]  
(15)

From the perspective of Bayesian learning, the regularization term of combination norm has such a function that if the neural unit of the neural network relates to the unit, it is prominent, otherwise, it is 0. Therefore, a combination norm is pushed under the condition of prior knowledge, which can make the model’s weight generalization ability stronger. To sum up, the new objective function of ISAE is defined as
\[ J_{\text{ISAE}} = \min \frac{1}{N} \sum_{i=1}^{N} \left\| x'_i - x_i \right\| + \beta M_{l_1/l_2}; \]  
(16)

where $\beta$ is the norm adjustment parameter. In the study by Zhao et al.\textsuperscript{33} and Sun et al.\textsuperscript{35} the training process of the ISAE is the same as the RBM by using the greedy layer-by-layer pre-training. The ISAE (as described in Fig. 2) is a deep neural network composed of multiple-layer improved auto-encoders.

4. THE PROPOSED INTELLIGENT WEAK FAULT RECOGNITION FRAMEWORK

Combining the advantages of the ORVMD and the ISAE, an intelligent weak fault recognition framework is proposed and divided into two stages:

Stage1: Weak signal enhancement stage

**Step1:** Several sensors are applied to capture the vibration signals from rotating machinery.

**Step2:** Initializing combination parameter and the fitness function, the optimized parameter combination is obtained.

**Step3:** The raw signals are decomposed and reconstructed via the ORVMD.

**Step4:** The enhanced signals are converted into spectral signals, which are divided into training samples and testing samples, respectively.

Stage2: Intelligent recognition stage

**Step5:** Initializing the parameters of the ISAE.

**Step6:** Using training samples to train the ISAE.


5. EXPERIMENTAL VALIDATION AND ANALYSIS

In this section, the availability of the proposed fault recognition framework is validated with two diagnostic cases of rolling bearing data set from the electrical engineering laboratory (Case Western Reserve University (CWRU)) and the full-life testing bench of Southeast University (Accelerated Bearing Life Test (ABLT-1A)), respectively.

5.1. Experimental Analysis with Rolling Bearing Weak Failure of CWRU

Generally, rolling bearing fault signals have strong non-stationary characteristics in practical industrial engineering. Thus, their fault characteristic information is weak. Consequently, the development and research of intelligent fault diagnosis technology have a positive practical significance.

For further validate the availability of the proposed framework in fault recognition, the experimental data used in this section originated from the rolling bearings experimental data of Case Western Reserve University (CWRU) in the United States. Specifically, the selective rolling bearing was a 6205-JEM SKF deep groove ball bearing. It is well-known that its internal diameter, external diameter, and thickness rolling element diameter are 25 mm, 52 mm, 15 mm and 8.18 mm, respectively. Subsequently, vibration signals of rolling bearing were collected at a load of 1 hp, a rotational speed of 1750 r/min, and a sampling frequency of 12 kHz. The failure of rolling bearing in this section was manufactured and its testbed is described in Fig. 4. Moreover, 7 kinds of rolling bearing health conditions were studied in this case study. They were the outer ring slight fault; outer ring serious fault; inner ring slight fault; inner ring serious fault; rolling element slight fault; rolling element serious fault and normal condition. Accordingly, the statistical summary of the data set is illustrated in Table 1. Among them, 50 groups were randomly taken as training samples for each health condition, while the remaining 50 groups were taken as testing samples.

To reveal different degrees of failure, the waveform of different kind of vibration signals in the time domain and unilateral spectrum frequency domain are drawn in Fig. 5. Weak fault signals are significantly affected by noise and irrelevant interference, and their impact characteristics are not obvious. Also, the resonance band in the spectrum was not obvious. Thus, weak faults were difficult to be observed.

According to Fig. 3, the proposed method can be roughly divided into two stages: weak signal enhancement and intelligent recognition. To decompose and reconstruct the weak signals and other signals, the proposed ORVMD was applied to reconstruct the raw signals. Limited by the length of this study, the vibration signals of inner ring weak failure were regarded as an example to decompose and enhance signals using the ORVMD algorithm.

### 5.1.1. Signal Enhancement by the Proposed ORVMD

To be specific, ORVMD algorithm can be devoted to decomposed and reconstruct the weak failure signals of rolling bearing inner ring. According to Yan et al. and Chenglin et al., the parameters of particle swarm optimization algorithm were set up as Table 2. The local minimum envelope entropy in the process of particle swarm optimization with the

<table>
<thead>
<tr>
<th>Table 1. The established rolling bearing data set under different health condition of CWRU.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fault types</td>
</tr>
<tr>
<td>Rolling element slight fault</td>
</tr>
<tr>
<td>Rolling element serious fault</td>
</tr>
<tr>
<td>Inner ring slight fault</td>
</tr>
<tr>
<td>Inner ring serious fault</td>
</tr>
<tr>
<td>Outer ring slight fault</td>
</tr>
<tr>
<td>Outer ring serious fault</td>
</tr>
<tr>
<td>Normal condition</td>
</tr>
</tbody>
</table>

### 5.1.2. Intelligent Recognition of Rolling Bearing Weak Failure

The parameter setting of the proposed ORVMD and ISAE algorithms.

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum evolutionary algebra (ORVMD)</td>
<td>$c_{max} = 10$</td>
</tr>
<tr>
<td>Population size $M$ (ORVMD)</td>
<td>$M = 10$</td>
</tr>
<tr>
<td>Maximum number of chaotic search (ORVMD)</td>
<td>$max_k = 10$</td>
</tr>
<tr>
<td>Acceleration constant (ORVMD)</td>
<td>$c_1 = 2$</td>
</tr>
<tr>
<td>Acceleration constant (ORVMD)</td>
<td>$c_2 = 2$</td>
</tr>
<tr>
<td>Optimization objective (ORVMD)</td>
<td>Envelope entropy</td>
</tr>
<tr>
<td>Local minimum envelope entropy (ORVMD)</td>
<td>Sixth generation</td>
</tr>
<tr>
<td>Best parameter (ORVMD)</td>
<td>$a = 830$</td>
</tr>
<tr>
<td>Best parameter (ORVMD)</td>
<td>$k = 7$</td>
</tr>
<tr>
<td>Learning rate (ISAE)</td>
<td>0.0146</td>
</tr>
<tr>
<td>Number of iterations (ISAE)</td>
<td>0.0391</td>
</tr>
<tr>
<td>Batch size (ISAE)</td>
<td>0.0424</td>
</tr>
<tr>
<td>Regularization (ISAE)</td>
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<tr>
<td>Sparsity Target (ISAE)</td>
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</tr>
<tr>
<td>Weight Penalty (ISAE)</td>
<td>0.2522</td>
</tr>
<tr>
<td>Norm parameter (ISAE)</td>
<td>0.3855</td>
</tr>
<tr>
<td>Activation function (ISAE)</td>
<td>tanh</td>
</tr>
</tbody>
</table>

<p>| Table 2. The parameter setting of the proposed ORVMD and ISAE algorithms. |</p>
<table>
<thead>
<tr>
<th>Parameter name</th>
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</tr>
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<tbody>
<tr>
<td>Maximum evolutionary algebra (ORVMD)</td>
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<tr>
<td>Acceleration constant (ORVMD)</td>
<td>$c_2 = 2$</td>
</tr>
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<td>Optimization objective (ORVMD)</td>
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</tr>
<tr>
<td>Norm parameter (ISAE)</td>
<td>0.3855</td>
</tr>
<tr>
<td>Activation function (ISAE)</td>
<td>tanh</td>
</tr>
</tbody>
</table>

<p>| Table 3. The index for IMF component signals. |</p>
<table>
<thead>
<tr>
<th>Component signals</th>
<th>Relevant energy ($Re$)</th>
<th>Relevant coefficient ($Re$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMF1</td>
<td>0.0154</td>
<td>0.2522</td>
</tr>
<tr>
<td>IMF2</td>
<td>0.0512</td>
<td>0.3855</td>
</tr>
<tr>
<td>IMF3</td>
<td>0.0674</td>
<td>0.4471</td>
</tr>
<tr>
<td>IMF4</td>
<td>0.1182</td>
<td>0.5122</td>
</tr>
<tr>
<td>IMF5</td>
<td>0.0424</td>
<td>0.3750</td>
</tr>
<tr>
<td>IMF6</td>
<td>0.1059</td>
<td>0.4942</td>
</tr>
<tr>
<td>IMF7</td>
<td>0.0391</td>
<td>0.3912</td>
</tr>
</tbody>
</table>
Figure 5. The waveform of vibration signals in the time domain and frequency domain.

Figure 6. The minimum envelope entropy of inner ring weak fault signal (CWRU) via ORVMD.

Evolution algebra is decomposed in Fig. 6. Apparently, the local minimum envelope entropy value was in the sixth generation. Thus, the best parameter combination can be selected as \([a, k] = [830, 7]\). Naturally, its inner ring weak failure vibration signals was decomposed and reconstructed via the ORVMD.

Specifically, component signals via the ORVMD are drawn in Fig. 7. Compared with other component signals, the regular impact in the waveform of the IMF4 was more obvious, and its frequency band center was just at 3000 Hz. For the vicinity of the resonance frequency, the envelope entropy of the IMF4 for all the components was also the smallest. Fur-
thermore, the envelope spectrum via envelope demodulation is described in Fig. 8 and Fig. 9, in which the line amplitude of the spectrum at the frequency multiplier and the characteristic frequency were prominent. The results demonstrated that the characteristic frequency features were accurately extracted.

The ORVMD with parameter optimization can decompose fault signals from high frequency to low frequency. Hence, its fault vibration signal feature information was expanded. Afterwards, the weak signal was reconstructed by the new components screened index (relevant energy, $R_e$). According to the concept of $R_e$ index, the $R_e$ index and relevant coefficient ($R_c$) between 7 components (IMF) and the original signal was calculated in Table 3.

The comparison with the original relevant coefficient and energy index is illustrated in Table 2. The new index ($R_e$) highlighted the correlation degree between different components and the original signal. The cumulative Relevant Energy index was equal to 0.8, which fully covered the main information for signal reconstruction. The reconstructed signal is displayed in Fig. 9. The envelope spectrum showed that its fault frequency is more prominent, which is observed in the figure. The reconstructed signals can be directly observed at the frequency component of 158.2 Hz at the generation. However, the above-mentioned signal processing methods are in the signal enhancement stage, but the enhanced signal processing methods still needed an effective intelligent recognition algorithm. In other words, the traditional vibration signal analysis based on time domain and frequency domain cannot quantify the characteristics of the mechanical fault information. Therefore, an effective fault recognition framework was required.
5.1.2. Intelligent Fault Recognition by the Proposed ISAE

According to Zhao et al.,\textsuperscript{44} ISAE was set as two-layer AE, and the number of nerve units was \(1024-400-100-6\). To sum up, the detailed parameters of ISAE are described in Table 2. For an experimental comparison, three kinds of signal processing methods, namely, the original signal, signal processed by VMD, and signal processed by ORVMD (regard as OS-ISAE, VMD-ISAE, and ORVMD-ISAE, respectively) were selected for comparison. Thus, the fault recognition results of the three methods are obtained in Fig. 10.

Compared with other frameworks, the proposed framework can achieve a higher diagnostic accuracy.

5.1.3. Feature Visualization via t-SNE

To visualize the above-mentioned diagnosis results, the last layer of extracted features was visualized by using t-SNE (t-distributed stochastic neighbor embedding).\textsuperscript{45} At present, t-SNE technology is widely used in data visualization, to understand the diagnosis model \{original signal + ISAE = FD1; VMD + ISAE = FD2; ORVMD + ISAE = FD3\}. T-SNE is applied to convert high-dimensional representation into a two-dimensional scatter plot. The above-mentioned three diagnostic methods can obtain the two-dimensional scatter distribution diagram of characteristic points through t-SNE as described in Fig. 11.

As illustrated in Fig. 11, only the ORVMD-ISAE based-diagnostic model was able to divide the different samples of various conditions, which indicates the effectiveness of the proposed framework.

5.2. Experimental Analysis with Rolling Bearing Weak Failure of ABLT-1A

5.2.1. Data Collection and Parameter Setting

In this section, the rolling bearing data collected via ABLT-1A were applied to validate the effectiveness of the proposed diagnosis framework. In view of this point, the wire cutting was employed to simulate different rolling bearing faults in this section, and the damaged bearings were installed on the accelerated bearing life tester (ABLT-1A provided by Hangzhou Bearing Test Research Center, its structure diagram is shown in the Fig. 12). Also, the corresponding vibration signals were collected under different health conditions. Correspondingly, the failure condition of rolling bearing is drawn in the Fig. 13.

The combination of the measuring instrument with data acquisition card and PCB acceleration sensor was used to pick up the five kinds of rolling bearing fault vibration signals under different fault conditions (i.e. inner ring fault, inner-outter ring compound fault, outer ring-ball compound fault, and outer ring-ball weak compound fault, which were marked as F1, F2, F3, F4, and F5, respectively) were used for signal acquisition. More precisely, its specific parameters of the experiment are represented in Table 4. At the same time, to facilitate the subsequent work, every 1024 points of each type of fault signals were intercepted as one fault sample, and 100 samples were obtained for each fault condition. 50 groups were randomly used as training samples, and the remaining groups are used as testing samples.
5.2.2. Signal Processing Results of Experimental Data

The waveform of the time domain and unilateral spectrum frequency domain are drawn in Fig. 14. Weak fault signals were greatly affected by noise and irrelevant interference, and their impact characteristics were not obvious. Also, the resonance band in the spectrum was not obvious. Thus, weak faults were difficult to be detected.

Moreover, Fig. 15 analyzed the simulation signal by using the proposed framework. Namely, the local minimum envelope entropy value located in the 7-th generation, the best parameter combination is \([a, k] = [905, 8]\).

The waveform of different components signals via ORVMD are described in Fig. 16. Compared with other components, the regular impact in the waveform of IMF6 was more obvious. The frequency band center was located at 3000 Hz. Near the
Figure 11. 2-dimensional feature visualization of testing samples a) OS-ISAE based-diagnostic model; b) VMD-ISAE based-diagnostic model; c) ORVMD-ISAE based-diagnostic model.
Figure 12. The illustration of ABLT-1A: a) the perspective view; b) the schematic diagram.

Figure 13. The failure view of the inner-outer ring compound fault.

Table 4. The experimental condition of ABLT-1A.

<table>
<thead>
<tr>
<th>Experimental conditions</th>
<th>Specific parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing type</td>
<td>HRB6205 single row deep groove ball bearings</td>
</tr>
<tr>
<td>Number of test bearing</td>
<td>4 sets</td>
</tr>
<tr>
<td>Load</td>
<td>0 kg</td>
</tr>
<tr>
<td>Data acquisition card</td>
<td>NB9234</td>
</tr>
<tr>
<td>Sensor</td>
<td>PCB acceleration sensor</td>
</tr>
<tr>
<td>Sampling frequency</td>
<td>10240 Hz</td>
</tr>
<tr>
<td>Rotating speed</td>
<td>1050 r/min</td>
</tr>
<tr>
<td>Length of vibration signal</td>
<td>102400</td>
</tr>
</tbody>
</table>

Table 5. The $Re$ index of IMF component signals.

<table>
<thead>
<tr>
<th>Component signals</th>
<th>Relevant energy ($Re$)</th>
<th>Relevant coefficient ($Rc$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMF1</td>
<td>0.0661</td>
<td>0.3786</td>
</tr>
<tr>
<td>IMF2</td>
<td>0.2255</td>
<td>0.6042</td>
</tr>
<tr>
<td>IMF3</td>
<td>0.0335</td>
<td>0.3677</td>
</tr>
<tr>
<td>IMF4</td>
<td>0.0157</td>
<td>0.3044</td>
</tr>
<tr>
<td>IMF5</td>
<td>0.0140</td>
<td>0.2847</td>
</tr>
<tr>
<td>IMF6</td>
<td>0.0158</td>
<td>0.2937</td>
</tr>
<tr>
<td>IMF7</td>
<td>0.0370</td>
<td>0.3565</td>
</tr>
<tr>
<td>IMF8</td>
<td>0.0343</td>
<td>0.3344</td>
</tr>
</tbody>
</table>

5.2.3. Fault Classification and Recognition Results

For experimental comparison, three kinds of signals, namely, the original signal after FFT transformation, VMD signal after FFT transformation and ORVMD after FFT transformation, were input into ISAE algorithm, respectively. The classification results are obtained in Fig. 19. Figure 19 indicates the fault confusion matrix of ABLT-1A rolling bearing fault data set based on ORVMD-ISAE, VMD-ISAE and OS-ISAE. The diagnosis results illustrated that the proposed framework can fully diagnose the five types of faults.

5.2.4. Feature Visualization via t-SNE

To validate that the proposed method can learn the effective feature representation from raw vibration signals, the extracted last layer of features via ISAE were visualized through the t-SNE. To remember the diagnosis model method {original signal+ISAE; VMD+ISAE; ORVMD+ISAE}, t-SNE was used to convert the high dimensional representation into a 3-dimensional graph. The above three diagnostic methods obtained the 3-dimensional scatter distribution diagram of characteristic points through t-SNE as shown in the Fig. 20. Only the ORVMD-SAE diagnostic model was able to divide the samples of various conditions, which indicated the effectiveness of the proposed framework.

5.2.5. Compared with Other Diagnostic Methods

Specifically, different combination fault diagnostic was selected as comparative testing to validate the effectiveness of the proposed method. The selected various combination methods were displayed as follows: {ORVMD+ISAE ($Re$), ORVMD+DBN ($Re$), VMD+SAE ($Re$), VMD+DBN ($Re$), ORVMD+SAE ($Re$), ORVMD+DBN ($Re$), VMD+ISAE ($Re$)} = Method1, Method2, Method3, Method4, Method5, Method6, Method7, Method8. The combination fault diagnosis framework of cross-training and testing samples was adopted by changing the proportion of training and testing samples, each group was set to be 10/90, 20/80, 60/40, 40/60, 20/80 and 10/90, respectively. Afterwards, the average recognition accuracy of different methods is shown in Fig. 21.

Generally, the recognition accuracy of the above-mentioned eight methods were increased with the increasing of the number of training samples. Owing to the increasing of the number of training samples, the more prior information is obtained. It can be observed from Fig. 21 the proposed ORVMD-ISAE method.
6. CONCLUSIONS

To improve the recognition accuracy of the weak fault diagnosis of rotating machinery, this paper proposes a new weak fault recognition framework. The core of the fault identification framework is to design the ORVMD algorithm for weak signal enhancement and the ISAE algorithm for weak fault identification. The designed ORVMD algorithm can comprehensively utilize the PSO and $Re$ index to overcome the problem of large error of conventional VMD in signal decomposition. At the same time, the designed ISAE algorithm can utilize the advantages of two types of regularization to improve the generalization performance of fault recognition. Finally, the experimental signals of the two types of rolling bearings validate the effectiveness of the proposed fault diagnosis framework and algorithm. The fault diagnosis framework proposed in this paper can provide a promising reference for the effective detection and diagnosis of weak faults in rotating machinery. In addition, we will use actual industrial field data to validate the effectiveness of the proposed fault diagnosis framework in future work.
Figure 16. Time-domain and frequency-domain waveform for different IMF components.

Figure 17. The envelope spectra of IMF6 from bearing inner-outer ring compound fault signal.
Figure 18. Time-frequency domain waveform and envelope spectrum of the reconstructed inner-outer ring weak fault signal.

ACKNOWLEDGEMENTS

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REFERENCES


Figure 19. The confusion matrix for fault diagnosis based: a) ORVMD-ISAE; b) VMD-ISAE; c) OS-ISAE.
Figure 20. 3-dimensional feature visualization of testing samples based: a) ORVMD-ISAE; b) VMD-ISAE; c) OS-ISAE.


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Shawki A. Abouel-seoud is a professor of vehicle design and dynamics at Helwan University. He obtained his BS degree from Helwan University in 1969 and his Ph.D. in mechanical engineering from Birmingham University, the U.K. in 1979. Shawki, a distinguished university professor emeritus at Helwan University in Cairo, Egypt, is recognized for promoting international collaboration, education, and the dissemination of knowledge in vibration and acoustics through the formation of professional organizations, the establishment of journals and congress series, and the creation of reference volumes for practitioners. Dr. Shawki joined Helwan University in 1991 as head of the Automotive Engineering Department at Minia University, Egypt. He was promoted to a distinguished university professor in 1990 and has been an emeritus professor since 2016. Dr. Shawki has made significant contributions in the acoustical and vibration fields including condition-based maintenance analysis for most of the vehicle elements (Gearbox, Engine, etc.), vehicle elements design, agricultural machinery design analysis, vehicle ride and handling characteristics analysis, and vibration and acoustic control for most of the vehicle elements.

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