The Exact Frequency Equations for the Rayleigh and Shear Beams with Boundary Damping

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While the Euler-Bernoulli beam is the most commonly utilized model in studying vibrating beams, one often requires a model that captures the additional effects of rotary inertia or deformation due to shear. The Rayleigh beam improves upon the Euler-Bernoulli by including the former effect, while the shear beam is an improvement that includes the latter. While all of these problems have been well studied when subject to energy-conserving boundary conditions, none have been solved for the case of boundary damping. We compute the exact frequency equations for the Rayleigh and shear beams, subject to boundary damping and, in the process, we find interesting connections between the two models, despite their being very different.

1. INTRODUCTION

The two most commonly utilized linear models for studying the transverse vibration of beams are the Euler-Bernoulli and Timoshenko beams. The Euler-Bernoulli is, by far, the simpler of the two, including only the kinetic energy due to lateral displacement and the strain energy due to bending. The Timoshenko beam improves upon the Euler-Bernoulli by including the effects of rotary inertia and deformation due to shear. (There are many, many references for both models in the literature. See Han et al. \(^1\) and Traill-Nash and Collar\(^2\) for detailed descriptions and comparisons, for the case of energy-conserving boundary conditions.) As such, the Timoshenko model is more complex and unwieldy than the Euler-Bernoulli. It is fortunate that, in so many engineering applications, the aspect ratio (i.e., the ratio of length to vertical thickness) usually is much greater than 10, in which case the shear effects are negligible and the Euler-Bernoulli model is more appropriate.\(^3,4\)

In some cases, though, the Euler-Bernoulli model may not be sufficient for a given situation, and one may need to consider, in addition, the effect of rotary inertia or of shear deformation, but not both. Thus, there are two standard intermediate beam models, the Rayleigh and shear beams.\(^1\)

The Rayleigh beam is the improvement on the Euler-Bernoulli model that, additionally, considers the effect of rotary inertia.\(^1,5\) The shear beam, on the other hand, improves on the Euler-Bernoulli by including the effect of shear deformation.\(^1,2\) (We note that there are two other “shear” models – the simple shear beam, which includes only the lateral displacement and shear deformation; and the pure shear beam, which includes only the shear deformation and rotary inertia.\(^1,6\) As such, neither of these is a direct improvement on the Euler-Bernoulli model. We also note that the shear beam treated herein does not include the effect of warping due to shear.)

For each of these four models — Euler-Bernoulli, Rayleigh, shear and Timoshenko — the exact equations for the wave numbers for the homogeneous problem, subject to all combinations of energy-conserving boundary conditions, have been derived, and these results have been gathered together in the excellent study by Han et al.\(^1\) Indeed, this seems to be the only reference to include the roller-supported (or sliding) boundary condition. As for problems subject to energy-dissipative boundary conditions, however, the only solutions in the literature are asymptotic approximations for the Euler-Bernoulli and Timoshenko.\(^7,8\)

Here, then, we present the exact frequency equations for the Rayleigh and shear beams, subject to boundary damping, and compare the results with each other, and with the corresponding energy-conserving problems. We choose to treat these models together here because they are of similar complexity and, moreover, because of the surprising and compelling similarity of the results, given the obvious differences between the two models. This paper will be useful for those interested in studying the damped Rayleigh and shear beam problems numerically, and as a springboard to the corresponding Timoshenko beam problem.

The paper, then, is organized as follows. In Section 2, we use a standard energy method to derive the partial differential equations for the Rayleigh and shear beams, along with...