The Exact Frequency Equations for the Euler-Bernoulli Beam Subject to Boundary Damping

Angela Biselli and Matthew P. Coleman

Fairfield University, Department of Mathematics, Fairfield, CT 06824 U.S.A.

(Received 25 October 2018; accepted 15 February 2019)

The Euler-Bernoulli (E-B) beam is the most commonly utilized model in the study of vibrating beams. The exact frequency equations for this problem, subject to energy-conserving boundary conditions, are well-known; however, the corresponding dissipative problem has been solved only approximately, via asymptotic methods. These methods, of course, are not accurate when looking at the low end of the spectrum. Here, we solve for the exact frequency equations for the E-B beam subject to boundary damping. Numerous numerical examples are provided, showing plots of both the complex wave numbers and the exponential damping rates for the first five frequencies in each case. Some of these results are surprising.

1. INTRODUCTION

The Euler-Bernoulli beam is, by far, still the most commonly implemented beam element in engineering applications, because most beams in these applications possess aspect ratios that allow one to neglect the effects of rotary inertia and shear deformation.\(^1\) It follows that the homogeneous E-B equation subject to energy-conserving boundary conditions has been well-studied. The exact frequency equations for the problem, subject to various combinations of the four standard energy-conserving boundary conditions — clamped, simply supported, roller-supported and free — are presented in Pilkey\(^2\) and Han et al.\(^3\) (the latter which seems to be the first reference to include the roller-supported condition). The E-B beam subject to boundary damping also has been well-studied; however, to date, the problem has only been solved approximately, using asymptotic methods.\(^4\)\(^-\)\(^7\) (The same, of course, can be said for the Timoshenko beam, for which numerous references can be found using an asymptotic approach.) By their nature, asymptotic methods are not accurate for the lowest modes, the modes possessing the greatest energies of vibration - thus, the most important modes, both mathematically and physically. For example, for the cantilever E-B beam, the first wave number is 1.875,\(^3\) while an asymptotic estimate for their number is 1.571 (which we have computed using the methodology from Chen et al.\(^5\)\)), the error being about 16%.

Here, then, we derive the exact frequency equations for the complex wave numbers of the E-B beam subject to standard energy-dissipative boundary conditions, then solve several special cases and present the results graphically, using Mathematica. Along the way, we find that the dissipative results do match the conservative results, when the system is subjected to the appropriate zero or infinite damping at each end.

This paper, then, is organized as follows. In Section 2, the partial differential equation and energy-dissipative boundary conditions are derived using a standard energy method, the problem is recast in dimensionless form, and diagrams for the physical implementation of the boundary conditions are presented. In Section 3, the system’s exact frequency equation is derived and compared with the exact equations for the ten cases involving energy-conserving boundary conditions. In Section 4 we present numerical results, providing, for a somewhat exhaustive set of special cases, plots of the complex wave numbers as functions of the damping parameters, which include some interesting results, for which we provide a physical explanation. In addition, we plot the (real) exponential damping rates for these same examples; some of these results are surprising, as we shall see.

2. THE PROBLEM

We consider a uniform E–B beam of length \(L\), with the following constant physical parameters: \(\sigma\), the mass-per-unit-volume; \(A\), the cross-sectional area; \(E\), Young’s modulus; and \(I\), the area moment of inertia. Letting \(W = W(x, t)\), \(0 \leq x \leq L\), \(t \geq 0\), be the lateral deflection, the energy of the beam at time \(t\) is:

\[
E(t) = \frac{1}{2} \int_0^L [\sigma AW_x(x, t)^2 + EIW_{xx}(x, t)^2]dx; \quad t \geq 0; \quad (1)
\]

(see Han et al.,\(^3\) for example). Here and below we use the notation \(W_t = \frac{\partial W}{\partial t}\), etc. We require boundary dissipation of energy, so that the boundary conditions must be such that \(E'(t) < 0\). But:

\[
E'(t) = \int_0^L [\sigma AW_t(x, t)W_{tt}(x, t) + EIW_{xx}(x, t)W_{xxt}(x, t)]dx;
\]

and performing integration by parts twice on the second term results in:

\[
E'(t) = EI[W_{xx}(x, t)W_{tt}(x, t) - W_{xxt}(x, t)W_t(x, t)]_{x=0}^L
+ \int_0^L W_t(x, t)[\sigma AW_t(x, t) + EIW_{xxt}(x, t)]dx.
\]

(3)