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EDITORIAL

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Ricardo E. Musafir

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Sound is generated by movement: the same way that whenever an object at the air-water interface is subject to unsteady movement, surface waves are generated and propagate away from its source, any unsteady movement in a real fluid gives origin to pressure waves which, although we cannot see, we can eventually hear — in which case we call them sound waves. When sound is generated in a homogeneous medium at rest by the movement or deformation of an ‘external’ body, the flow field, which comprises both the ‘accommodation’ movement close to the source and the medium wave-like reaction, which carries energy away, is indistinguishable from the sound field — they are one and the same thing. In this case, as fluctuations of pressure, density and velocity are typically small when compared to the average values of ambient pressure, density and sound speed, respectively, it is usually sufficient to describe the sound field using linearized equations. This means that all terms in the mass, momentum and energy equations involving products of fluctuation can be abandoned. Effectively, combining the linearized form of these three basic equations for a homogeneous medium at rest and neglecting gravity, viscosity and heat conduction (effects which cause negligible influence on the free field propagation of sound over ‘moderate’ distances) leads to the standard wave equation, also called the d’Alembert equation, as it was first obtained by Jean-le-Rond d’Alembert in 1747, for describing waves on a string.

Now, when sound is produced inside a moving fluid, like in the turbulent flow which exhausts from a jet turbine, there is certainly a flow field which is, frequently, independent of the sound field while, far away from the flow, only the sound field remains. Although this picture suggests that the separation between flow and sound may be straightforward, this separation is far from simple. In fact, there is no unambiguous answer to this issue because, while there is not a well defined frontier through which sound is being imparted to the medium, both the ‘flow’ and the ‘propagation’ are described by the same set of basic equations, even if different hypotheses may be used in each case.

The first general approach to tackle this problem was proposed by Sir James Lighthill (1924–1998) in 1952 (Proc. Royal Soc. A, 211 (1107)). Lighthill rearranged the momentum equation so that the difference between the stresses due to the flow in the real situation and those that would exist due to the propagation of sound waves in a reference homogeneous medium at rest could be interpreted as ‘external’ (or ‘equivalent’) sources acting on such a reference medium. With this very elegant trick, the mass and momentum equations could be combined into an inhomogeneous d’Alembert equation for density fluctuations, known as the Lighthill equation. This is, in fact, valid for any continuous medium, with the appropriate stress tensor considered. A significant advantage is that the d’Alembert equation has a known, simple solution.

The approach introduced by Lighthill establishes an acoustic analogy by replacing the real problem by another considering a hypothetical homogeneous medium at rest. In this approach, all complications are transferred to the source term which, if known, provides the required input for obtaining the sound field. However, to adequately predict the acoustic field in the presence of a mean flow, an excessive amount of detail is required for the ‘source’ description, because the wave operator in Lighthill’s equation (the d’Alembert operator) “does not know” that there is flow and consequently, all details regarding the interaction of sound waves with the mean flow (and with turbulence as well) are left in the source function and have to be modeled. In the incompressible flow approximation, for instance, these details are lost.

The way to escape this is to consider a different rearrangement of the basic equations so that the new wave operator considers the existence of a mean flow. In doing this, however, the charming simplicity of Lighthill’s equation has to be abandoned. Two important approaches were developed which constitute different acoustic analogies: one by G. M. Lilley (1919–2015; given first in a 1971 Lockheed-Georgia Report) and another by M. S. Howe (J. Fluid Mech., 71(4), 1975). In Lilley’s approach, terms non-linear in the fluctuations are considered as equivalent sources while linear terms are seen as describing sound propagation. Thus, at the cost of a more complex equation (which considers a parallel mean flow with transverse shear and whose solution, except in specific limiting cases, has to be obtained numerically), a more detailed description is provided.

Lilley’s approach can be generalized for an arbitrary mean flow by considering the system of basic equations with all terms non-linear in the fluctuations regarded as equivalent sources at the respective equation (Béchara et al., AIAA J. 32(3), 1994; Goldstein, J. Fluid Mech. 488, 2003). Howe’s approach, on the other hand, considers as sources of sound, vorticity and entropy inhomogeneities, treating the propagation as an irrotational and homentropic process. In short, these approaches consider different reference situations and thus, different criteria to separate flow and sound. All of them have provided — and keep providing — quite important results. Each of these (and of other non-cited approaches as well) is more appropriate to specific situations and this is partially related to the type of approximation that is adequate to a given problem. If one wants to compute, for instance, effects of nonlinear steepening on jet noise, a Lilley-type approach (based on linear equations) is unlikely to yield adequate predictions.

The Lighthill analogy is surely a masterpiece, which has been applied even to Cosmology (Lilley, IJAV, 8(3), 2003). However, it is also, not infrequently, applied in ways which, given the approximations involved, the solution has limited applicability. It seems that, despite the numerous attempts made in this direction so far, no single aeroacoustic theory can be regarded as providing the optimal solution to all problems in the field. Whenever we have to introduce approximations (and we always have to do this) we have to check carefully which approach is more likely to give the desired results.

Ricardo E. Musafir
IHAV Director
In the present study, the nonlinear flexural vibration behavior of a double layered prestressed viscoelastic nanoplate under shear in-plane load is investigated based on nonlocal elasticity theory. Using nonlinear strain-displacement relations, the geometrical nonlinearity is modeled. Both nonlocal plate theory and Hamilton’s principle are utilized for deriving the governing equations. The differential quadrature method (DQM) is employed for the computation of nonlinear frequency of the nanoplate. The detailed parametric study is conducted, focusing on the influences of small scale, aspect ratio of the plate, Winkler and Pasternak effects, van der Walls (vdW) interaction, temperature, the effect of pre-stress under shear in-plane load, and the viscosity of the plate. The influence of the viscoelastic coefficient is also discussed. The plots for the ratio of nonlinear to linear frequencies versus maximum transverse amplitude for double layered viscoelastic nanoplate are presented.

1. INTRODUCTION

In recent years, due to the excellent electronic, chemical, and mechanical characteristics of carbon, nanotubes and nanoplates have been widely considered by researchers in different fields such as biosensors and atomic-force microscope in micro/nano electromechanical systems (MEMS/NEMS). Development of these nanostructures requires a good understanding of their properties, such as vibration behavior. For instance, Alizada and Sofiyev studied the mechanics of deformation, stability and stress analysis of a substrate coated by nanomaterials. Up to now, several studies have been conducted on the vibration characteristics of nanoplates. In many of these studies, the free vibration of plates is computed by classical plate theory without considering the nonlinear terms in the governing equations. There are several methods for considering vibration of nanostructures including analytical and numerical. In numerical methods, finite element method (FEM) and differential quadrature method (DQM) have a good agreement with analytical methods. Pradhan and Phadikar used Navier’s approach to solve the governing equations of classical plate theory (CLPT) and to obtain linear natural frequencies of the nanoplates. To obtain the vibration response of single-layered graphene sheets (SLGS), Murmu and Pradhan employed DQM. Using these numerical results, they showed the small size effects in natural frequency SLGS. Ansari et al. studied the vibrational behavior of SLGS using DQM. The mechanical transverse deformation of bilayer graphene sheets (BLGS) under central loading is simulated by Scarpa et al. using a mixed atomistic continuum-FE technique. Their results provided good agreement with experimental and theoretical data available in open literature. Based on nonlocal elasticity theory, the free vibration behavior of a rectangular graphene sheet under shear in-plane load is studied by Goodarzi et al. Furthermore, Murmu et al. considered vibration and buckling of nanostructures under initial compressive pre-stressed condition. Recently, higher-order shear deformation plate and beam theories for wave propagation, bending and vibration behavior in functionally graded plates and beams have been developed, including work by Bourada et al. and Zemri et al. In these open literatures, behaviors of functionally graded (FG) nanostructures (beam and plate) using the nonlocal differential constitutive relations of Eringen are developed. Using a four variable refined plate theory, Zidi et al. illustrated the transverse shear strains across the thickness of FG plate. Tounsi et al. investigated thermoelastic bending of functionally graded sandwich plates using shear deformation theory. In their paper, they studied influences of the transverse shear deformation, thermal load and stresses in functionally graded metal-ceramic plates. To study bending of exponential graded plates, a new sinusoidal higher-order plate theory is developed. Numerical results of present theory are compared with three-dimensional elasticity solutions and other higher-order theories reported in the literature. The employment of nonlocal elasticity theory in nano mechanical systems (NMSs) has been reported by various researchers. Various beam models are developed based on nonlocal elasticity theory such as Timoshenko beam model, higher-order nonlocal beam model and single and couple beam models. Most of the previous studies on the vibration characteristics of nanoplates are based on the linear analysis. There are a few studies about nonlinear analysis of the nanoplate considering the nonlocal effect. In 1961, Yamaki studied the basic governing equations of nonlinear vibration of plates under different boundary conditions. For a solution to this equation, he used an approximate solution. Shen et al. investigated nonlinear vibration of SLGS using nonlocal plate model. Using DQM, nonlinear free vibration of Mindlin plates was investigated by Malekzadeh. Jomehzadeh and Saidi investigated nonlinear vibration behavior and the effect of small length scale based on the von Karman geometrical model and Eringen theory of nonlocal continuum. Also, Setoodeh et al. investigated the small-scale effect on the nonlinear free vibration of orthotropic SLGS using the nonlocal elasticity plate theory. In other more recent work, nonlinear free vibration of magneto-electro-elastic laminated rectangular plates with simply supported boundary condition was studied by Razavi and Shooshartari. Arani et al. investigated transverse nonlinear vibration of orthotropic double-layered graphene sheets (DLGS) using nonlocal elasticity orthotropic plate theory without con-
sidering the effect of the viscoelastic damping coefficient of the material.\textsuperscript{34} Also, Arani et al. demonstrated the nonlinear vibration of the coupled system of double-layered annular graphene sheets (DLAGSSs) including a Visco-Pasternak foundation.\textsuperscript{35} However, no study was conducted for the nonlinear vibration behavior of viscoelastic nanoplates with pre-stress effect. Wang et al. investigated the nonlinear vibration behavior of double layered viscoelastic nanoplates using an analytical method.\textsuperscript{36} In this work, they reported nonlinear vibration characteristics of double layered viscoelastic nanoplates without considering the pre-stressed effect. With respect to developmental works on nonlinear vibration of the nanoplates, it should be noted that none of the research mentioned above, has studied the pre-stressed in nonlinear vibration behavior viscoelastic nanoplates.

In this paper, for the first time, the pre-stressed under in-plane load (normal and shear) in nonlinear vibration of double layered viscoelastic nanoplates is investigated using DQM. Also, the influences of small scale, aspect ratio of the plate, Winkler and Pasternak effects, vdW interaction, thermal effect, and the viscoelastic coefficient is discussed. The plots for the ratio of nonlinear to linear frequencies versus maximum transverse amplitude for double layered viscoelastic nanoplate are presented. Additionally, some new conceptions from the nonlinear vibration behavior are observed.

2. PROBLEM FORMULATION

Figure 1 illustrates the schematic diagram of double layered nano-plates with the length $a$, width $b$ and thickness $h$. In this work, the nanoplates are assumed to be homogeneous and isotropic. According to the nonlocal continuum theory proposed by Eringen, the stress at a reference point $X$ is considered to be a function of the strain field at every point $X'$ in the body.\textsuperscript{37, 38} The nonlocal stress tensor $\sigma$ at point $X$ can be expressed as

$$\sigma_{ij} = \int K(|X' - X|, \tau) C_{ijkl} \varepsilon_{kl}(X') dX';$$

where $C_{ijkl}$ and $\varepsilon_{kl}$ are the elastic module tensor of classical isotropic elasticity and strain tensors, respectively. $K(|X' - X|)$ is the Kernel function and represents the nonlocal modulus. Eringen demonstrated that it is possible to represent the integral constitutive relation in an equivalent differential form as

$$(1 - \mu \nabla^2) \sigma_{ij} = C_{ijkl} \varepsilon_{kl};$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator and $\mu = (\nu a^2)^2$ is the nonlocal parameter that demonstrates the nanoscale effect on the response of nanoplates, in which, $a$ is an internal characteristic length and $\nu$ is a constant that approximate to each material.\textsuperscript{38} The value of the nonlocal parameter depends on the boundary conditions, chirality, mode shapes, number of walls, and the nature of motion.\textsuperscript{39} In the investigation of the nonlocal parameter effect, the value of $\nu$ needs to be determined from experiments or by matching the dispersion relation of plane waves with those of atomic lattice dynamics. So far, no experiments have been conducted to predict the magnitude of $\nu$ for carbon nanotubes. In the open literature, it is suggested that the value of the nonlocal parameter can be determined by using a comparison of dispersion curves from the nonlocal continuum mechanics and molecular dynamics simulation.\textsuperscript{40–42}

Based on Kirchhoff’s plate theory,\textsuperscript{43, 44} the displacement fields can be expressed as

$$u_1 = u_0(x, y, t) - \frac{\partial w}{\partial x};$$

$$u_2 = v_0(x, y, t) - \frac{\partial w}{\partial y};$$

$$u_3 = w(x, y, t).$$

In Eq. (3), $u_1$ and $u_2$ are the in-plane displacements of the plate along the $x$ and $y$ directions. $u_3$ is the transverse displacement along the $z$ direction, and $u_0$ and $v_0$ are the middle surface displacements along the $x$ and $y$ directions. Since the nonlinear vibration is assumed to have large amplitude motion, the von Karman type strain displacement relations are employed as

$$\varepsilon = \varepsilon_0 + 2\kappa;$$

where $\varepsilon$ is the strain vector and $\varepsilon_0$ and $\kappa$ are the nonlinear strain vector and the variation of curvature vector respectively. They can be expressed as

$$\varepsilon_0 = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial v}{\partial y} \right)^2 & \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 & \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial v}{\partial y} \right)^2 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix};$$

$$\kappa = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}.$$

The stress resultants, $N_{ij}$, and the moment resultants, $M_{ij}$, for the nonlocal nanoplate can be defined as

$$\{N_{ij}, M_{ij}\} = \int \sigma n_{ij} (1, z) dz; \quad i = x, y; \quad j = x, y, z.$$ (7)

Using Eq. (2), the plane stress constitutive relation of a nonlocal plate becomes

$$\begin{bmatrix} \sigma_{xx}^{nl} \\ \sigma_{yy}^{nl} \\ \sigma_{xy}^{nl} \end{bmatrix} - \mu \nabla^2 \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E \frac{1 - \nu}{1 + \nu} \left( 1 + \frac{g \mu}{\sigma_{xx}} \right) & \frac{E \nu}{1 + \nu} \left( 1 + \frac{g \mu}{\sigma_{yy}} \right) & 0 \\ \frac{E \nu}{1 + \nu} \left( 1 + \frac{g \mu}{\sigma_{xx}} \right) & E \frac{1 - \nu}{1 + \nu} \left( 1 + \frac{g \mu}{\sigma_{yy}} \right) & 0 \\ 0 & 0 & \frac{E}{1 + \nu} \left( 1 + \frac{g \mu}{\sigma_{xy}} \right) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} - \alpha \Delta T \\ \varepsilon_{yy} - \alpha \Delta T \\ \varepsilon_{xy} \end{bmatrix};$$

where $E$, $\nu$, $g$, $\alpha$, and $\Delta T$ denote the Young’s modulus, the Poisson’s ratio, the viscoelastic structural damping coefficient, the coefficient of thermal expansion, and the temperature difference between the top and bottom layers of the nanoplate respectively. The following equilibrium equations can be ex-
pressed

\[
\begin{align*}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= m_0 \frac{\partial^2 u_0}{\partial t^2}, \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= m_0 \frac{\partial^2 v_0}{\partial t^2}; \\
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} &= f + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} \right) \\
&= m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4 \partial t^2} \right); \\
\frac{1}{1 - \mu} \nabla^2 M_{xx} &= -D \left( \frac{\partial^2 w}{\partial x^2} + g \frac{\partial^3 w}{\partial t \partial x^2} \right) - \\
&= \nu D \left( \frac{\partial^2 w}{\partial y^2} + g \frac{\partial^3 w}{\partial t \partial y^2} \right); \\
\frac{1}{1 - \mu} \nabla^2 M_{yy} &= -D \left( \frac{\partial^2 w}{\partial y^2} + g \frac{\partial^3 w}{\partial t \partial y^2} \right) - \\
&= \nu D \left( \frac{\partial^2 w}{\partial x^2} + g \frac{\partial^3 w}{\partial t \partial x^2} \right); \\
(1 - \mu \nabla^2) M_{xy} &= -D (1 - \nu) \left( \frac{\partial^2 w}{\partial x \partial y} + g \frac{\partial^3 w}{\partial t \partial x \partial y} \right); \\
N_{xx} &= P_1; \quad N_{yy} = P_2; \quad N_{xy} = P_3. \quad (10)
\end{align*}
\]

In the present study, we consider the following relations for the vibration analysis under in-plane load:

where \( f \) is the transverse load, \( m_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \, dz \), and \( m_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho z^2 \, dz \), in which \( \rho \) denotes the density of the material.  45

\[ \begin{align*}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= m_0 \frac{\partial^2 u_0}{\partial t^2}, \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= m_0 \frac{\partial^2 v_0}{\partial t^2}; \\
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} &= f + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xy} \frac{\partial w}{\partial x} \right) \\
&= m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4 \partial t^2} \right); \\
\frac{1}{1 - \mu} \nabla^2 M_{xx} &= -D \left( \frac{\partial^2 w}{\partial x^2} + g \frac{\partial^3 w}{\partial t \partial x^2} \right) - \\
&= \nu D \left( \frac{\partial^2 w}{\partial y^2} + g \frac{\partial^3 w}{\partial t \partial y^2} \right); \\
\frac{1}{1 - \mu} \nabla^2 M_{yy} &= -D \left( \frac{\partial^2 w}{\partial y^2} + g \frac{\partial^3 w}{\partial t \partial y^2} \right) - \\
&= \nu D \left( \frac{\partial^2 w}{\partial x^2} + g \frac{\partial^3 w}{\partial t \partial x^2} \right); \\
(1 - \mu \nabla^2) M_{xy} &= -D (1 - \nu) \left( \frac{\partial^2 w}{\partial x \partial y} + g \frac{\partial^3 w}{\partial t \partial x \partial y} \right); \quad (11)
\end{align*} \]
where $D = \frac{Eh^3}{12(1-\nu^2)}$ denotes the bending stiffness of the nanoplate. For the large amplitude bending vibration of thin-walled plates, the influence of in-plane inertia can be negligible.\textsuperscript{46} Applying the linear operator $(1 - \mu \nabla^2)$ on each side of the equilibrium equation, third Eq. (9), the equation of motion for the nonlocal viscoelastic plate can be written as

$$D \left[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + g \frac{\partial^2 w}{\partial t \partial x} + 2g \frac{\partial^2 w}{\partial t \partial y} \right] + (1 - \mu \nabla^2) \left[ m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2} \right) \right] = (1 - \mu \nabla^2) \left[ -k_{1w} w_1 - k_{2w} w_3 \right] + k_s \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2} \right) - C \left( w_1 - w_2 \right) + (1 - \mu \nabla^2) \left[ P_1 \frac{\partial^2 w}{\partial x^2 \partial y} + P_2 \frac{\partial^2 w}{\partial x \partial y^2} \right].$$

(15a)

In Eq. (12), the transverse load $f$ can be defined as

$$f_1 = -k_{1w} w_1 - k_{2w} w_3 + k_s \left( \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2} \right) - C \left( w_1 - w_2 \right); \quad f_2 = -k_{1w} w_2 - k_{2w} w_3 + k_s \left( \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} \right) - C \left( w_2 - w_1 \right);$$

(15b)

where $k_{1w}, k_{2w}, k_s, k_G$ and $C$ are linear, nonlinear, and shear coefficient of Winkler, Pasternak and vdW interaction coefficients respectively. The vdW interaction between the layers of multi walled nanostructures can be modeled as the Winkler type foundation and $C$ that is the vdW interaction coefficient.\textsuperscript{47}

Based on Eqs. (12), (13), and (14), the governing equations of flexural vibration for double layered nanoplates can be derived as

$$D \left[ \frac{\partial^4 w_1}{\partial x^4} + 2 \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} + g \frac{\partial^2 w_1}{\partial t \partial x} + 2g \frac{\partial^2 w_1}{\partial t \partial y} \right] + (1 - \mu \nabla^2) \left[ m_0 \frac{\partial^2 w_1}{\partial t^2} - m_2 \left( \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^2 \partial t^2} \right) \right] + (1 - \mu \nabla^2) \left[ -k_{1w} w_1 - k_{2w} w_3 \right] + k_s \left( \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^2 \partial t^2} \right) - C \left( w_1 - w_2 \right) + (1 - \mu \nabla^2) \left[ P_1 \frac{\partial^2 w_1}{\partial x^2 \partial y} + P_2 \frac{\partial^2 w_1}{\partial x \partial y^2} \right].$$

(15a)
3. DQM AND SOLUTION PROCEDURE

The DQM method is an efficient and accurate numerical approach in comparison with the weighted residual methods such as FE method. The DQM has been widely used for solving various dynamic and stability problems. In DQM a derivative of a function $F$ is assumed as the weighted linear sum of all functional values within the computational domain and at a given grid point $(x_i, y_j)$ and is approximated as

$$\frac{d^n F}{dx^n}_{x=x_i} = \sum_{j=1}^{N} C^{(n)}_{ij} F(x_j); \quad (16)$$

where

$$C^{(1)}_{ij} = \frac{\pi(x_i)}{(x_i - x_j)\pi(x_j)}; \quad i, j = 1, 2, \ldots, N; \quad i \neq j. \quad (17)$$

And $\pi(x_i)$ is defined as

$$\pi(x_i) = \prod_{j=1}^{N} (x_i - x_j); \quad i \neq j. \quad (18)$$

And when $i = j$,

$$C^{(1)}_{ij} = C^{(1)}_{ii} = -\sum_{k=1}^{N} C^{(1)}_{ik}; \quad i = 1, 2, \ldots, N; \quad i \neq k; \quad i = j. \quad (19)$$

The weighting coefficients for the second, third, and fourth derivatives are determined using matrix multiplication:

$$C^{(2)}_{ij} = \sum_{k=1}^{N} C^{(1)}_{ik} C^{(1)}_{kj}; \quad (20)$$

$$C^{(3)}_{ij} = \sum_{k=1}^{N} C^{(1)}_{ik} C^{(2)}_{kj} = \sum_{k=1}^{N} C^{(2)}_{ik} C^{(1)}_{kj}; \quad (21)$$

$$C^{(4)}_{ij} = \sum_{k=1}^{N} C^{(1)}_{ik} C^{(3)}_{kj} = \sum_{k=1}^{N} C^{(3)}_{ik} C^{(1)}_{kj}; \quad i, j = 1, 2, \ldots, N. \quad (22)$$

Using the following rule, the distribution of grid points based on Gauss-Chebyshev-Lobatto points in domain is calculated as

$$x_i = a \left[ 1 - \cos \left( \frac{i - 1}{N - 1} \pi \right) \right]; \quad i = 1, 2, \ldots, N;$$

$$y_j = b \left[ 1 - \cos \left( \frac{j - 1}{M - 1} \pi \right) \right]; \quad j = 1, 2, \ldots, M. \quad (23)$$

To solve the time derivatives of Eqs. (15a) and (15b), we can assume it in the form

$$w(x, y, t) = W(x, y)e^{\omega t}; \quad (24)$$

in which $\omega$ is the complex eigenvalue. $W(x, y)$ is the displacement of the mid-plane in the nanoplate along the $z$-direction at point $(x, y)$ calculated using CLPT. In other words, $W(x, y)$ is the same mode shape. The effects of boundary condition should be applied on weighting coefficients; here we consider simply supported edges at all sides. The boundary conditions are incorporated in the analyses during the formulation of the

weighting coefficients based on the rule of modified weighting coefficient matrix. As a result, we have

$$\bar{C}^{(1)}_{ij} = \bar{C}^{(1)}_{N,j} = 0; \quad j = 1, 2, \ldots, M;$$

$$\bar{C}^{(1)}_{i,1} = \bar{C}^{(1)}_{1,M} = 0; \quad i = 1, 2, \ldots, N. \quad (25)$$

And the weighting coefficients for the simply supported boundary condition can be written as

$$\bar{C}^{(2)}_{ij} = \bar{C}^{(2)}_{ij}; \quad (26)$$

$$\bar{C}^{(3)}_{ij} = \sum_{k=1}^{N} \bar{C}^{(1)}_{ik} \bar{C}^{(1)}_{kj};$$

$$\bar{C}^{(4)}_{ij} = \sum_{k=1}^{N} \bar{C}^{(1)}_{ik} \bar{C}^{(3)}_{kj}. \quad (27)$$

Substituting Eq. (23) into the governing Eqs. (15a) and (15b), we have Eqs. (25a) and (25b).

In Eqs. (25a) and (25b), $\bar{C}$ and $\bar{A}$ denote the weighting coefficients in $x$ and $y$ directions respectively. In order to carry out the matrix multiplication, two mathematical products (Hadamard and Kronecker) are employed. Using Hadamard and Kronecker products of matrices, defined in Appendix 1, the coupled nonlinear formulations are expressed by Eqs. (26a) and (26b).

Equations (26a) and (26b), can be explained in matrix form which is called nonlinear eigenvalue problem.

$$\left[ \omega^2 [M] + \omega [D] + [K_L + K_{NL}] \right] \{ W \} = 0;$$

or

$$\omega^2 \left[ \begin{array}{cc} [M_1] & 0 \\ 0 & [M_2] \end{array} \right] \left[ \begin{array}{c} W_1 \\ W_2 \end{array} \right] + \omega \left[ \begin{array}{cc} [D_1] & 0 \\ 0 & [D_2] \end{array} \right] \left[ \begin{array}{c} W_1 \\ W_2 \end{array} \right] + \left[ \begin{array}{cc} [K_{L1}] & [K_{L2}] \\ [K_{NL1}] & [K_{NL2}] \end{array} \right] \left[ \begin{array}{c} W_1 \\ W_2 \end{array} \right] = 0; \quad (28)$$

where $[M]$ and $[D]$ are the mass matrix and damping matrix respectively. $[K_L]$ is the linear stiffness matrix and $[K_{NL}]$ is the nonlinear stiffness matrix which is a function of $W$. This nonlinear equation can be solved using a direct iterative process as follows:

a. First, nonlinearity is ignored by taking $K_{NL} = 0$ to obtain the eigenvalue problem demonstrated in Eq. (27). This yields the linear eigenvalue $\omega_L$ and corresponding eigenvector $(w)$.

b. Using linear $W$, $K_{NL}$ could be evaluated. Eigenvalue problem is then solved by substituting $K_{NL}$ into Eq. (27). This would give the nonlinear eigenvalue $\omega_{NL}$ and the new eigenvector.

c. The above process is repeated iteratively until the frequency values from the two subsequent iterations $\omega^r$ and $\omega^{r+1}$ satisfy the prescribed convergence criteria as

$$|\omega_{NL}^{r+1} - \omega_{NL}^r| < \varepsilon_0; \quad (29)$$

where $\varepsilon_0$ is a small value number and in this present $\varepsilon_0$ is $10^{-4}$. 

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\[
D \left[ \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(1)} W_{1,k,j} + 2 \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{c}_{j,k_2}^{(2)} W_{1,k_1,k_2} + \sum_{k=1}^{M} \mathbf{a}_{i,k}^{(4)} W_{1,k,j} + g \omega \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(4)} W_{1,k,j} + \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{1,k_1,k_2} + \sum_{k=1}^{M} \mathbf{a}_{i,k}^{(4)} W_{1,k,j} \right) \right] \\
+ (1 - \mu \nabla^2) \omega^2 \left[ m_0 - m_2 \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{1,k,j} + \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{1,k,j} \right) \right] = (1 - \mu \nabla^2) \left[ -k_{1w} W_{1} - k_{2w} W_{1}^3 + \right]
\]
\[
k_{s} \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{1,k,j} + \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{1,k,j} \right) - k_{g} \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{1,k,j} + \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{1,k,j} \right) - C(W_{1} - W_{2}) \\
+ (1 - \mu \nabla^2) \left( -P_{1} \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{1,k,j} - P_{2} \sum_{k=1}^{N} \mathbf{a}_{j,k}^{(2)} W_{1,k,j} - 2P_{3} \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{1,k_1,k_2} + \right)
\]
\[
\left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{1,k,j} \right) \left( \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{1,k,j} \right) + 4C_{66} \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{1,k,j} \right) \left( \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{1,k_1,k_2} \right) \right) \right],
\]
\[
\left( \sum_{k=1}^{N} \mathbf{a}_{i,k}^{(2)} W_{2,k,j} \right) + C_{66} \left( \sum_{k=1}^{N} \mathbf{a}_{i,k}^{(2)} W_{1,k,j} \right) \left( \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{1,k_1,k_2} \right) \right) \right],
\]
\[
D \left[ \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(4)} W_{2,k,j} + 2 \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{c}_{j,k_2}^{(2)} W_{2,k_1,k_2} + \sum_{k=1}^{M} \mathbf{a}_{i,k}^{(4)} W_{2,k,j} + g \omega \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(4)} W_{2,k,j} + \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{2,k_1,k_2} + \sum_{k=1}^{M} \mathbf{a}_{i,k}^{(4)} W_{2,k,j} \right) \right] \\
+ (1 - \mu \nabla^2) \omega^2 \left[ m_0 - m_2 \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{2,k,j} + \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{2,k,j} \right) \right] = (1 - \mu \nabla^2) \left[ -k_{1w} W_{2} - k_{2w} W_{2}^3 + \right]
\]
\[
k_{s} \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{2,k,j} + \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{2,k,j} \right) - k_{g} \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{2,k,j} + \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{2,k,j} \right) - C(W_{2} - W_{1}) \\
+ (1 - \mu \nabla^2) \left( -P_{1} \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{2,k,j} - P_{2} \sum_{k=1}^{N} \mathbf{a}_{j,k}^{(2)} W_{2,k,j} - 2P_{3} \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{2,k_1,k_2} + \right)
\]
\[
\left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{2,k,j} \right) \left( \sum_{k=1}^{M} \mathbf{a}_{j,k}^{(2)} W_{2,k,j} \right) + 4C_{66} \left( \sum_{k=1}^{N} \mathbf{c}_{i,k}^{(2)} W_{2,k,j} \right) \left( \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \mathbf{c}_{i,k_1,k_2}^{(2)} \mathbf{a}_{j,k_2}^{(2)} W_{2,k_1,k_2} \right) \right) \right] \right],
\]
\[
(25a)
\]
\[
(25b)
\]
4. RESULTS AND DISCUSSION

In this section, the nonlinear vibration behaviors including thermal effect in viscoelastic nanolayers are discussed. Firstly, the results are compared with the available literatures to verify the accuracy of the present formulations. The natural frequencies obtained by the local and nonlocal models are presented in Table 1. Also, those results are shown in Fig. 2. In another work, Razavi and Shooshtari modeled a plate to analyze the nonlinear free vibration of laminated rectangular plates with simply supported boundary condition subjected to magneto-electro-elastic loads.33

Table 1 and Fig. 2 demonstrate the influence of nonlocal parameter, that frequency ratio is defined as the ratio of the frequency of the nonlocal theory ωnl to the corresponding local theory ωl. The elastic modulus E = 1.06 Tpa, length or breadth L = 10 nm, thickness of each plate h = 0.34 nm, the Poisson’s ratio ν = 0.3 and density ρ = 2250 kg/m³ are employed.

Also, in rest analysis, the vdW interaction coefficient C = 108 Gpa/Å is assumed. Characteristics and properties of material are taken from Pradhan and Phadikar and Razavi and Shooshtari.5,33 It’s noted that in Table 1 we compared the results of the frequency ratio ωnl/ωl of isotropic square plate with simply supported boundary conditions with the available literatures. These results also are obtained without considering the nonlinear terms, effects of elastic medium, pre-stressed, and thermal effect in governing equation. Since the literatures about nonlinear vibration for double layered nanoplates are
and Winkler coefficients \( k_g \) Figs. 4–12, the dimensionless viscosity coefficient for simple support in all sides. In this investigation and in a method based on the number of grid points is plotted in Fig. 3 the convergence. The convergence and accuracy of the DQ method based on the number of grid points makes the convergence. The convergence and accuracy of the DQ method based on the number of grid points is plotted in Fig. 3 for simple support in all sides. In this investigation and in Figs. 4–12, the dimensionless viscosity coefficient \( g = 0.01 \) and Winkler coefficients \( k_{1w} = k_{2w} = k_w, k_x = 5 \) and \( k_{w} = 1.13 \times 10^{18} \) Pa/m are taken from Ghorbanpour Arani et al. and Wang et al. In whole solutions, using the expression \( P = \frac{P_a}{D} \), the load \( P \) is employed to non-dimensional state.

Table 2 demonstrates the nonlinear frequency ratio \( \omega_{NL}/\omega \) for amplitude-to-thickness \( (a_{\max}/h) \) ratio of isotropic square plate \( (a/h = 100, \nu = 0.3) \) with simply supported boundary conditions. The results presented in Table 2 are found to be reasonably in agreement with those of Chu and Herrman, Wah, Mei and Decha-Umphai, Chia and Prabhakara, and Manoj et al. obtained by different numerical schemes. In the present analysis the Van der Waal forces are neglected. For isotropic simply supported square plates, Chu and Herrman investigated nonlinear vibration with an analytical method. Also, for an investigation of vibration of rectangular plates with large amplitudes, Wah used an approximate formulation. Mei and Decha employed a finite element method for solution of nonlinear vibration characteristic. Besides, to study nonlinear vibration behavior, are employed an analytical method in Chia and Prabhakara and Manoj et al.

Table 3 illustrates the nonlinear frequency ratio \( \omega_{NL}/\omega \) of double layered viscoelastic square \( (\beta = 1) \) nanoplates considering the foundation parameters \((k_G, k_w, \Delta T)\) against different values of load \( (P) \) for versus maximum transverse amplitude \( (u_{\max} = 1) \), and nonlocal parameter \( (\mu = 1 \text{ nm}) \). It is noted that the nonlinear frequency ratio increases with increasing \( (k_G, k_w, \Delta T) \) and decreases with increasing values of load \( (P_1 = P_2 = P_3 = P) \). The values of the nonlinear frequency ratio for different values of load \( (P_1, P_2, P_3) \) are presented in Table 4. It is noted that with increasing load in smaller nonlocal parameters, the nonlinear frequency ratio decreases, but in larger nonlocal parameters the nonlinear frequency ratio increases. The nonlinear frequency ratio for any temperature variation increases with the elastic foundation parameters.

Figure 4 demonstrates the effects of nonlinear parameter on the nonlinear frequency ratio versus the in-plane load \( (P_1) \) for temperature field \( (\Delta T = 200^\circ\text{C}) \). From Fig. 4 it is observed that for each load, there is a specific direction in which nonlinear frequency ratio first decreases and also increases.

The effects of pre-stressed could be studied from Fig. 5, in which the nonlinear frequency ratio \( (\omega_{NL}/\omega) \) of versus nonlocal parameters \( (\mu) \) is plotted for viscoelastic nanoplate with simply supports in all sides. In Fig. 5, to illustrate the effect of the in-plane load on nonlinear vibration response, we plotted nonlocal parameter for values \( (0 - 2 \text{ nm}) \) and also the temperature value \( (\Delta T = 200^\circ\text{C}) \). It also can be seen in Fig. 5 that with increasing load \( (P) \) and nonlocal parameter, the nonlinear frequency ratio decreases.

From Fig. 6 it can be observed that the effects of length aspect ratio \( (\beta = h/a) \) on the nonlinear vibrations are considerable for various nonlocal parameters and in-plane loads in the temperature value \( (\Delta T = 200^\circ\text{C}) \). Also, by observing Fig. 6 it can be seen that the effect of length aspect ratio on the nonlinear vibrations is more significant than nonlocal parameters.
Table 2. The frequency ratio ($\omega_{NL}/\omega$) for the amplitude-to-thickness ($w_{\text{max}}/h$) ratio of isotropic square plate with simply supported boundary conditions.

<table>
<thead>
<tr>
<th>Amplitude ratio</th>
<th>Chu and Herrman\cite{25}</th>
<th>Wah\cite{52}</th>
<th>Mei and Decha-Unphar\cite{53}</th>
<th>Chia and Prabhakara\cite{30}</th>
<th>Manoj et al.\cite{56}</th>
<th>Present work</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>1.0195</td>
<td>1.0222</td>
<td>1.0182</td>
<td>1.0185</td>
<td>1.0195</td>
<td>1.0088</td>
</tr>
<tr>
<td>0.40</td>
<td>1.0756</td>
<td>1.0858</td>
<td>1.0709</td>
<td>1.0717</td>
<td>1.0757</td>
<td>1.0785</td>
</tr>
<tr>
<td>0.60</td>
<td>1.1625</td>
<td>1.1833</td>
<td>1.1530</td>
<td>1.1534</td>
<td>1.1625</td>
<td>1.1649</td>
</tr>
<tr>
<td>0.80</td>
<td>1.2734</td>
<td>1.3067</td>
<td>1.2589</td>
<td>1.2566</td>
<td>1.2734</td>
<td>1.2760</td>
</tr>
<tr>
<td>1.00</td>
<td>1.4023</td>
<td>1.4491</td>
<td>1.3826</td>
<td>1.3753</td>
<td>1.4024</td>
<td>1.4267</td>
</tr>
</tbody>
</table>

Table 3. Nonlinear frequency ratio for various values of $P$, $k_G$, $k_w$, and $\Delta T$ against maximum transverse amplitude ($w_{\text{max}}$ = 1), nonlocal parameter ($\mu = 1$) and with simply supported boundary conditions.

<table>
<thead>
<tr>
<th>$k_G$</th>
<th>$k_w$</th>
<th>$\Delta T = 0$</th>
<th>$\Delta T = 200^\circ C$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$P_1 = P_2 = P_3 = P$</td>
<td>$P_1 = P_2 = P_3 = P$</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>1.3902</td>
<td>1.3913</td>
<td>1.3925</td>
<td>1.3936</td>
</tr>
<tr>
<td>0.3989</td>
<td>0.3991</td>
<td>0.3922</td>
<td>0.3933</td>
</tr>
<tr>
<td>1.3906</td>
<td>1.3917</td>
<td>1.3928</td>
<td>1.3940</td>
</tr>
<tr>
<td>0.3901</td>
<td>0.3912</td>
<td>0.3924</td>
<td>0.3935</td>
</tr>
<tr>
<td>1.3904</td>
<td>1.3905</td>
<td>1.3916</td>
<td>1.3928</td>
</tr>
<tr>
<td>0.3915</td>
<td>0.3926</td>
<td>0.3938</td>
<td>0.3949</td>
</tr>
<tr>
<td>1.3912</td>
<td>1.3923</td>
<td>1.3935</td>
<td>1.3946</td>
</tr>
<tr>
<td>0.3907</td>
<td>0.3919</td>
<td>0.3930</td>
<td>0.3942</td>
</tr>
<tr>
<td>1.3900</td>
<td>1.3911</td>
<td>1.3923</td>
<td>1.3934</td>
</tr>
</tbody>
</table>

Figure 5. The effects of nonlocal parameter on the nonlinear frequency ratio versus the in-plane load ($P_1 = P_2 = P_3 = P$) for temperature field ($\Delta T = 200^\circ C$).

and in-plane loads.

To illustrate the effect of load, the nonlinear frequency ratio is displayed against the in-plane loads for various length aspect ratios in Fig. 7. As revealed in Fig. 7, this frequency increases by a growth in load. It should be noted that this growth is slight.

To see the effect of the Pasternak coefficient on the nonlinear frequency ratio ($\omega_{NL}/\omega_L$), the results from maximum transverse amplitude ($w_{\text{max}} = 1$), with various in-plane load are shown in Fig. 8. It can be clearly seen from Fig. 8 that the effect of the in-plane load in nonlinear vibration behavior is more important than the effect of the Pasternak coefficient. The Pasternak coefficient ($K_g = 1.13$ pa.m) for the viscoelastic nanoplate is taken from similar values of modulus coefficients that were taken by Wang et al.\cite{56}

The effect of the in-plane load on the nonlinear frequency ratio versus maximum transverse amplitude for the Pasternak coefficient is demonstrated in Fig. 9. Variation of frequency against increase of the Pasternak coefficient and maximum transverse amplitude is displayed in Fig. 9.

In Fig. 10, to illustrate the effects of the in-plane load and temperature field on the nonlinear frequency ratio, we plot the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) versus maximum transverse amplitude. This is shown that the effect of the shear in-plane load in nonlinear vibration behavior is more important than the effect of the temperature fields. It also can be seen that in Fig. 10 with increasing the shear in-plane load, the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) of viscoelastic double layered nanoplate decreases and for larger values of maximum transverse amplitude ($W_{\text{max}}$), the frequencies rise.

Figure 11 demonstrates the effects of uniform temperature fields on the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) versus maximum transverse amplitude ($W_{\text{max}}$) for the in-plane load. This is obvious that with the increasing of maximum transverse amplitude ($W_{\text{max}}$), the temperature effect increases, but the frequency curves show that this increasing is low. Moreover, from this figure it is seen that the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) decreases with increasing in temperature and the in-plane load.

Figure 12 presents the influence of nonlocal parameter ($\mu$) on the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) versus in-plane loads for different temperatures ($\Delta T = 0$, $200^\circ C$). From Fig. 12, it can be seen that the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) decreases with increasing the in-plane load for all nonlocal parameter (Fig. 12a and in Fig. 12b) for smaller nonlocal parameter but for larger nonlocal parameters, with increasing nonlocal parameter and in-plane load, the nonlinear frequency ratio ($\omega_{NL}/\omega_L$) increases. It should be noted that in the whole of the plots, the first mode of the nonlinear frequency is considered. Also, numerical results of Figs. 12a and 12b are shown in Table 5.

5. CONCLUSIONS

In this paper, for the first time, the nonlinear thermo-mechanical vibration behavior of prestressed viscoelastic nanoplates is investigated based on nonlocal elasticity theory. Employing nonlinear strain-displacement relations, the geometrical nonlinearity is modeled while governing equations are derived through Hamilton’s principle and they are solved applying semi-analytical DQM. Eringen’s nonlocal elasticity
Acknowledgements

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Figure 6. The effects of length aspect ratio ($\beta = b/a$) on the nonlinear frequency ratio versus maximum transverse amplitude for various nonlocal parameters and the in-plane load ($P_1, P_2, P_3$).

Figure 7. The effects of the in-plane load ($P_1, P_2, P_3$) on the nonlinear frequency ratio for various length aspect ratio ($\beta = b/a$).


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Figure 8. The effects of the in-plane load ($P_3$) on the nonlinear frequency ratio for various the constant of the Pasternak coefficient ($K_G$).

Figure 9. The effects of the in-plane load ($P_1 = P_2 = 50$, $P_3 = 0$) on the nonlinear frequency ratio versus maximum transverse amplitude for the Pasternak coefficient ($K_G$).

Figure 10. The effects of the in-plane load ($P_1$, $P_2$, $P_3$) on the nonlinear frequency ratio versus maximum transverse amplitude for various nonlocal parameters and temperature fields.


Figure 11. The effects of uniform temperature field on the nonlinear frequency ratio versus maximum transverse amplitude for various nonlocal parameters and the in-plane load ($P_1 = P_2 = P_3 = P$).


Figure 12. Effect of nonlocal parameter on the nonlinear frequency ratio versus the in-plane load ($P_1, P_2, P_3$).


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### APPENDIX 1

Consider matrices $A = (a_{ij})$ and $B = (b_{ij})$ of order $m \times n$ and $C = (c_{ij})$ of order $p \times q$. The definition of the matrix products are given as follows:

Hadamard product

$$A \circ B = (a_{ij}b_{ij})_{ij};$$  \hspace{1cm} (A.1)

where $a_{ij}b_{ij}$ is a scalar and $A \circ B$ is of order $m \times n$.

Kronecker product

$$A \otimes B = (a_{ij}C)_{ij};$$  \hspace{1cm} (A.2)

where $a_{ij}C$ is of order $p \times q$ and $A \otimes B$ is of order $mp \times nq$. 
Analysis on the Aeroelastic Stability of Open Cylindrical Shells in Subsonic Airflow Using the Theoretical and Two-way CFD/CSD Coupled Methods

Yu-Yang Chai
P. O. Box 137, School of Astronautics, Harbin Institute of Technology, Harbin 150001, China.

Feng-Ming Li
P. O. Box 137, School of Astronautics, Harbin Institute of Technology, Harbin 150001, China.
College of Aerospace and Civil Engineering, Harbin Engineering University, Harbin 150001, China.

Zhi-Guang Song
College of Aerospace and Civil Engineering, Harbin Engineering University, Harbin 150001, China.

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The aeroelastic stability of an open cylindrical shell in subsonic airflow is analyzed. The equation of motion of the open cylindrical shell whose one surface is subjected to the subsonic airflow is established based on the Donnell shell theory and transformed into ordinary differential equations using Galerkin’s method. The linear potential flow theory is applied to derive the aerodynamic pressure. The natural frequencies of the aeroelastic system are obtained, from which the flow velocity for the open cylindrical shell under instability can be calculated. The effects of the material and geometric parameters on the critical instability velocity are discussed. Furthermore, the open cylindrical shell is modelled using the finite element software ANSYS. The time domain responses of the structure in subsonic airflow are calculated using the two-way CFD/CSD (computational fluid-structure dynamics) coupled method. From the results, it can be seen that with the increase of the thickness and elastic modulus of the shell, the critical instability velocity increases also. The open cylindrical shell with a smaller radius shows better aeroelastic properties than that with a larger radius. The time domain responses obtained by the CFD/CSD method are compared with those calculated by the theoretical method, and the results of these two methods have a good agreement with each other.

NOMENCLATURE

\( C_p \) Aerodynamic damping matrix

\( D = \frac{Eh^3}{12(1-\mu)} \) bending stiffness

\( E \) Elastic modulus

\( f \) Airy stress function

\( F \) Airy stress shape function

\( g, q \) Generalized coordinates

\( g_0 \) Eigenvector

\( h \) Thickness of the open cylindrical shell

\( K \) Modified Bessel function

\( K_1, K_2, K_3, K_4 \) Structural stiffness matrices

\( K_p \) Aerodynamic stiffness matrix

\( l \) Length of the open cylindrical shell

\( M_1 \) Structural mass matrix

\( M_p \) Aerodynamic mass matrix

\( q \) External force

\( R \) Radius of the open cylindrical shell

\( t \) Time

\( W \) Displacement shape function

\( T \) Transpose symbol

\( U \) Airflow velocity

\( w \) Transverse displacement of the panel

\( \bar{W} \) Alternative vector

\( \rho, \rho_a, \rho_f \) Mass densities of the open cylindrical shell, airflow and fluid, respectively

\( \Delta P \) Aerodynamic pressure

\( \varphi \) Opening angle

\( \mu \) Poisson’s ratio

\( \lambda \) Eigenvalue

\( \nabla^2 \) Laplace operator

Subscripts

\( m, n \) Modal indices

\( a \) Air

\( f \) Fluid

1. INTRODUCTION

Aeroelastics is an important branch in the field of fluid-structure coupling, and it is a typical dynamic theory which investigates the interaction effects among the aerodynamic force, the elastic force, and the inertial force. The open cylindrical shells are widely used in aircraft structures, high-speed trains, civil engineering structures, and so on. Under the subsonic airflow circumstances, aeroelastic problems of open cylindrical shells occur constantly. In order to prevent these structures from causing damage, the research on aeroelastic stability of open cylindrical shells in subsonic airflow is of great significance.
In recent years, there has been much literature in the study of the aeroelastic characteristics of cylindrical shells. Chen et al. investigated the aeroelastic characteristic of an anisotropic cylindrical shell.\(^1\) The effects of material and geometric parameters on the natural frequencies were analyzed. Karagiozis et al. studied the nonlinear vibration of clamped cylindrical shells subjected to axial aerodynamic force using theoretical and experimental methods.\(^2\) Based on the linear potential flow theory and Galerkin’s method, Pellicano and Amabili analyzed the nonlinear vibration and the aeroelastic stability of cylindrical shells filled with fluid.\(^3,4\) The influences of geometric nonlinearity and the damping were considered. Haddadpour et al. researched the aeroelastic stability of cylindrical shells with simply supported boundaries using the classical methods.\(^5\) Pellicano and Barbieri investigated the complex dynamics of circular cylindrical shells subjected to inertial axial loads using the nonlinear Sanders-Koiter theory.\(^6\) Zhou studied the vibration and stability of ring-stiffened thin-walled cylindrical shells conveying fluid.\(^7\) The effects of the airflow velocity, elastic modulus and geometric parameters on the critical instability velocity were considered. Jansen investigated the effect of boundary conditions on the nonlinear vibration and flutter of laminated cylindrical shells.\(^8\) Uğurlu and Ergin researched the dynamics and stability of circular cylindrical shells containing and submerged in flowing fluid using a higher order boundary element method.\(^9,10\) Amabili et al. systematically studied the non-linear dynamics and stability of circular cylindrical shells containing flowing fluid.\(^11-14\) Bochkarev and Matveenko carried out the numerical analysis of stability of a stationary or rotating circular cylindrical shell containing axially flowing and rotating fluid.\(^15\) Shen et al. studied the stability of fluid-conveying periodic shells on an elastic foundation with external loads applying the transfer matrix method.\(^16\) Paak et al. researched the nonlinear dynamics and stability of cantilevered circular cylindrical shells conveying fluid.\(^17,18\) Li and Yao analyzed the 1/3 subharmonic resonance of a nonlinear laminated cylindrical shell in subsonic air flow.\(^19\)

The dynamics and stability of open cylindrical shells were also investigated in the past decades. Yao and Li conducted the investigations on the aeroelastic stability and active control of a composite laminated open cylindrical shell in subsonic airflow.\(^20\) The variations of the natural frequencies with the airflow velocity were investigated using the generalized eigenvalue method. Selmane, and Lakis investigated the nonlinear vibration of anisotropic open cylindrical shells subjected to a flowing fluid.\(^21\) The finite element method and the classical cylindrical shell theory were used and the variations of the vibration frequency with the amplitude of structure were studied. Selmane, and Lakis also studied aeroelastic stability of anisotropic open cylindrical shells with different boundary conditions.\(^22\) The effects of the elastic modulus of the orthotropic material on the aeroelastic stability were considered. Tooran and Lakis performed a study on the dynamics and stability of anisotropic laminated open cylindrical shells filled with or subjected to a flowing fluid considering shear deformation using the finite element method.\(^23\) Pilgun and Amabili investigated the non-linear vibrations of shallow circular cylindrical panels with complex geometry using the meshfree technique based on the R-function theory.\(^24\) Ribeiro studied the nonlinear free periodic vibrations of open cylindrical shallow shells.\(^25\) Ribeiro and Jansen researched the nonlinear vibrations of laminated cylindrical shallow shells under thermomechanical loading.\(^26\) The influences of the temperature change, initial curvature, and panel thickness were analyzed. Li and Narita investigated the aeroelastic flutter of laminated shallow shells considering variable flow angles.\(^27\)

Many researchers used the two-way CFD/CSD coupled method which is also called the two-way FSI (fluid-structure interaction) method to calculate the aeroelastic properties of structures. Farhat et al. researched the pressure distribution and the response of the wing based on the CFD/CSD method in subsonic airflow.\(^28\) Carrion et al. carried out the aeroelastic analysis of wind turbines using a tightly coupled CFD/CSD method.\(^29\) Mian et al. conducted a comprehensive numerical investigation of geometric nonlinearity effect in high-aspect-ratio wing using CFD/CSD coupled approach.\(^30\) Liu et al. studied the aeroelastic characteristics of two- and three-dimensional wings.\(^31\) The flutter boundary calculated by the two-way CFD/CSD coupled method was compared with the experimental data.

Through the above literature review, it is noted that many of them have investigated the complex dynamics of open cylindrical shells under different loads, but few researchers have carried out the analysis on the aeroelastic stability of open cylindrical shells in subsonic airflow. In addition, few investigations based on both the theoretical and two-way CFD/CSD coupled methods have been conducted. Because of these reasons, the aeroelastic stability of open cylindrical shells is studied in the present work. The aerodynamic pressure model is derived from the linear potential flow theory. The Galerkin’s method is used to transform the partial differential equations into the ordinary differential equations. The variations of the fundamental frequency with the airflow velocity are investigated using the generalized eigenvalue method. The effects of material and geometric parameters on the critical instability velocity are discussed. Furthermore, the validity of the critical instability velocity calculated by the theoretical method is validated by the two-way CFD/CSD coupled method.

2. FORMULATION FOR THE EQUATION OF MOTION

The geometry of an open cylindrical shell in subsonic airflow as shown in Fig. 1, is considered. The length and radius of the open cylindrical shell are \(l\) and \(R\). The opening angle and thickness which is far less than the radius are \(\varphi\) and \(h\). In this model, one surface is subjected to the subsonic airflow, and the velocity of the airflow is \(U\). According to the Donnell theory, the equation of motion contains only normal displacement \(w\). Considering the effect of aerodynamic pressure, the equation of motion of the open cylindrical shell can be given as:

\[
D\nabla^2 \nabla^2 w + \frac{1}{R} \frac{\partial^2 f}{\partial x^2} + \rho h \frac{\partial^2 w}{\partial t^2} = \Delta P + q; \quad (1)
\]

\[
\frac{Eh}{R} \frac{\partial^2 w}{\partial x^2} - \nabla^2 \nabla^2 f = 0; \quad (2)
\]

where \(w\) is the displacement in the \(z\)-direction, \(E\) is the elastic modulus of the material, \(\rho\) is the density of the open cylindrical shell, \(f\) is the Airy stress function, \(q\) is the external force, \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}\) is the Laplace operator, \(D = \frac{Eh^3}{12(1-\mu^2)}\) is the bending stiffness of the open cylindrical shell, \(\mu\) is the Poisson’s ratio, and \(\Delta P\) is the aerodynamic pressure calculated using the linear potential flow theory.
For incompressible, inviscid, and irrotational subsonic airflow, the aerodynamic pressure $\Delta P$ can be determined from the linear potential flow theory, and it is written as:

$$\Delta P = \rho_0 \sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{\psi}_{m,n}(R) \left[ (\frac{\partial}{\partial t} + U \frac{\partial}{\partial x})^2 W_{m,n}(x,\theta) g_{m,n}(t) \right];$$

(3)

where $\rho_0$ is the density of the airflow and $\tilde{\psi}_{m,n}(R)$ is a function which is expressed as:

$$\tilde{\psi}_{m,n}(R) = K \left( \frac{2 \pi m \pi}{\varphi} l R \right);$$

(4)

where $K$ is the modified Bessel function of the second kind.

For the sake of simplification, we introduce the following alternative vector:

$$\tilde{W}_{m,n}(x,\theta) = \frac{\psi_{m,n}(R)}{\tilde{\psi}_{m,n}(R)} W_{m,n}(x,\theta).$$

(5)

So, the aerodynamic pressure can be represented as:

$$\Delta P = \rho_\infty \tilde{W}^T(x,\theta) \bar{g}(t) + 2U \frac{\partial \tilde{W}^T(x,\theta)}{\partial x} (\tilde{g}(t) + U \frac{\partial^2 \tilde{W}^T(x,\theta)}{\partial x^2} g(t));$$

(6)

where $\tilde{W}$ is defined as $\tilde{W} = [\tilde{W}_{11}, \tilde{W}_{12}, \ldots, \tilde{W}_{1N}, \tilde{W}_{21}, \tilde{W}_{22}, \ldots, \tilde{W}_{2N}, \ldots, \tilde{W}_{MN}]^T$.

The normal displacement $w$ and Airy stress function $f$ in Eqs. (1) and (2), are expressed as:

$$w(x,\theta,t) = \sum_{m=1}^{M} \sum_{n=1}^{N} W_{m,n}(x,\theta) g_{m,n}(t) = \tilde{W}^T(x,\theta) \bar{g}(t);$$

(7)

$$f(x,\theta,t) = \sum_{m=1}^{M} \sum_{n=1}^{N} F_{m,n}(x,\theta) \dot{g}_{m,n}(t) = \tilde{F}^T(x,\theta) \dot{q}(t);$$

(8)

where $\tilde{W} = [W_{11}, \ldots, W_{1N}, W_{21}, \ldots, W_{2N}, \ldots, W_{MN}]^T$ is the displacement shape function, $\tilde{F} = [F_{11}, \ldots, F_{1N}, F_{21}, \ldots, F_{2N}, \ldots, F_{MN}]^T$ is the Airy stress shape function, and $\dot{g} = [\dot{g}_{11}, \ldots, \dot{g}_{1N}, \dot{g}_{21}, \ldots, \dot{g}_{2N}, \ldots, \dot{g}_{MN}]^T$ and $\dot{q} = [\dot{q}_{11}, \ldots, \dot{q}_{1N}, \dot{q}_{21}, \ldots, \dot{q}_{2N}, \ldots, \dot{q}_{MN}]^T$ are the generalized coordinates.

The boundary conditions of the open cylindrical shell with four simply supported edges can be expressed as:

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad (x = 0, x = l);$$

(9)

$$w = 0, \quad \frac{\partial^2 w}{\partial \theta^2} = 0, \quad (\theta = 0, \theta = \varphi).$$

(10)

So, the Airy stress function in Eq. (8), can be assumed as:

$$W_{m,n}(x,\theta) = \sin \left( \frac{m \pi x}{l} \right) \sin \left( \frac{2 n \pi \theta}{\varphi} \right)$$

(11)

where $m$ is the number of the half waves along the axial direction and $n$ is the number of the circumferential waves.

Substituting Eqs. (7), (9), (10), and (11), into Eq. (2), the following equation can be obtained:

$$\nabla^2 \nabla^2 f = 0, \quad (x = 0, x = l, \theta = 0, \theta = \varphi).$$

(12)

So, the Airy stress shape function in Eq. (8), can be assumed as:

$$F_{m,n}(x,\theta) = \sin \left( \frac{m \pi x}{l} \right) \sin \left( \frac{2 n \pi \theta}{\varphi} \right).$$

(13)

Substituting Eqs. (6), (7), and (8), into Eqs. (1), and (2), we can obtain the following formulations:

$$D \nabla^2 \nabla^2 \tilde{W}^T \tilde{g}(t) + \frac{1}{R} \frac{\partial^2 \tilde{W}^T}{\partial x^2} q + \rho \frac{\partial^2 \tilde{W}^T}{\partial t^2} g(t) = \rho_\infty \tilde{W}^T \bar{g}(t) + 2U \frac{\partial \tilde{W}^T}{\partial x} \tilde{g}(t) + U^2 \frac{\partial^2 \tilde{W}^T}{\partial x^2} \tilde{g}(t);$$

(14)

$$E h \frac{\partial^2 \tilde{W}^T}{\partial x^2} g(t) = \nabla^2 \nabla^2 T \tilde{q}(t) = 0.$$  

(15)

By substituting Eqs. (5), (11), and (13), into Eqs. (14), and (15), and performing the Galerkin’s method, one can obtain the following ordinary differential equations:

$$M_1 \ddot{g} + K_1 \dot{g} + K_2 \dot{q} = M_p \ddot{g} + C_p \dot{g} + K_p g;$$

(16)

$$K_3 \ddot{g} - \dot{K}_3 = 0;$$

(17)

where $K_1, \ldots, K_4$ are the structural stiffness matrices, $M_1$ is the mass matrix, and $M_p$, $C_p$ and $K_p$ are the aerodynamic mass, damping, and stiffness matrices which are induced by the external subsonic airflow.

Based on Eqs. (16), and (17), a standard state equation can be written as:

$$M \ddot{g} + C \dot{g} + K g = 0;$$

(18)

where:

$$M = M_1 - M_p;$$

(19)

$$C = -C_p;$$

(20)

$$K = K_1 + K_2 K_4^{-1} K_3 - K_p.$$  

(21)

The general solution of Eq. (18), can be expressed as:

$$\dot{g} = g_0 e^{\lambda t};$$

(22)

where $\lambda$ and $g_0$ are the eigenvalue and eigenvector of the system. Substituting Eq. (22), into Eq. (18), we can obtain the following generalized eigenvalue equation:

$$\left( M \lambda^2 + C \lambda + K \right) g_0 = 0.$$  

(23)
The sufficient and necessary conditions for obtaining a nonzero solution of Eq. (23) require the determinant of coefficient to be zero, which can be written as:

$$[M\lambda^2 + C\lambda + K] = 0 \quad (24)$$

The natural frequencies under different airflow velocities can be obtained by solving Eq. (24), and the critical instability flow velocity can be determined by the fundamental frequency (first order natural frequency), which will be illustrated in the following section.

3. NUMERICAL SIMULATIONS AND DISCUSSION

3.1. Verification with Open Literature

The formulations presented in this paper and the MATLAB codes in the numerical simulations will be verified through comparing the present natural frequencies with and without the outside fluid with those described by Selmane and Lakis. The structural parameters of the open cylindrical shell are as follows: $R = 0.235 \text{ m}$, $l = 0.94 \text{ m}$, $h = 0.00235 \text{ m}$, and $\varphi = 360^\circ$. The material parameters of the cylindrical shell are: $E = 219.81 \text{ GPa}$, $\rho = 7850 \text{ kg/m}^3$, and $\mu = 0.3$. The density of the static external flow is $\rho_f = 1000 \text{ kg/m}^3$.

The results are shown in Table 1. The natural frequencies obtained by using the present method agree well with the results from Selmane and Lakis.

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>In vacuum</th>
<th>Outside fluid</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>$(1,4)$</td>
<td>$(1,9)$</td>
</tr>
<tr>
<td>Selmane and Lakis &amp;</td>
<td>659   &amp; 2187</td>
<td></td>
</tr>
<tr>
<td>Present results</td>
<td>692.4 &amp; 2168.6</td>
<td></td>
</tr>
</tbody>
</table>

The natural frequencies of the open cylindrical shell made of fiberglass for different radii are shown in Fig. 3. It is seen from Fig. 3(a) that for $R = 3 \text{ m}$ the fundamental frequencies for the thickness $h = 0.001 \text{ m}$ or $h = 0.0015 \text{ m}$ are determined for $m = 1$ and $n = 3$, respectively. And from Fig. 3(b) it is found that for $R = 2 \text{ m}$ the fundamental frequency for the thickness $h = 0.001 \text{ m}$ can be obtained when $m = 1$ and $n = 3$, and the fundamental frequency corresponds to $m = 1$ and $n = 2$ when the thickness is enlarged to $h = 0.0015 \text{ m}$.

The material parameters of carbon fiber plastic are considered firstly. For different radii of the open cylindrical shell, the varying curves of the fundamental frequency with the airflow velocity are obtained for different thicknesses and shown in Figs. 4 and 5. It is seen from Figs. 4 and 5 that the fundamental frequency of the open cylindrical shell decreases with the increase of the airflow velocity. Consequently, the critical instability flow velocity can be determined when the fundamental frequency decreases to zero which indicates that the structural system is in the instability state. For $R = 3 \text{ m}$, it is observed in Fig. 4 that the critical instability velocity is $U = 68 \text{ m/s}$ when the thickness is $h = 0.001 \text{ m}$, but the critical instability velocity increases to $U = 117 \text{ m/s}$ when the thickness increases to $h = 0.0015 \text{ m}$. It can be concluded that for the same radius, the critical instability velocity increases with the increase of the thickness of the open cylindrical shell, which indicates that...
the open cylindrical shell with thicker thicknesses exhibits better aerodynamic properties than that with thinner thicknesses.

The varying curves of fundamental frequency of the open cylindrical shell for $R = 2$ m is shown in Fig. 5. It is found that the critical instability velocity is $U = 107$ m/s when the thickness is $h = 0.001$ m and the critical instability velocity increases to $U = 140$ m/s when the thickness augments to $h = 0.0015$ m.

The influences of the radius on the aeroelastic properties of the open cylindrical shell are also studied. For $h = 0.001$ m, the variation of the fundamental frequency with the flow velocity is displayed in Fig. 6. It is observed that the critical instability velocity decreases with the increase of the radius. So, the open cylindrical shell with a smaller radius shows better aeroelastic properties than that with a larger radius.

Now, the fiberglass material is considered. The varying curves of the fundamental frequency of the open cylindrical shell with airflow velocity are calculated and shown in Figs. 7 and 8. Compared with Figs. 4 and 5, the similar phenomena are observed. It is seen from Fig. 7 that for $R = 3$ m, the critical instability velocity is $U = 96$ m/s when the thickness is $h = 0.001$ m and the critical instability velocity increases to $U = 163$ m/s when the thickness augments to $h = 0.0015$ m. And from Fig. 8 it is found that for $R = 2$ m the critical instability velocity is $U = 152$ m/s when the thickness is $h = 0.001$ m and the critical instability velocity increases to $U = 196$ m/s when the thickness is enlarged to $h = 0.0015$ m.

The influences of the material properties on the aeroelastic behaviors of the open cylindrical shell are analyzed. For these two kinds of materials, the variations of the fundamental frequency with the flow velocity are computed for $R = 3$ m and $h = 0.001$ m and shown in Fig. 9. It can be concluded that generally, the open cylindrical shell with a larger elastic modulus shows better aeroelastic properties than that with a smaller elastic modulus.

4. VALIDATION WITH CFD/CSD METHOD

4.1. Modelling

In order to prove the correctness of the theoretical analysis results, the identical case in section 3 is solved in ANSYS simulation. Moreover, most aircraft structures are so complicated that it is difficult to find out the theoretical solution. However, the two-way CFD/CSD numerical method possesses advantages on dealing with complex structures. That is to say, it can be used to analyze the aeroelastic stability of complex structures. Therefore, in this section, the validation of the CFD/CSD method used in the analysis of the aeroelastic stability is carried out. The open cylindrical shell is modelled using the finite element software ANSYS. Figure 10 shows the open cylindrical shell model and the fluid field. It is seen that the fluid field is close to a rectangular solid and the structural hexahedron mesh can be used to divide the fluid field. In this
model, the upper surface of the open cylindrical shell is subjected to the subsonic airflow. The left and the right ends of the fluid field are the entrance and exit of the subsonic airflow. The middle surface which contacts with the open cylindrical shell is considered to be the fluid-structure coupling surface.

The fluid-structure coupled calculation is transient and the computing time in the solid and fluid fields is \( t = 1 \) s. For the time settings, the step length is set to be \( 10^{-4} \) s and the number of iterations is 60 times. At the beginning of computing, the uniform pressure is imposed on the structure and the action time is \( 10^{-2} \) s.

The structural parameters and material parameters of the open cylindrical shell as those in section 3 are considered. The density of air is \( \rho_a = 1.29 \) kg/m\(^3\). The critical instability velocities of the open cylindrical shell have been theoretically determined as shown in Figs. 4–9 for different structural and material parameters. In this section, the critical instability velocities calculated in section 3 as shown in Figs. 4–9 are selected to calculate the time domain responses of the structure using the two-way CFD/CSD coupled method to verify the validity of the present theoretical method.

4.2. Results and Verification

Firstly, the material parameters of carbon fiber plastic are considered. The pressure nephograms of fluid-structure interaction surface at different times are shown in Fig. 11 for the critical instability airflow velocity \( U = 68 \) m/s as shown in Fig. 4. It is observed that the fluid-structure interaction surface has obvious deformation, which indicates that displacement of the open cylindrical shell is induced by the airflow.

As shown in Fig. 11(a), the displacement of the open cylindrical shell is upward and the whole pressure field is negative. The distribution of pressure in the fluid field is relatively complicated when \( t = 1 \) s as shown in Fig. 11(b). The left side of the shell is in a positive pressure area, and the middle of the open cylindrical shell is in a negative pressure area. It can be observed from Fig. 11(b) that the absolute value of the maximum negative pressure is larger than the maximum positive pressure.

The time domain responses at the center point of the open cylindrical shell with \( R = 3 \) m and \( h = 0.001 \) m are calculated using the ANSYS software for the flow velocities \( U = 60 \) m/s, \( U = 65 \) m/s, \( U = 68 \) m/s, and \( U = 70 \) m/s and displayed in Fig. 12. When the flow velocity is \( U = 60 \) m/s, the time domain response is convergent quickly as shown in Fig. 12(a). It can be seen from Fig. 12(b) that the time domain responses decrease gradually with the increase of the time for the flow velocity \( U = 65 \) m/s (less than but close to the critical instability velocity \( U = 68 \) m/s calculated theoretically in Fig. 4), but the extent of decay is not great. It is obviously found from
Fig. 12(c) that the vibration amplitude at the center point of the open cylindrical shell remains unchanged for the airflow velocity \( U = 68 \text{ m/s} \). Consequently, the structural response does not decay, and it can be concluded that \( U = 68 \text{ m/s} \) corresponds to the critical instability velocity which is the same as that obtained in Fig. 4 by using the theoretical method. Fig. 12(d) shows the time-domain response at the center point of the open cylindrical shell for \( U = 70 \text{ m/s} \) which is a little larger than the critical instability velocity \( U = 68 \text{ m/s} \). It is noted that the vibration response is divergent, which means that the divergence instability occurs.

The time domain responses of the structure with \( R = 3 \text{ m} \) and \( h = 0.0015 \text{ m} \) are also obtained by the ANSYS software for the airflow velocities \( U = 110 \text{ m/s}, U = 115 \text{ m/s}, U = 117 \text{ m/s} \), and \( U = 120 \text{ m/s} \) and shown in Fig. 13. It can be seen that the time domain response of the structure changes from convergence to divergence with the increase of the airflow velocity. Moreover, the vibration amplitude at the center point of the open cylindrical shell remains unchanged basically for the airflow velocity \( U = 117 \text{ m/s} \), and it can be concluded that \( U = 117 \text{ m/s} \) corresponds to the critical instability velocity which is also the same as that obtained in Fig. 4 by using the theoretical method.

The time responses at the center point of the open cylindrical shell with \( R = 2 \text{ m} \) are also calculated as shown in Fig. 14. It is found from Fig. 14 that the vibration amplitude at the center point of the open cylindrical shell remains unchanged for the airflow velocity \( U = 107 \text{ m/s} \) when the thickness is \( h = 0.001 \text{ m} \). It is also observed that the equal amplitude vibration happens for the airflow velocity \( U = 140 \text{ m/s} \) when the thickness augments to \( h = 0.0015 \text{ m} \). The results calculated by the ANSYS are also coincident with the results obtained by the theoretical method as shown in Fig. 5.

Now, the fiberglass material is considered. Figure 15 shows the time responses at the center point of the open cylindrical shell made of fiberglass for \( R = 3 \text{ m} \) and \( R = 2 \text{ m} \). The vibration amplitude remains unchanged for the airflow velocity \( U = 96 \text{ m/s} \) when the thickness is \( h = 0.001 \text{ m} \) as shown in Fig. 15(a). That is to say, \( U = 96 \text{ m/s} \) is the critical instability velocity calculated by ANSYS when the thickness is \( h = 0.001 \text{ m} \), and the critical instability velocity increases to \( U = 163 \text{ m/s} \) when the thickness augments to \( h = 0.0015 \text{ m} \) as shown in Fig. 15(b). And from Figs. 15(c) and (d) it is found that for \( R = 2 \text{ m} \) the critical instability velocity is \( U = 152 \text{ m/s} \) when the thickness is \( h = 0.001 \text{ m} \) and the critical instability velocity increases to \( U = 196 \text{ m/s} \) when the thickness is increased to \( h = 0.0015 \text{ m} \). The critical instability velocities calculated by ANSYS are also basically coincident with those obtained in Figs. 7 and 8 by using the theoretical method.

After the above calculations, it can be concluded that the results obtained by the theoretical method and the two-way CFD/CSD coupled method have a very good agreement with each other, which verifies the validity of the present investigation.

5. CONCLUSIONS

The aeroelastic characteristics of an open cylindrical shell in subsonic airflow are studied theoretically, and the time domain responses of the shell are also calculated using the two-way CFD/CSD coupled method and compared with the results of the theoretical method to verify the validity of the present work. The aeroelastic stability of the shell is analyzed by solving the eigenvalue problem. The effects of the material and geometric parameters on the fundamental frequency are analyzed. From the investigation results, the following conclusions can be drawn:

1. The fundamental frequency of the open cylindrical shell decreases with the increase of the airflow velocity, and the aeroelastic system is in the instability state when the structural fundamental frequency decreases to zero.

2. With the increase of the thickness of the shell, the critical instability velocity increases also. The open cylindrical shell...
Figure 12. Responses of the structure made of carbon fiber plastic in the time domain with different airflow velocities for $R = 3 \text{ m}$ and $h = 0.001 \text{ m}$.

Figure 13. Responses of the structure made of carbon fiber plastic in the time domain with different airflow velocities for $R = 3 \text{ m}$ and $h = 0.0015 \text{ m}$.
with thicker thicknesses exhibits better aeroelastic properties than that with thinner thicknesses.

(3) The open cylindrical shell with a smaller radius shows better aeroelastic properties than that with a larger radius, and the shell with a larger elastic modulus also displays better aeroelastic behaviors than that with a smaller elastic modulus.

(4) The results obtained by the two-way CFD/CSD coupled method have a very good agreement with those calculated by the theoretical analysis method.

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Energy Distribution Analysis Regarding the Crack Size in a Rotating Shaft

Cristina Castejón, María Jesús Gómez, Juan Carlos García-Prada and Eduardo Corral

MAQLAB Group, Mechanical Dept. University Carlos III of Madrid, Av. de la Universidad, 30, 28911, Leganés, Madrid, Spain.

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Maintenance is critical to avoid catastrophic failures in rotating machinery, and the detection of cracks plays a critical role because they can originate failures with costly processes of reparation, especially in shafts. Vibration signals are widely used in machine monitoring and fault diagnostics. The most critical issue in machine monitoring is the suitable selection of the vibration parameters that represent the condition of the machine. Discrete Wavelet Transform, and one of its recursive forms, called Wavelet Packet Transform, provide a high potential for pattern extraction. Several factors must be selected and taken into account in the Wavelet Transform application such as the level of decomposition, the suitable mother wavelet, and the level basis or features. In this work, the dynamic response of a shaft with different levels of crack is studied. The evolution of energy of the vibration signals obtained from the rotating shaft and the frequencies where maximum increments of energy appear with the crack are analyzed. The results allow the conclusion that changes in energies computed by means of the Wavelet Packet Transform can be successfully used for crack detection.

1. INTRODUCTION

Great efforts have been made to understand the dynamic behavior of rotating machinery, most times using numerical models, as in the research of Felix et. al. Special interest has been focused on the dynamics of cracked rotating machinery because crack detection in rotors is a very important subject in maintenance. Some states of art are dedicated to the effects of cracks in these elements. Studies about the dynamics of cracked rotors have been carried out with different methods with the aim of selecting patterns able to detect fault conditions. It was highlighted by Bachschmid and Penacci that despite the high number of papers published in this area, it was not usual that they present experimental results.

A review of crack identification in rotating shafts was made by Papadopoulos. The diagnosis techniques were classified by applying either model-based or vibration methods.

Model-based methods considered loads in the place of the crack that generates the same effects as the crack does. These types of models were mainly based on information extracted from directly measured signals, signal models, and process models. That is the case of works such as Refs. and.

On the other hand, vibration methods were based in detecting variations in the response assigned to a crack. Sabnavis et. al. affirmed that 1x and 2x components experience very important changes at the steady state with the presence of a crack, however they were very hard to be detected in practice due to different factors, such as noise or assembly. Signals analyzed by vibration methods can come from real systems as in the case of Ref. 2 or models, as in the cases of Refs. 10 and 1. The run up/down of a rotor has been studied in other works. It was affirmed in Ref. that in a cracked rotor there was no certain critical speed, but a wide frequency band covering the critical speed where violent oscillations were observed. These oscillations can be used for crack detection.

In the last years, different technologies have been used to process signals coming from dynamical systems. Most authors used three approaches for the analysis of vibration signature: time-domain approaches, based on statistical parameters such as mean, root mean square, variance, kurtosis, frequency-domain approaches (based on Fast Fourier Transform (FFT)), and its variations, and finally time-frequency analysis approaches, such as Wavelet Transform (WT).

FFT and Hilbert transform have been traditionally used to observe changes in the response in eigenfrequencies when a crack appears. However in the last years new techniques classified by Sabnavis et. al. as “non-traditional methods” have appeared. Nowadays the most used technique to treat the signals is WT analysis. WT was used due to its effectiveness treating non-stationary signals. A review of the applications of WT to fault diagnosis in machines was shown in Ref. WT has been successfully applied for the case of bearings, gears, and beams.

WT is the next step of FFT, because it gives information both in time and frequency domain, offering the proper treatment both for stationary and for non-stationary signals. However, WT has a disadvantage: the incapability for decomposing high frequency bands where information about faults can be located. The Wavelet Packet Transform (WPT) constitutes an improvement of WT, allowing the decomposition of all frequency bands and the location of the differences of cracked and healthy elements. The applications of WPT are highly increasing and nowadays its use of diagnosing cracked rotating elements is spreading. WPT coefficients can be directly used as features as they contain reliable information about faults, nevertheless WPT normalized energy parame-
ters were proposed as crack indicators in Ref. 26. The use of WPT is not a straightforward issue; great diversity of mother wavelets and decomposition levels can be used. Besides, there is not a standard methodology to select them.

In this work, experimental vibration signals are obtained from a rig. An energy distribution analysis of the signals, computed using the WPT, is performed. The WPT energy is suitable for feature extraction because it offers simple parameters easy and clear to handle. WPT energies of all signals are calculated by packets, and their evolution with the crack size are analyzed. Special attention is paid to frequencies related to the first harmonics of the rotation speed, since they are considered to be the best indicators of crack.

2. WAVELET PACKETS TRANSFORM (WPT)

WT is especially efficient to carry out local analysis of non-stationary signals. The same way as FFT obtains the correlation coefficients of a signal with a sinusoidal function, WT obtains the correlation coefficients of the signal analyzed with the mother wavelet selected. Coefficients obtained depend on the scale and position of the mother wavelet, getting information both of time and frequency domain. WT can be applied in a continuous way, Continuous Wavelet Transform (CWT), or discrete, Discrete Wavelet Transform (DWT).

CWT allows the analysis of structures of signals through their correlation coefficients, instead of using the whole signal. The mathematic formulation of CWT is represented as:

\[ T(a, b, \varphi) = w \int_{-\infty}^{\infty} x(t) \psi^*(\frac{t-b}{a}) dt; \]  

where \( x(t) \) represents the temporary signal and \( \varphi \) the mother wavelet. The weight function is represented by \( w, b \) is related to the position, and \( a \) is the scale of the mother wavelet. \( T(a, b, \varphi) \) are the coefficients of the wavelets that depend on \( a, b \) and \( \varphi \). The product of the conjugate mother wavelet with the signal is integrated in the whole range of the signal. This operation is known mathematically as a convolution. 27

Starting from the definition of CWT, working with discrete wavelets is common because signals comprise discrete data and DWT improves the computational cost. To discretize data, the dyadic grid is used, where \( a = 2^k \). Then, the equation of DWT results in:

\[ T(2^k, b) = \frac{1}{2^k} \int_{-\infty}^{\infty} x(t) \psi^*(\frac{t-b}{2^k}) dt; \]  

where \( k \) is the decomposition level, is called octave, and consists in each level in which the signal is decomposed.

However, in computational terms, DWT is more suitable using the definition of Ref. 15 based on the use of filter banks. The decomposition is performed by filtering the signal using a low pass filter \( g \) getting approximation \( (A) \) information and using a high pass filter \( h \) getting detail \( (D) \) information, represented as:

\[ A[j] = \sum_{n=-\infty}^{\infty} x[n]g[2j-n]; \]

\[ D[j] = \sum_{n=-\infty}^{\infty} x[n]h[2j-n]. \]

The decompositions halve the frequency band of the input, so according to the Nyquist rule, it is necessary to downsample by two. 26

Both the Multiresolution Analysis (MRA) and WPT are extensions of DWT that consist on the application of DWT in a recursive way. WPT solve the disadvantage of the MRA, where the downsampling process can not be performed for the \( D \) information. 28 This disadvantage of the MRA causes a lower spectral resolution at high frequencies and a low temporary resolution at low frequencies. On the other hand, using WPT, both information \( A \) and \( D \) can be decomposed recursively until the desired resolution. Using WPT decomposition, if \( W(k, j) \) represents the coefficients of the signal in each packet, \( k \) the decomposition level, and \( j \) the position of the packet within the decomposition level, each correlation vector \( W(k, j) \) has the structure of:

\[ W(k, j) = \{w_1(k, j), ..., w_N(k, j)\} = \{w_i(k, j)\}; \]

where \( i \) is the position of the coefficient within its packet.

2.1. Energy of the WPT Coefficients

The concept of energy used in WPT analysis is very close to the Fourier Theory. 25 The energy of packet \( E_{k,j} \) represents the energy contained in the frequency band covered by the packet. The energy of a packet is obtained from the sum of all the squares of its coefficients \( w_i(k, j) \) as in:

\[ E_{k,j} = \sum_i \{w_i(k, j)\}^2. \]

2.2. Frequency contents using WPT

The decomposition level selected determines the frequency resolution offered by a single packet \( f_r \) (the same for all packets at each level). Using decomposition level \( k \), the number of packets obtained is \( 2^k \). Considering the sampling frequency \( f_s \), the global frequency resolution of the signal is \( F = f_s / 2 \), according to the Nyquist theorem. The frequency resolution of each node \( f_s \) is given by:

\[ f_r = \frac{F}{2^k}. \]

The use of WPT causes reflections in the frequencies related to sub-harmonics power of two of the sampling frequency \( (\frac{f_s}{4}, \frac{f_s}{8}, ...). \) 28 Reflections appear due to the use of real filters instead of ideal filters. The use of real filters causes duplication of certain frequencies due to the crossing of the low-pass filters and the high-pass filter at each decomposition iteration.

3. EXPERIMENTAL WORK

The energy distribution analysis performed in this manuscript was carried out using real data obtained from
Machine Simulation Fault. The experimental setup consisted of a rig with an aluminum shaft, two bearings, and a motor that drove the shaft through an elastic coupling. The rig is shown in Fig. 1. The properties of the shaft are shown in Table 1.

The shaft was tested in healthy condition, and tests were initially designed to include 9 different crack depths. Later, in practice, crack depth 4 could not be machined; thus, 8 crack depths were tested, and crack depth 4 is missing in the results. All defects were induced with a saw cut. The saw had a width of 1 mm, the minimum available. The values of crack depths, expressed as relation of depth of the crack \( a \) vs diameter of the shaft \( D \), are shown in Table 2.

A scheme of the shaft with dimensions and the location of the flat face cracks (representing a crack depth 9) and a real induced defect (crack depth 2) are shown in Fig. 2.

Tests were performed at a steady state at three different conditions of rotating speed: 20 Hz, 40 Hz and 60 Hz. The number of points measured was \( 2^{14} \) with a sampling frequency \( f_s = 6 \) kHz. The measurement chain was composed first by a uniaxial accelerometer B&K 4383 measuring accelerations in the vertical direction. Later, a signal conditioner B&K NEXUS and an acquisition card Keithley KUSB-3100 were used.

For each condition of fault and speed, 1,500 measures were taken. To avoid random noise, signals were averaged by 100. Then, the number of measures to handle was 15 by condition. Figure 3 shows individual time domain signals obtained at healthy shaft and shaft cracked with depth 9, both rotating at 60 Hz.

Figure 4 shows the average FFT of 100 signals obtained consecutively. The figure shows FFT at healthy shaft and shaft cracked with depth 9, both rotating at 60 Hz.

### 3.1. Modal analysis

An experimental modal analysis of the rig was carried out to identify the natural frequencies of the rig and the shaft. First, the modal analysis was performed with the shaft dismounted and later with the shaft mounted on it at the different depths of crack. Thus, frequencies related to the shaft and to the rig could be clearly distinguished.

The structural frequencies of the rig are shown in Table 3. The first structural frequency of the shaft is shown in Table 4. That frequency could be theoretically calculated and was confirmed using CAE simulations. The speeds of the test were far from this value, so the rotor could be considered as rigid.
4. RESULTS AND DISCUSSION

An analysis of WPT energies of all the signals measured is performed. The aim is to select the best features with the aim of locating the packets or nodes that experiment larger changes when a crack appears.

“Daubechies 6” mother wavelet is used due to its proved effectiveness in this area. Considering a single-level basis (packets from a unique decomposition level), only terminal nodes are selected. The value of $f_r$ must be small enough to avoid influences of other frequencies that can contaminate the effects of the crack. After preliminary studies, the decomposition level selected is 9. Thus, the frequency resolution of each node $f_r$ is 5.85 Hz, according to Eq. (7).

The WPT energy is processed for all the packets obtained at decomposition level 9. The relative energy of each packet $i$, $\Delta E_{ri}$, calculated regarding the value of the healthy condition $(E_{hi})$, is calculated for all the cases:

$$\Delta E_{ri} = \frac{E_i - E_{hi}}{E_{hi}}; \quad (8)$$

where $E_i$ is the energy of the packet $i$ for the specified decomposition level, for all the crack depths tested (from 1 to 8) and for all the speeds (20, 40 and 60 Hz). The parameter $E_{hi}$ represents the energy of the packet $i$ for the same speed condition and in the healthy case.

The evolution of $\Delta E_{ri}$ is analyzed for the different cases of speed and crack. Special attention is paid to the packets $i$ related to the first three harmonics of the rotation speed (1x, 2x and 3x) because, according to the bibliography, they are the best indicators of cracks in shafts. Packets with maximum relative energy increments are searched.

4.1. Energy Analysis for Harmonics of Rotation Speed

Figures 5, 6, and 7 show the evolution of mean values of $\Delta E_i$ for the packets related to the first three harmonics of the

### Table 4. Value of first structural frequency of the shaft.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$f_{s1}$</th>
<th>Value (Hz)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td>316.4</td>
</tr>
</tbody>
</table>

Figure 3. Time domain individual signals rotating at 60 Hz for (a) healthy shaft and (b) shaft cracked depth 9.

Figure 4. FFT averaged across 100 individual signals rotating at 60 Hz for (a) healthy shaft, and (b) shaft cracked depth 9.
rotation speed.

At the speed of 20 Hz, as in Fig. 5, \( \Delta E_r \) do not show any correlation with the crack size for any harmonic; relative energy increments are very similar excepting for the defect depth 1. However, energy increments regarding the healthy conditions are significant, and above crack depth 2 all the defects could be easily detected. At 40 Hz, as in Fig. 6, a negative correlation of the energy increments with the crack size is observed; however there is atypical behaviour for defect levels 1 and 5. The same atypical behaviour in crack depths 1 and 5 is observed at the speed of 60 Hz as in Fig. 7, where a clear positive correlation is observed. \( \Delta E_r \) are, in general, higher for the third harmonic than for the first and second.

A clear influence of the speed is observed. When the speed increases, \( \Delta E_r \) values are also higher reaching the maximum values at 60 Hz for the third harmonic, where the relative energy increments are almost 700% for the crack depth 9.

4.2. Energy Analysis for Packets with Maximum \( \Delta E_r \)

After analyzing the evolution of \( \Delta E_r \) for the packets related to harmonics of the rotation speed, the same study is applied to the whole signal. The mean values of \( \Delta E_r \) are calculated for all the packets obtained for the 8 crack conditions. The 9 packets that offer maximum values of \( \Delta E_r \) for each crack depth (from 1 to 8) are selected. A total number of 72 packets (9 maximum values multiplied by 8 crack depths) could be selected for each speed; nevertheless, most of the packets are coincident at different crack depths, so the number of packets selected is reduced when the packets more excited are the same, independently of the crack size.

Tables 5, 6 and 7 show the selection of packets, ordered from highest to lowest values of \( \Delta E_r \) for the crack depth 9. Analyzing the results, most of the frequency bands that increase their energy with the crack size are related to structural frequencies of the rig, called \( f_1 \), \( f_2 \) and \( f_3 \) (see Table 3).

In most cases, packets selected match also sub-harmonics power of two of the sampling frequencies \( f_s \). The reflections caused by the use of WPT amplify the phenomena caused by the crack in the structural frequencies.

Figures 8, 9 and 10 show \( \Delta E_r \) at the three different speeds. These figures show \( \Delta E_r \) of the defect depths 3, 6 and 9 for all the significant packets shown in Tables 5, 6 and 7 in the same
Values of $\Delta E_r$ are very significant at the speed of 20 Hz as shown in Fig. 8. Even for the case of crack depth 3, a minimum relative increasing of energy of 47% is observed for the packet number 9, and a maximum of almost 90% is found for the packet 1. Above crack depth 3, the $\Delta E_r$ values are very similar, and they do not depend on the crack size; therefore, a threshold value could be established for detecting crack sizes tested with reliability.

At 40 Hz as shown in Fig. 9, values of $\Delta E_r$ above 15% are found even for the crack depth 3 for all packets selected. For packets from 1 to 7, where reflections of the sampling frequency are found, crack depths 3, 6, and 9 experiment energy increasing above 50%, and the values are very similar independently of the crack size. For the rest of packets, the maximum value of energy increment is commonly found at the crack depth 3.

On the other hand, at 60 Hz as shown in Fig. 10, the maximum value of $\Delta E_r$ is found at the crack depth 6 for all packets except 13 and 15. Thus, a correlation between the energy increments for each packet versus the crack size is not de-

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**Table 6.** Selection of frequency bands with maximum $\Delta E_r$ at 40 Hz, ordered from highest to lowest $\Delta E_r$ in crack depth 9.

<table>
<thead>
<tr>
<th>Speed (Hz)</th>
<th>Order</th>
<th>Frequency band (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>1</td>
<td>1054.7-1060.5 (15f₁)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>978.5-984.4 (21f₁ and 21f₉/128)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>943.4-949.2 (close to 20f₁)</td>
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<tr>
<td></td>
<td>4</td>
<td>931.6-937.5 (20f₁ and 5f₉/32)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>937.5-943.4 (20f₁ and 5f₉/32)</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>984.4-990.2 (21f₁ and 21f₉/128)</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1037.1-1043 (close to 22f₁)</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>386.7-392.6</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>363.3-369.1 (close to 8f₁)</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>380.9-386.7 (close to 8f₁)</td>
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<tr>
<td></td>
<td>11</td>
<td>744.1-750 (16f₁ and f₉/8)</td>
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<tr>
<td></td>
<td>12</td>
<td>369.1-375 (8f₁ and f₉/16)</td>
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</tbody>
</table>

**Table 7.** Selection of frequency bands with maximum $\Delta E_r$ at 60 Hz, ordered from highest to lowest $\Delta E_r$ in crack depth 9.

<table>
<thead>
<tr>
<th>Speed (Hz)</th>
<th>Order</th>
<th>Frequency band (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>1</td>
<td>228.5-234.4 (5f₁)</td>
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<tr>
<td></td>
<td>2</td>
<td>234.4-240.2 (5f₁)</td>
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<tr>
<td></td>
<td>3</td>
<td>240.2-246.1 (close to 4f₁)</td>
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<td></td>
<td>4</td>
<td>293-298.8 (2f₁)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>351.6-357.4 (5f₂)</td>
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<tr>
<td></td>
<td>6</td>
<td>187.5-193.4 (4f₁ and f₉/32)</td>
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<td></td>
<td>7</td>
<td>855.5-890.6 (19f₁)</td>
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<td></td>
<td>8</td>
<td>181.6-187.5 (4f₁ and f₉/32)</td>
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<td></td>
<td>9</td>
<td>1500-1505.9 (32f₁ and f₉/4)</td>
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<td></td>
<td>10</td>
<td>1494.1-1500 (32f₁ and f₉/4)</td>
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<td>11</td>
<td>375-380.9 (8f₁ and f₉/16)</td>
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<td>369.1-375 (8f₁ and f₉/16)</td>
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<td>13</td>
<td>984.4-990.2 (21f₁ and 21f₉/128)</td>
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<td>14</td>
<td>744.1-750 (16f₁ and f₉/8)</td>
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<td>15</td>
<td>978.5-984.4 (21f₁ and 21f₉/128)</td>
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<td></td>
<td>16</td>
<td>2244.1-2250 (48f₁ and 3f₉/8)</td>
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<td></td>
<td>17</td>
<td>363.3-369.1 (close to 8f₁ and f₉/16)</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>380.1-386.7 (close to 8f₁ and f₉/16)</td>
</tr>
</tbody>
</table>
tected. However, the values of $\Delta E_r$ are very high, finding in all cases (except packets 17 and 18) relative increments higher than 100%.

At 20 Hz the number of packets selected and analyzed is 9, the minimum one. That means that the packets with maximum $\Delta E_r$ match for all the crack sizes. At 40 and 60 Hz there are more variations in the packets with maximum changes in energy (12 packets selected at 40 Hz and 18 at 60 Hz). For the cases of 40 and 60 Hz, there are also more variations in the values of $\Delta E_r$; however, there are not correlation of $\Delta E_r$ with the crack size. When the speed increases, the increasing of energy is more chaotic. The packets selected correspond to structural frequencies, as proved when matching the significant frequencies to the structural ones. Results show that when a crack appears, not only frequencies related to breathing phenomena are observed (1, 2, and 3 harmonics of rotating speed), but also structural frequencies are excited. Crack effects are modulated in structural frequencies and duplicated in some cases due to reflections of the sampling frequency $f_s$. Changes in energy are more significant in these frequencies than in the harmonics of the rotation speed. This phenomenon can be assigned to a modulation of the crack effects to the structural frequencies, amplified by the reflections of the sampling frequency due to the use of WPT. Specifically, harmonics $8 \times f_1$, $16 \times f_1$, $32 \times f_1$ and $48 \times f_1$, that also match sub-harmonics of the sampling frequency, experience very significant increasing at all speeds, for all crack conditions.

5. CONCLUSIONS

An energy analysis of vibrations signals obtained from a rotating machine is performed for eight different crack depths. First, a modal analysis of the rig is carried out and experimental vibration signals are obtained in the housing bearing, at different fault conditions and at different rotation speeds at steady state. A WPT energy analysis is carried out.

The rotation speed has a strong influence in the relative increment of energy $\Delta E_r$ with the crack size, and at 60 Hz a clear correlation between energy increments and the crack size is observed. Values of $\Delta E_r$ above 100% are detected even for the smallest crack sizes for frequencies related to harmonics of the rotation speed.

However, frequencies different from the harmonics and related to structural frequencies of the rig seem to be more affected by the crack than them. In this case, the energy increments are more clearly observed when the speed increases. At 60 Hz, relative increments of more than 400% even for the smallest crack sizes are found. Due to the reflections caused by the use of WPT in sub-harmonics power of two of the sampling frequency, phenomena appearing at those frequencies are amplified. Then, when structural frequencies of the rig meet sub-harmonics power of two of the sampling frequency in the same packet, increments of energy due to the crack are amplified and clearly observed. Thus, experiments can be designed in order to force this coincidence.

It can be concluded that WPT can be used for crack detection with reliability by establishing a threshold value for the energy of certain packets.

6. ACKNOWLEDGMENTS

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REFERENCES


Thermo-Mechanical Vibration Analysis of Imperfect Inhomogeneous Beams Based on a Four-Variable Refined Shear Deformation Beam Theory Considering Neutral Surface Position

Farzad Ebrahimi and Ali Jafari
Department of Mechanical Engineering, Faculty of Engineering, Imam Khomeini International University, Qazvin, Iran.

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In this disquisition, an exact solution method is developed for analyzing the vibration characteristics of porous functionally graded (FG) beams by considering neutral surface position and different thermal loadings via a four-variable shear deformation refined beam theory. Four types of environmental conditions through the z-axis direction are assumed: uniform (UTR), linear (LTR), nonlinear (NLTR) and sinusoidal (STR) temperature rises. Mechanical properties of porous FG beams are supposed to vary through the thickness direction and are modeled via the modified power-law. The modified power-law is formulated using the concept of even and uneven porosity distributions. Since the variation of pores along the thickness direction influences the mechanical properties, porosity plays a key role in the mechanical response of FG structures. The governing differential equations and related boundary conditions of porous FG beams are subjected to temperature field that is derived by Hamilton’s principle based on a four-variable refined theory which verifies shear deformation regardless of any shear correction factor. The Navier-type solution procedure is used to achieve the natural frequencies of porous-FG beams supposed to various thermal loadings which satisfies the simply-simply boundary condition. A parametric study is led to carry out the effects of material graduation exponent, porosity volume fraction, different porosity distribution, and thermal effect on dimensionless frequencies of porous FG beams. It is concluded that these parameters play noticeable roles in the vibration behavior of imperfect FG beams. Presented numerical results can be applied as benchmarks for future designs of imperfect FG structures with porosity phases.

1. INTRODUCTION

Technology development in the field of making materials with functional properties introduces Functionally Graded Materials (FGMs) as a new class of smart composite structures. FGMs have led many researchers to analyze the mechanical specifications of these materials with engineering structures like beams, plates and shells. Due to the high strength and temperature resistance of FGMs, they are increasingly utilized in mechanical, civil, and aerospace engineering as structural components.1–3 FGMs are advanced types of composite materials with inhomogeneous micromechanical structure where the concentration, shape and orientation of constituent phases vary in one or more directions for optimizing the performance. Typically, FGMs are composed of two different parts such as ceramics with excellent characteristics in heat and corrosive resistances and metal with good toughness. Functional grading of the material properties is often in one direction. However, grading can be implemented in several directions. These materials have been developed for general purpose structural components such as rocket engine components or turbine blades where the components are exposed to extreme temperatures. The FGMs were introduced by Japanese scientists in mid-1980s as aerospace application for the first time. FGMs possess various advantages in comparison with traditional composites. Some of these advantages are: multi functionality, the ability to control deformation, corrosion resistance, dynamic response, the minimization or removal of stress concentrations, smoothing the transition of thermal stress and resistance to oxidation. Recently, due to the need for new materials for engineering applications in modern industries including aerospace, nuclear energy, turbine components, rocket nozzles, critical furnace parts, FGMs have received wide attention.4–7 The advantages of using FGM structures in general engineering structures have been increasingly recognized in recent decades. Therefore it is important to understand the behaviors of engineering structures made of FGMs. For example, the vibration, static and dynamic behavior of FG beams and plates are often found in general engineering structures.8,9 A large number of investigations are documented in literature dealing with the static, buckling and dynamic characteristics of FGM structures.10,11 Pradhan and Chakraverty have presented free vibration FG beam characteristics using Euler and Timoshenko beam theories.12 The Rayleigh-Ritz method was used to obtain frequencies in their analysis. Thai and Vo investigated the bending and free vibrations of FG beams within the framework of different higher-order shear deformation beam theories.5 Due to the huge application of FG structures in different fields such as civil, marine and aerospace engineering, and the difference between the making and working temperatures of FG structures, it is important to take into account the thermal effect when designing FG structures. Mahi et al. studied the temperature-dependent vibration behavior of FG beams.13 The important influence of temperature change on the vibration re-
response of FG beams is also taken into account. Recently, thermal post-buckling and the vibration of thermally post-buckled FG beams were analyzed by Esfahani et al. The generalized differential quadrature method was employed to solve the equations of motion.

With the rapid development in technology of structural elements, structures with graded porosity can be introduced as one of the latest developments in FGMs. The structures treat pores as microstructures by taking the local density into account. Researches focus on the development of preparation methods for FGMs such as powder metallurgy, vapor deposition, self-propagation, centrifugal casting, and magnetic separation. These methods have their own ineffectiveness such as the complexity of the technique and its high costs. An efficient way to manufacture FGMs is sintering process in which, due to difference in solidification of the material constituents, porosities or micro-voids through material can be created. An investigation has been carried out on porosities that exist in FGMs that have been fabricated by a multi-step sequential infiltration technique. According to this information, for building more secure and accurate structures, it’s important to consider the porosity impact on designing FGM structures. Porous FG structures have many interesting combinations of mechanical properties, such as high stiffness in conjunction with very low specific weight. Since porous FG structures have reached remarkable attention by many engineers, recent papers in the field of FG structures discuss the mechanical responses of structural ingredients made of porous FGMs. Wattanasakulpong and Unghakhorn examined the linear and non-linear vibration of porous FGM beams with elastically restrained ends. Ebrahimi and Mokhtari provided differential transformation methods to examine the vibration behavior of rotating Timoshenko FG beams with even porosities. They reported that porosity volume fraction plays a key role in the vibrational response of the FG beams. In order to predict flexural vibration of porous FG Timoshenko beams, Wattanasakulpong and Chaikittiratana used the Chebyshev collocation method. Ebrahimi and Zia applied the Galerkin and multiple scales methods to solve nonlinear vibration of porous FG beams. Ebrahimi and Zia presented that the thermo-mechanical vibration response of temperature-dependent porous FG beams are subject to various temperature risings based on the classic beam theory (CBT). CBT disregards the influence of shear deformation. In other words, CBT is unable to model thick beams and higher modes of vibration. Hereupon, the first order shear deformation theory (FSDT) is suggested to overcome the defects of CFT with supposition a shear correction factor in the thickness direction of beam. As respects FSSTD isn’t able to evaluate the zero-shear stress on the top and bottom surfaces of the beam, there appeared a need to develop new theory. In order to bypass these defects, higher order shear deformation theory (HSDST) was introduced. This theory predicts transverse shear stresses without need of any shear correction factors. Many papers are published which framework HSDST to investigate mechanical response of FG structures. Moreover, Ait Yahia et al. studied the porosity effect on the wave propagation of FG plates by using various higher-order shear deformation theories. Recently, Mechab et al. developed the nonlocal two-variable refined plate theory for the free vibration of FG porous nanoplates that are resting on elastic foundations. Most recently Ebrahimi et al. studied the vibration of porous FG Euler beams subjected to thermal loadings.

It is worth mentioning that, although various inclusion-related studies about the vibration of FG beams have been conducted in recent years, no published works consider the porosity and thermal effect on the vibration response of imperfect FG beams with the concept of exact neutral axis position and different porosity distributions based on the four-variable refined shear deformation theory. This complicated problem is not well-investigated and there is a need for further studies.

The present research developed a four-variable refined shear deformation theory for the thermo-mechanical vibration of FG beams with porosities. Reduced beam theory considers a constant transverse displacement and higher-order variation of axial displacement through the depth of the beam so that there is no need for any shear correction factors. Two kinds of porosity distribution, namely even and uneven through the thickness directions, are considered. Four types of environmental conditions through the z-axis direction are supposed as: uniform (UTR), linear (LTR), nonlinear (NLTR) and sinusoidal (STR) temperature rises. The modified power-law model is used to describe the gradual variation of the material properties of porous beams. By applying Hamilton’s principle, governing equations of higher order MEE-FG beams are obtained together based on four-variable refined shear deformation theory and are solved by applying an analytical solution method. Several numerical exercises indicate that various parameters such as thermal environment, power-law exponent, porosity parameters and types of porosity distribution have remarkable influence on fundamental frequencies of porous FG beam.

2. THEORETICAL FORMULATIONS

2.1. Power-low Functionally Graded Beams with Porosities

Figure 1 shows a uniform functionally graded beam with a rectangular cross-section of length \( L \), width \( b \) and thickness \( h \). The beam was made up of homogeneous and isotropic functionally graded materials in which the volume fraction and micro-structural morphology of the material compositions varied continuously in thickness direction. Functionally graded materials are the new generation of composite materials. In this case, composite materials were usually produced from two or more different materials. For this study, FG materials were made from a mixture of ceramic and metal. The effective material properties of FG beams including Young’s modulus \( E \), shear modulus \( G \), mass density \( \rho \) and thermal expansions \( \alpha_{\text{exp}} \) change continuously in the thickness direction (z-axis) based on power-law distribution. Poisson’s ratio was assumed to be constant in the z-axis direction. In this paper, FG beams were assumed to have porosities spreading within the thickness due to defects during production.

The effective material properties \( P_f \) of FG beams with two kind of porosities that distributed identical in two phases of ceramic and metal were expressed by using the modified rule of mixture: \(^{22}\)

\[
P_f = P_c(V_c - \frac{\alpha}{2}) + P_m(V_m - \frac{\alpha}{2}).
\]

Where \( \alpha \) denotes the volume fraction of porosities \( \alpha \ll 1 \), for perfect FGM \( \alpha \) was set to zero, \( P_c \) and \( P_m \) were the ma
terial properties of ceramic and metal. \( V_c \) and \( V_m \) were the volume fractions of ceramic and metal. The compositions represented in relation to:36

\[
V_c + V_m = 1. \tag{2}
\]

Then the volume fraction of ceramic phase \( (V_c) \) was defined as follows:

\[
V_c = \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n; \tag{3}
\]

where \((0 \leq n)\) was the volume fraction index that determined the material distribution through the thickness of the beam. According to this distribution, we had a metal beam for the large value of \( n \) and when \( n \) equals zero, a ceramic beam remained and \( Z \) was the distance from the mid-plane of the graded beam.

According to Eqs. (1), (2), the effective material properties of FG beams with even porosities (FGM-I) were expressed in the following forms.36

\[
P(z_{ns}) = (P_c - P_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{P_m - \alpha}{2} (P_c + P_m); \tag{4a}
\]

\[
E(z_{ns}) = (E_c - E_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{E_m - \alpha}{2} (E_c + E_m); \tag{4b}
\]

\[
\rho(z_{ns}) = (\rho_c - \rho_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{\rho_m - \alpha}{2} (\rho_c + \rho_m); \tag{4c}
\]

\[
G(z_{ns}) = (G_c - G_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{G_m - \alpha}{2} (G_c + G_m); \tag{4d}
\]

\[
\alpha(z_{ns}) = (\alpha_c - \alpha_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{\alpha_m - \alpha}{2} (\alpha_c + \alpha_m). \tag{4e}
\]

Here, it should be noted that the FGM-I had porosity phases with even distributions of volume fraction over the cross section. The FGM-II had porosity phases that spread frequently near the middle zone of the cross-section and the amount of porosity seemed to decrease linearly to zero at both the top and bottom of the cross-section. Figur 2 shows examples of cross-section areas of FGM-I and-II with porosity phases. For the second type, uneven distribution of porosities (defined as FGM-II), the effective material properties were replaced by the form below.36

\[
P(z_{ns}) = (P_c - P_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{P_m - \alpha}{2} (P_c + P_m) \left( 1 - \frac{2 |z_{ns} + c|}{h} \right); \tag{5a}
\]

\[
G(z_{ns}) = (G_c - G_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{G_m - \alpha}{2} (G_c + G_m) \left( 1 - \frac{2 |z_{ns} + c|}{h} \right); \tag{5b}
\]

\[
E(z_{ns}) = (E_c - E_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{E_m - \alpha}{2} (E_c + E_m) \left( 1 - \frac{2 |z_{ns} + c|}{h} \right); \tag{5c}
\]

\[
\rho(z_{ns}) = (\rho_c - \rho_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{\rho_m - \alpha}{2} (\rho_c + \rho_m) \left( 1 - \frac{2 |z_{ns} + c|}{h} \right); \tag{5d}
\]

\[
\alpha(z_{ns}) = (\alpha_c - \alpha_m) \left( \frac{z_{ns} + c}{h} + \frac{1}{2} \right)^n + \frac{\alpha_m - \alpha}{2} (\alpha_c + \alpha_m) \left( 1 - \frac{2 |z_{ns} + c|}{h} \right). \tag{5e}
\]

The position of the neutral axis of the imperfect FG beams were determined to satisfy the first moment with respect to elastic stiffness being zero as follows:

\[
\int_{-h/2}^{h/2} E(z_{ms})(z_{ms} - C)dz_{ms} = 0. \tag{6}
\]

Consequently, the position of neutral surface can be obtained as:

\[
C = \int_{-h/2}^{h/2} E(z_{ms})z_{ms}dz_{ms} = 0. \tag{7}
\]

For more precise anticipation of FGMs behavior under extreme temperature fields, material properties must be dependent on temperature. Therefore, temperature-dependent coefficients of material phases were expressed according to the non-linear equation below.37

\[
P = P_0 (P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3). \tag{8}
\]

In this case, \( P_0, P_{-1}, P_1, P_2 \) and \( P_3 \) were the temperature dependent coefficients that were tabulated in the table materials properties (Table 3) for Si\(_3\)N\(_4\) and SUS304. The bottom and top surface of FG beams were supposed to be fully metal (SUS304) and fully ceramics Si\(_3\)N\(_4\), respectively.
Deformation beam theory was expressed as follows:

$$\gamma_{zz} = \left(1 - \frac{\partial f}{\partial x}\right) \frac{\partial w_s}{\partial x} = g \frac{\partial w_s}{\partial x};$$  (14)

where $\epsilon_{xx}$, $\gamma_{zz}$ was the normal and shear strains and $g(z) = \left(1 - \frac{1}{2} \frac{z}{h}\right)$ was the shape function of the transverse shear strains as following:

$$g(z_n) = 1 - \left[1 - \tanh^2 2 \left(\frac{z_n c}{h}\right) - \frac{4}{\cosh^2(1)} \left(\frac{z_n c}{h}\right)^2\right].$$  (15)

The Euler Lagrange equations have been used to derive the equation of motion by using Hamilton’s principle, in which the motion of an elastic structure in the time interval $t_1 < t < t_2$ is so that the integral with respect to time of the total potential energy is extremum:

$$\int_{t_1}^{t_2} \delta(U - T + V)dt = 0;$$  (16)

where $U$ was strain energy, $V$ was work done by external forces and $T$ was kinetic energy. The virtual variation of strain energy $\delta U$ was calculated as:

$$\delta U = \int_v \sigma_{ij} \delta \epsilon_{ij} dV = \int_A^{L} \left(\sigma_{xx} \delta \epsilon_{xx} + \sigma_{xz} \delta \gamma_{xz}\right) dx dA;$$  (17)

where $\delta$ was the variation symbol, $A$ was the cross-section area of the uniform beam, $\sigma_{xx}$ the axial stress and $\sigma_{xz}$ was the shear stress, by substituting the expressions for $\epsilon_{xx}$ and $\gamma_{xz}$ into Eq. (15) as:

$$\delta u = \int_0^L \int_A^{L} \sigma_{xx} \left[\frac{\partial u}{\partial x} - z \frac{\partial^2 w_b}{\partial x^2} - f \frac{\partial^2 w_s}{\partial x^2}\right] dAdx +$$

$$\int_0^L \int_A^{L} \sigma_{xz} \left[\frac{\partial w_s}{\partial x}\right] dAdx;$$  (18)

$$\delta u = \int_0^L \int_A^{L} \left[\frac{\partial N}{\partial x} - \frac{\partial^2 M_b}{\partial x^2} \delta w_b - \frac{\partial^2 M_s}{\partial x^2} \delta w_s +$$

$$\frac{\partial Q_{xz}}{\partial x} \delta w_s\right] dAdx.$$  (19)

In which the variables introduced in arriving at the last expression were defined as follows:

$$(N, M_b, M_s) = \int_A^{L} \sigma_{xx}(1, z_n, f) dA, \quad (Q) = \int_A^{L} \sigma_{xz}(g) dA.$$  (20)

The first variation of the virtual kinetic energy was written in the form:

$$\delta T = \frac{1}{2} \int_0^L \left[ I_0 \left(\frac{\partial^2 u}{\partial t^2}\right) dx + I_2 \left(\frac{\partial^2 w_b}{\partial t^2}\right) dx + k_2 \left(\frac{\partial^2 w_s}{\partial t^2}\right) \delta u +$$

$$I_1 \left[\frac{\partial^2 u}{\partial t \partial x}\right] dx + J_1 \left[\frac{\partial^2 w_b}{\partial t \partial x}\right] dx - J_2 \left[\frac{\partial^2 w_s}{\partial t \partial x}\right] dx +$$

$$I_0 \left[\frac{\partial^2 w_b}{\partial t \partial x}\right] dx + \left(\frac{\partial^2 w_s}{\partial t \partial x}\right) dx + \left(\frac{\partial^2 w_s}{\partial t^2}\right) \delta u\right] dx.$$  (21)
where \((I_0, I_1, I_2, J_1, J_2, k_2)\) was the mass inertia that can be defined as:

\[
(I_0, I_1, I_2) = \int_A \rho(z_{ns}, T)(1, z_{ns}, z_{ns}^2) dA;
\]

\[
(J_1, J_2, K_2) = \int_A \rho(z_{ns}, T)(f, f z_{ns}, f^2) dA.
\] (22)

Also the first variation of potential energy was written in the form:

\[
\delta V = \int_0^L \left[ f(x)\delta u + q(x)\delta (w_x + w_b) + N^T \frac{\partial (w_x + w_b)}{\partial x} \right] dx.
\] (23)

In this study, for analyzing vibration of porous FG beam in thermal environment, the first variation of external loadings due to thermal loadings was obtained as:

\[
\delta V = \int_0^L \left[ N^T \frac{\partial (w_x + w_b)}{\partial x} \right] dx.
\] (24)

where \(N^T\) was defined as following:

\[
N^T = \int_{-h/2-C}^{h/2+C} E(z_{ns}, T)\alpha_{exp}(z_{ns}, T)\Delta T dz.
\] (25)

In which \(\alpha_{exp}\) is the coefficient of thermal dilatation that is typically positive and very small (0 \(\ll\) \(\alpha < 1\)). By inserting Eqs. (18), (21) and (24) into Eq. (14) and setting the coefficients of \(\delta u\), \(\delta w_b\) and \(\delta w_s\) to zero, the following Euler-Lagrange equations were obtained:

\[
(\delta u : 0), \frac{\partial N}{\partial x} = I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 w_b}{\partial t^2 \partial x} - J_1 \frac{\partial^2 w_s}{\partial t^2 \partial x};
\] (26)

\[
(\delta w_b : 0), \frac{\partial^2 M_b}{\partial x^2} - K_0 \frac{\partial^2 (w_x + w_b)}{\partial x^2} =
I_0 \left( \frac{\partial^2 w_x}{\partial t^2} + \frac{\partial^2 w_b}{\partial t^2} \right) + I_1 \frac{\partial^3 u}{\partial t^2 \partial x^2} - J_1 \frac{\partial^3 w_x}{\partial t^2 \partial x^2};
\] (27)

\[
(\delta w_s : 0), \frac{\partial^2 M_s}{\partial x^2} + \frac{\partial Q_{xx}}{\partial x} - K_0 \frac{\partial^2 (w_x + w_b)}{\partial x^2} =
I_0 \left( \frac{\partial^2 w_x}{\partial t^2} + \frac{\partial^2 w_b}{\partial t^2} \right) - k_2 \frac{\partial^4 w_x}{\partial t^2 \partial x^2} + J_1 \frac{\partial^3 u}{\partial t^2 \partial x^2} - J_2 \frac{\partial^4 w_x}{\partial t^2 \partial x^2};
\] (28)

For a material that is linearly elastic and obeys the 1D Hooke’s law, the relation between stress-strain was described as:

\[
\sigma_{xx} = E(z_{ns})\varepsilon_{xx};
\] (29)

\[
\sigma_{xz} = G(z_{ns})\gamma_{xz};
\] (30)

where \(G\) was the shear modulus and \(E\) was the Young’s modulus, by substituting the Eqs. (11), (12) into Eqs. (31) and (32) and subsequent results into Eq. (17) and integrating over the beam’s cross-section, stress resultants were derived as:

\[
N = A_x \frac{\partial u}{\partial x} - B_{xx} \frac{\partial^2 w_b}{\partial x^2} - C_{xx} \frac{\partial^2 w_s}{\partial x^2};
\]

\[
M_b = B_{xx} \frac{\partial u}{\partial x} - D_{xx} \frac{\partial^2 w_b}{\partial x^2} - E_{xx} \frac{\partial^2 w_s}{\partial x^2};
\]

\[
M_s = C_{xx} \frac{\partial u}{\partial x} - E_{xx} \frac{\partial^2 w_b}{\partial x^2} - F_{xx} \frac{\partial^2 w_s}{\partial x^2};
\]

\[
Q = A_x \frac{\partial w_s}{\partial x}.
\] (31)

In which the cross-sections rigidities were calculated as follows:

\[
(A_{xx}, B_{xx}, C_{xx}, D_{xx}, E_{xx}, F_{xx}) =
\int_A \left( 1, z, z_{ns}^2, f(z_{ns}, T) \right) dA.
\] (32)

\[
A_{xz} = \int_A g^2 G(z_{ns}, T) dA.
\] (33)

The last form of Euler-Lagrange equations for porous FG beam subjected to thermal loading based on new hyperbolic shear deformation beam theory in terms of displacement \(u, w_b\) and \(w_s\) were derived as:

\[
A_{xx} \frac{\partial^2 u}{\partial x^2} - B_{xx} \frac{\partial^3 w_b}{\partial x^4} - C_{xx} \frac{\partial^3 w_s}{\partial x^4} =
I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 w_b}{\partial t^2 \partial x} - J_1 \frac{\partial^2 w_s}{\partial t^2 \partial x};
\] (34)

\[
B_{xx} \frac{\partial^3 u}{\partial x^3} - D_{xx} \frac{\partial^4 w_b}{\partial x^4} - E_{xx} \frac{\partial^4 w_s}{\partial x^4} - N^T \frac{\partial^2 (w_b + w_s)}{\partial x^2} =
I_0 \frac{\partial^2 (w_b + w_s)}{\partial t^2} + I_1 \frac{\partial^2 w_b}{\partial t^2 \partial x} - J_1 \frac{\partial^2 w_s}{\partial t^2 \partial x} - J_2 \frac{\partial^4 w_s}{\partial t^2 \partial x^2};
\] (35)

\[
C_{xx} \frac{\partial^3 u}{\partial x^3} - F_{xx} \frac{\partial^4 w_b}{\partial x^4} - G_{xx} \frac{\partial^4 w_s}{\partial x^4} + N_T \frac{\partial^2 (w_b + w_s)}{\partial x^2} =
I_0 \frac{\partial^2 (w_b + w_s)}{\partial t^2} - k_2 \frac{\partial^4 w_s}{\partial t^2 \partial x^2} + J_1 \frac{\partial^3 u}{\partial t^2 \partial x^2} - J_2 \frac{\partial^4 w_s}{\partial t^2 \partial x^2};
\] (36)

3. SOLUTION METHOD

3.1. Analytical Solution

In this section, an analytical solution of the Euler-Lagrange equations for vibration of S-S porous functional grading beam based on Navier type method were provided. The displacement variables were adopted as combinations of non-significant coefficients and known trigonometric functions to satisfy Lagrange equation and boundary conditions. The following displacements variables were assumed to be formed as:

\[
u(x, t) = \sum_{m=1}^{\infty} \nu_m \cos \left( \frac{m\pi}{L} x \right) e^{i\omega_m t};
\] (37)
in which \((w_m, w_{bm}, w_{sm})\) are the unknown Fourier coefficient that will be calculated for each value of \(m\).

By substituting Eqs. (38), (39), (40) into Eqs. (35), (36), (37) respectively, leads to Eqs. (41), (42), (43):

The analytical solutions were obtained from the following equation:

\[
\begin{align*}
- & A_{xx} \left( \frac{m \pi}{L} \right)^2 + I_0 \omega^2 \left( \frac{m \pi}{L} \right) w_m + B_{xx} \left( \frac{m \pi}{L} \right)^3 - \\
& I_1 \omega^2 \left( \frac{m \pi}{L} \right) w_{bm} + C_{xx} \left( \frac{m \pi}{L} \right)^3 - \\
& - J_1 \omega^2 \left( \frac{m \pi}{L} \right) w_{sm} = 0;
\end{align*}
\]

\[
\begin{align*}
\left[ -B_{xx} \left( \frac{m \pi}{L} \right)^3 - I_1 \omega^2 \left( \frac{m \pi}{L} \right) \right] & u_m + \left[ -D_{xx} \left( \frac{m \pi}{L} \right)^4 + \\
& \dot{N}^2 \left( \frac{m \pi}{L} \right)^2 + \omega^2 (I_0 + I_2 \left( \frac{m \pi}{L} \right)^2) \right] w_m + \\
& \omega^2 (I_0 + J_2 \left( \frac{m \pi}{L} \right)^2) w_{sm} = 0;
\end{align*}
\]

\[
\begin{align*}
\left[ -C_{xx} \left( \frac{m \pi}{L} \right)^3 - J_1 \omega^2 \left( \frac{m \pi}{L} \right) \right] & u_m + \left[ -E_{xx} \left( \frac{m \pi}{L} \right)^4 + \\
& \dot{N}^2 \left( \frac{m \pi}{L} \right)^2 + I_0 \omega^2 + J_2 \omega^2 \left( \frac{m \pi}{L} \right)^2 \right] w_{bm} + \\
& \omega^2 (k_2 \left( \frac{m \pi}{L} \right)^2 + I_0) w_{sm} = 0;
\end{align*}
\]

4. THERMAL ENVIRONMENT AND TEMPERATURE DISTRIBUTIONS

4.1. Uniform Temperature Rise (UTR)

A porous FG beam that had a reference temperature equal to \(T_0 = 300\) and is free of stresses at \(T_0\) was used. The temperature of the FG beam was uniformly raised to a final temperature with the difference of \(\Delta T\) as:

\[
\Delta T = T - T_0.
\]

4.2. Linear Temperature Rise (LTR)

By assuming that the temperature of the top surface of the porous FG beam was \(T_t\) and varied linearly from \(T_t\) to \(T_b\), the bottom surface temperature and the temperature rise was finally able to be determined as:

\[
T = T_m + \Delta T \left( \frac{1}{2} + \frac{z_{ns} + c}{h} \right).
\]

And \(\Delta T\) was defined as:

\[
\Delta T = T_t - T_b.
\]

4.3. Nonlinear Temperature Rise (NLTR)

By solving the following equation, the steady-state one-dimensional heat conduction equation with the known temperature boundary conditions on bottom and top surfaces of the FG beam was achieved:

\[
\left[ 41 - \frac{d}{dz} \left( \kappa(z, T) \frac{dT}{dz} \right) \right] = 0;
\]

\[
T \left( \frac{h}{2} \right) = T_c, \quad T \left( \frac{h}{2} \right) = T_m.
\]

The following equation solved the solution of Eq. (51) subjected to the boundary conditions:

\[
T = T_m + \left( \Delta T \right) \int_{h/2}^{z_{ns} + c} \frac{1}{\kappa(z, T)} \frac{dz}{h}.
\]

where \(\Delta T = T_c - T_m\).

4.4. Sinusoidal Temperature Rise (STR)

The temperature field when FG beam was exposed to sinusoidal temperature rise across the thickness can be defined as:

\[
T = T_m + \left( \Delta T \right) \left( 1 - \cos \frac{\pi}{2} \left( \frac{1}{2} + \frac{z_{ns} - c}{h} \right) \right);
\]

where \(\Delta T = T_c - T_m\) is temperature change.

5. NUMERICAL EXAMPLE

In the following section, the validation of porous FG beams with S-S boundary conditions was confirmed in Table 3.

Then, the influence of porosity distributions, porosity volume fraction, power-law exponent and temperature rises on the non-dimensional frequencies of the porous FG beam were explored. The functionally graded porous beam was comprised of Steel (SUS304) and Silicon nitride (Si₃N₄) where its properties are given in Table 1. It was supposed that the temperature
Table 1. Temperature dependent coefficients of thermal expansion coefficient, Young’s modulus, mass density and Poisson’s ratio for Si₃N₄ and SUS304. 

| Material | Properties | $P_0$ | $P_{-1}$ | $P_1$ | $P_2$ | $P_3$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Si₃N₄</td>
<td>$E$ (MPa)</td>
<td>348.43e+9</td>
<td>-3.070e-4</td>
<td>2.160e-7</td>
<td>-8.946e-11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha$ (K⁻¹)</td>
<td>5.8723e-6</td>
<td>9.095e-4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho$ (Kg/m³)</td>
<td>2370</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\nu$</td>
<td>0.24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>SUS304</td>
<td>$E$ (MPa)</td>
<td>201.04e+9</td>
<td>3.079e-4</td>
<td>-6.534e-7</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha$ (K⁻¹)</td>
<td>12.330e-6</td>
<td>8.086e-4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\rho$ (Kg/m³)</td>
<td>8166</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\nu$</td>
<td>0.3262</td>
<td>-2.002e-4</td>
<td>3.797e-7</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

rise in fully metal surface to reference temperature $T_0$ of the FG beam is $T_m - T_0 = 5K$.\(^{30}\)

The non-dimensional natural frequencies ($\lambda$) were calculated by relations in Eq. (51).

$$\lambda = \omega \frac{L^2}{h} \sqrt{\frac{\rho n}{e_{n}}} \tag{51}$$

To verify the accuracy of the present method, the numerical results obtained were compared with those available in the literature to demonstrate the performance of the present study. For this purpose, non-dimensional frequencies of FG beams with S-S boundary condition were compared with those of Şimşek and Thai for different volume fraction indexes and slenderness.\(^{6,11}\) Hereupon, $f(x)$ was considered based on the Reddy beam theory as $\frac{4x^3}{L^3}$ and the FG beams made of alumina and aluminum were compared with the results from Şimşek and Thai which had been obtained by using Lagrange’s equations and the Navier type solution.\(^6\) Computations had been carried out for the following material and beam properties: ($E_{Al} = 70$GPa, $\rho_{Al} = 2702$kg/m³, $\nu_{Al} = 0.3$, $E_{Al_2O_3} = 380$GPa, $\rho_{Al_2O_3} = 3960$kg/m³, $\nu_{Al_2O_3} = 0.3$) for $L/h = (5, 20)$. Table 2 found that the non-dimensional frequency parameters obtained in the present investigation were in agreement with the results provided in these literatures and thus validated the proposed method of solution.

In the present section, results were extracted for various temperature changes, power-law indexes, porosity parameters with four temperature rises (UTR, LTR, NLTR, STR) and two porosity distributions (even, uneven) to present an adequate sensitivity analysis.

As a first verification and investigation example, effects of porosity volume fraction, thermal loading, porosity distribution and power-law exponents on the first non-dimensional natural frequency of the porous FG simply supported beams were assessed. The results were extracted for different porosity parameters ($a = 0.0, 0.1, 0.2$), gradient index ($n = 0.1, 0.2, 0.5, 1$), given for three temperature changes ($\Delta T = 20, 40, 80K$) and constant value of ($L/h = 20$) at Table 3. Two types of porosity distributions were considered (even and uneven), as well as temperature rises included uniform, linear, nonlinear and sinusoidal distributions. Present results are derived using the Navier type solution method.

Results given in Table 3 show that the growing of the power-law exponents provide lower non-dimensional frequencies of porous FG beams. In fact, the $n = 0$ beam is made from ceramics and has the greatest frequency. Increasing the power law exponents from 0 to 5 changes the composition of the FG beams from a fully ceramic beam to a beam with a combination of ceramic and metal. By increasing the metal percentage and having the smaller value of Young’s modulus in metal with respect to ceramic, the stiffness of the system decreases. Thus, as also known from mechanical vibrations, natural frequencies decrease as the stiffness of a structure decreases. In addition, it is obvious from this Table that increasing temperature change (UTR, LTR, NLTR, STR) yields decreasing of natural frequencies. This indicates that increasing changes in temperature yield a decrease in Young’s modulus E. This effect would increase if the temperature was raised. It is concluded that four temperature rises have a considerable effect on the vibration behavior of FG porous beams. It is found that the non-dimensional frequency of porous FG beams under sinusoidal temperature rise is higher than FG beam subjected to NLTR, and the frequency of FG beam subjected to NLTR is higher than that subjected to LTR, which is higher than that under UTR. The difference between non-dimensional frequencies of different temperature rises (UTR, LTR, NLTR, STR) becomes larger by increasing changes in temperature. The reason is that the rigidity of the FG beams for sinusoidal temperature rise is greater than the other cases of temperature rises. According to results of this table, it is beheld, when the power-law indexes are in the range of [0-1], the natural frequencies grow with the increase in the porosity parameters for every temperature increase and porosity distributions. Because of the internal pores in FG growth rigidity beams, this situation is more eminent for lower values of material graduation. Increasing porosity decreases fundamental frequencies when the percentage of the metal is higher than ceramic ($1 < n$) and temperature change is constant. However, this trend is opposite with increasing changes in temperature. Comparing the frequency of porous FGM beams with even and uneven porosity distribution revealed that when the power index is in the range of [0–0.5], natural frequencies of the even porosity are higher than uneven however, this treatment is vice versa in the range of power law index over 0.5. In addition, for a certain values of temperature change and gradient index, changes in the porosity parameter ($a$) leads to more variations in frequencies of even distribution in comparison of uneven. In other words, in FGM I, the porosity has more significant impact on natural frequency of beam than that of FGM II.

Table 4 presents the effect of various temperature changes, porosity parameters, material graduations on the non-dimensional frequency of the S-S imperfect FG beams subjected uniform temperature rise with both porosity distributions. Here again, it is seen that by increasing the material power law index, the non-dimensional frequencies decrease. This is due to the increment in flexibility of the FG beams, since the percentage of metal phase increases when power in-
It can be stated from Fig. 2 that the first dimensionless natural frequency variations of the FG porous beams with simply supported boundary condition subjected to uniform temperature rising for different values of porosity and gradient index parameters is plotted. Non-dimensional natural frequencies of FG beams that are near zero, where the critical point is, decrease with the increase of temperature change. This is because of the reduction in the total stiffness of the beam. Geometrical stiffness shows a decrease when temperature rises. We can get higher frequency results before the critical temperature if the porosity volume fraction is higher in value for a porous FG beam. On the other hand, after the critical temperature this behavior is vice versa. Furthermore, it can be stated that the temperature change can soften FG beam at pre-buckling region in a way that when the temperature rise this effect will be increased. Lower porosity indexes will cause to a decrease of stiffness of the structure. By consideration of the lower porosity parameter, this is the main reason for postponing of branching point of the FG beam. Also, it can be seen that increasing the material graduation exponents leads to reduction in the non-dimensional frequency for every type of porosity distribution. In fact, when power-law is equal to zero beam is made from fully Si₃N₄ and has the greatest frequency. Increasing the material graduation exponent from 0 to 10 changes the composition of the FG beams from a full Si₃N₄ beam to a beam with a combination of Si₃N₄ and SUS304. By increasing the metal percentage and having the smaller value of Young’s modulus of SUS304 with respect to Si₃N₄, the stiffness of the system decreases. Thus, natural frequencies decrease as the stiffness of a structure decreases. Moreover, it can be seen that, depending on an increase in the material gradient index and porosity parameter, the buckling temperatures can decrease.

In order to clearly understand the difference between different temperature risings, Fig. 3 displays the variations of the first dimensionless frequencies of simply supported FG porous beams under four cases of thermal loadings (UTR, LTR, NLTR and STR) for different fraction of porosity volume and constant of \((L/h = 50, n = 1)\). A comparison between Figs. 3(a–d) revealed that the difference of variant porosity volume fractions is more considerable under sinusoidal temperature rise. It can be found that critical temperature point of porous FG beams subjected to sinusoidal temperature rises is higher than the other temperature risings. Comparison of the first non-dimensional natural frequencies of the FG (I) beam respected to NLTR with the changing of porosity volume fraction and material graduation are presented in Fig. 4 at \((L/h = 20)\). Four types of temperature changes are considered as 0, 20, 40 and 80. It is observed from the results of Fig. 4 that if the power indexes increase, the non-dimensional natural frequencies of porous FG beam will decrease. When the \(n\) (power-law exponent) is in the range of 0 to 2, reducing is higher than where power exponent is in range between 2 to 10. The effect of temperature change is obvious, the non-dimensional natural frequencies will be decreased by increasing temperature changes for all gradient indexes, thus various thermal environments have an important effect on the non-dimensional frequency of the porous FG beam. The porosity effect in even distributions is more considerable under sinusoidal temperature rise. In Fig. 3 displays the variations of the first dimensionless frequencies of simply supported FG porous beams under four cases of thermal loadings (UTR, LTR, NLTR and STR) for different fraction of porosity volume and constant of \((L/h = 50, n = 1)\). A comparison between Figs. 3(a–d) revealed that the difference of variant porosity volume fractions is more considerable under sinusoidal temperature rise. It can be found that critical temperature point of porous FG beams subjected to sinusoidal temperature rises is higher than the other temperature risings. Comparison of the first non-dimensional natural frequencies of the FG (I) beam respected to NLTR with the changing of porosity volume fraction and material graduation are presented in Fig. 4 at \((L/h = 20)\). Four types of temperature changes are considered as 0, 20, 40 and 80. It is observed from the results of Fig. 4 that if the power indexes increase, the non-dimensional natural frequencies of porous FG beam will decrease. When the \(n\) (power-law exponent) is in the range of 0 to 2, reducing is higher than where power exponent is in range between 2 to 10. The effect of temperature change is obvious, the non-dimensional natural frequencies will be decreased by increasing temperature changes for all gradient indexes, thus various thermal environments have an important effect on the non-dimensional frequency of the porous FG beam. The porosity effect in even distributions is more considerable under sinusoidal temperature rise.

### Table 2

Comparison of the non-dimensional natural frequency \(\lambda = \omega \frac{L^2}{h} \sqrt{\frac{EA}{\rho m}}\) of perfect FG beam with different values of slenderness ratio and power-law Exponent for simply-simply boundary condition \(f(z) = \frac{L^2}{\lambda^2}\).

<table>
<thead>
<tr>
<th>Power-law Exponent</th>
<th>(L/h)</th>
<th>present Analytical</th>
<th>(\text{Şimşek}^{11} (2010)) Lagrange’s equations</th>
<th>(\text{Tah}^{6} (2012)) Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 0)</td>
<td>5</td>
<td>5.15274</td>
<td>5.1527</td>
<td>5.1527</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5.46032</td>
<td>5.4603</td>
<td>5.4603</td>
</tr>
<tr>
<td>(n = 0.2)</td>
<td>5</td>
<td>4.808074</td>
<td>4.80924</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5.0815249</td>
<td>5.08286</td>
<td>...</td>
</tr>
<tr>
<td>(n = 0.5)</td>
<td>5</td>
<td>4.4106620</td>
<td>4.41108</td>
<td>4.4111</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>4.6511086</td>
<td>4.65159</td>
<td>4.6516</td>
</tr>
<tr>
<td>(n = 1)</td>
<td>5</td>
<td>3.9904189</td>
<td>3.99042</td>
<td>3.9904</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>4.2050549</td>
<td>4.20503</td>
<td>4.2050</td>
</tr>
<tr>
<td>(n = 2)</td>
<td>5</td>
<td>3.6264396</td>
<td>3.6264</td>
<td>3.6264</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.8361340</td>
<td>3.83611</td>
<td>3.8361</td>
</tr>
<tr>
<td>(n = 5)</td>
<td>5</td>
<td>3.4012044</td>
<td>3.40120</td>
<td>3.4012</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.6484863</td>
<td>3.64850</td>
<td>3.6485</td>
</tr>
<tr>
<td>(n = 10)</td>
<td>5</td>
<td>3.2816047</td>
<td>3.28160</td>
<td>3.2816</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.5389891</td>
<td>3.53896</td>
<td>3.5390</td>
</tr>
<tr>
<td>Full metal</td>
<td>5</td>
<td>2.66086</td>
<td>2.67732</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2.83602</td>
<td>2.83716</td>
<td>...</td>
</tr>
</tbody>
</table>
The effect of porosity volume fraction, porosity distribution, temperature rise and power law exponent on the non-dimensional frequency of a S-S FG porous beam subjected to different temperature rises. ($L/h = 20$).

$(\Delta T = 20[K])$

<table>
<thead>
<tr>
<th>FG type</th>
<th>Load type</th>
<th>$n = 0$</th>
<th>$n = 0.1$</th>
<th>$n = 0.2$</th>
<th>$n = 0.5$</th>
<th>$n = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLL</td>
<td>UTR</td>
<td>6.30395</td>
<td>5.55820</td>
<td>3.07277</td>
<td>4.27880</td>
<td>3.72768</td>
</tr>
<tr>
<td>LTR</td>
<td>6.35810</td>
<td>5.61597</td>
<td>5.13246</td>
<td>4.3404</td>
<td>3.78915</td>
<td></td>
</tr>
<tr>
<td>NLT</td>
<td>6.35810</td>
<td>5.61702</td>
<td>5.13423</td>
<td>4.43432</td>
<td>3.79273</td>
<td></td>
</tr>
<tr>
<td>STR</td>
<td>6.39326</td>
<td>5.69226</td>
<td>5.16530</td>
<td>4.72722</td>
<td>3.82093</td>
<td></td>
</tr>
</tbody>
</table>

$(\Delta T = 40[K])$

<table>
<thead>
<tr>
<th>FG type</th>
<th>Load type</th>
<th>$n = 0$</th>
<th>$n = 0.1$</th>
<th>$n = 0.2$</th>
<th>$n = 0.5$</th>
<th>$n = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PLL</td>
<td>UTR</td>
<td>6.03315</td>
<td>5.29298</td>
<td>4.81098</td>
<td>4.02262</td>
<td>3.47635</td>
</tr>
<tr>
<td>LTR</td>
<td>6.21397</td>
<td>5.4807</td>
<td>5.00249</td>
<td>4.21818</td>
<td>3.67156</td>
<td></td>
</tr>
<tr>
<td>NLT</td>
<td>6.21397</td>
<td>5.48287</td>
<td>5.00616</td>
<td>4.22432</td>
<td>3.67897</td>
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</tr>
<tr>
<td>STR</td>
<td>6.28434</td>
<td>5.54925</td>
<td>5.07009</td>
<td>4.28458</td>
<td>3.73701</td>
<td></td>
</tr>
</tbody>
</table>

It is concluded that this behavior is dependent on temperature changes. By increasing temperature changes, the certain value of the $n$ has gone up. Comparison of the first non-dimensional frequencies of S-S FG (II) beam subjected to STR with the changing of porosity volume fraction and power-law exponent are presented in Fig. 5 at ($L/h = 20$). Figure 5 shows that the dimensionless frequency of FG beams with uneven poros-
Table 3 (continued). The effect of porosity volume fraction, porosity distribution, temperature rise and power law exponent on the non-dimensional frequency of a S-S FG porous beam subjected to different temperature rises. ($L/h = 20$).

<table>
<thead>
<tr>
<th>FGM type</th>
<th>Load type</th>
<th>$\alpha = 0$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 0.2$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>UTR</td>
<td>5.43500</td>
<td>4.70074</td>
<td>4.22118</td>
<td>3.43421</td>
<td>2.88841</td>
</tr>
<tr>
<td></td>
<td>LTR</td>
<td>5.90841</td>
<td>5.19383</td>
<td>4.72673</td>
<td>3.95845</td>
<td>3.42146</td>
</tr>
<tr>
<td></td>
<td>NLTR</td>
<td>5.90841</td>
<td>5.19848</td>
<td>4.73460</td>
<td>3.97164</td>
<td>3.43744</td>
</tr>
<tr>
<td></td>
<td>STR</td>
<td>6.05826</td>
<td>5.33961</td>
<td>4.87047</td>
<td>4.09788</td>
<td>3.56100</td>
</tr>
<tr>
<td>FGM I</td>
<td>UTR</td>
<td>6.10113</td>
<td>5.14338</td>
<td>4.55125</td>
<td>3.62459</td>
<td>3.00870</td>
</tr>
<tr>
<td></td>
<td>LTR</td>
<td>6.51537</td>
<td>5.57531</td>
<td>4.99417</td>
<td>4.08331</td>
<td>3.47576</td>
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<td>6.51537</td>
<td>5.5800</td>
<td>5.00203</td>
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<td>5.12292</td>
<td>4.23978</td>
<td>3.53803</td>
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<td>FGM II</td>
<td>UTR</td>
<td>5.43503</td>
<td>4.70074</td>
<td>4.22117</td>
<td>3.43419</td>
<td>2.88841</td>
</tr>
<tr>
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<td>5.19383</td>
<td>4.72673</td>
<td>3.95845</td>
<td>3.42146</td>
</tr>
<tr>
<td></td>
<td>NLTR</td>
<td>5.90841</td>
<td>5.19848</td>
<td>4.73460</td>
<td>3.97164</td>
<td>3.43744</td>
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<td>STR</td>
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<td>5.33961</td>
<td>4.87047</td>
<td>4.09788</td>
<td>3.56100</td>
</tr>
<tr>
<td></td>
<td>UTR</td>
<td>5.82320</td>
<td>4.98626</td>
<td>4.45321</td>
<td>3.59821</td>
<td>3.01751</td>
</tr>
<tr>
<td></td>
<td>LTR</td>
<td>6.26061</td>
<td>5.44201</td>
<td>4.92036</td>
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<td></td>
<td>NLTR</td>
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<td>5.49151</td>
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<td>4.12462</td>
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<tr>
<td></td>
<td>STR</td>
<td>6.39785</td>
<td>5.57484</td>
<td>5.05095</td>
<td>4.20932</td>
<td>3.63295</td>
</tr>
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<td>4.70395</td>
<td>3.76699</td>
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</tr>
<tr>
<td></td>
<td>LTR</td>
<td>6.68032</td>
<td>5.72549</td>
<td>5.13579</td>
<td>4.21250</td>
<td>3.59499</td>
</tr>
<tr>
<td></td>
<td>NLTR</td>
<td>6.68032</td>
<td>5.84659</td>
<td>4.16353</td>
<td>4.25462</td>
<td>3.64128</td>
</tr>
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<td></td>
<td>STR</td>
<td>6.80574</td>
<td>5.64576</td>
<td>5.25394</td>
<td>4.32791</td>
<td>3.70836</td>
</tr>
</tbody>
</table>

To indicate the influences of temperature change on the non-dimensional frequency of the FG (I) beam subjected to STR for various temperature changes and power-law indexes, Fig. 6 presents the frequency results versus material graduation for perfect ($\alpha = 0$) and porous ($\alpha = 0.2, \alpha = 0.1$) FG beam at ($L/h = 20$). It can be seen that an increase in temperature change gives rise to decrease in the non-dimensional frequency for every value of power-law exponent. Comparison of the non-dimensional frequencies of simply-simply FG(I) beam subjected to UTR and STR with the changing of porosity volume fractions and power exponents at $\Delta T = 40, L/h = 20$ are presented in Fig. 7. It is concluded that the frequency of the beam subjected to uniform is less than sinusoidal temperature rises and the difference will be increased by raising the power exponents. By comparing diagrams with different porosity parameters, it can be found that increasing of porosity parameter yields to increasing of frequencies.

6. CONCLUSIONS

In this research thermo-mechanical vibrational characteristic of FG porous beams are subjected to various thermal loadings with two porosity distributions. The equations of motion are derived by using four-variable refined shear deformation beam theory and simply-simply boundary condition is considered. The material’s properties are temperature-dependent and vary in the thickness direction established upon the modified rule of mixture. The governing equations are derived by using the new hyperbolic shear deformation theory and by using Hamilton’s principle. The Navier-based analytical model is used to solve governing partial differential equations. According to the numerical results, it is found that the proposed modeling can provide accurate frequency results of the FG beams as compared to the other solution results. As a result, the characteristics of vibration for FGM porous beams are significantly influenced by temperature field, volume fraction of porosity, power-law indexes and porosity distributions. The effects of the induced thermal environment, volume fraction of porosity, power-law index and porosity distribution on non-dimensional frequencies of porous FG beams are investigated. Numerical results show that:
Table 4. The effect of porosity volume fraction, temperature and power law exponent on the non-dimensional frequency of a S-S FG porous beam subjected to uniform temperature rise. \((L/h = 20)\).

| n = 0 |
|---|---|---|---|---|---|---|---|
| FGM type | \(\alpha\) | \(\Delta T = 0\) | \(\Delta T = 20\) | \(\Delta T = 40\) | \(\Delta T = 80\) | \(\Delta T = 100\) | \(\Delta T = 120\) |
| FGM-I | 0 | 6.55963 | 6.30395 | 6.03316 | 5.43503 | 5.09989 | 4.73312 | 4.32591 |
| 0 | 7.12552 | 6.88946 | 6.64120 | 6.10115 | 5.80425 | 5.48483 | 5.13800 |
| 0.2 | 8.04350 | 7.82487 | 7.59532 | 7.10366 | 6.83824 | 6.56993 | 6.25700 |
| FGM-II | 0 | 6.55963 | 6.30395 | 6.03316 | 5.43503 | 5.09989 | 4.73312 | 4.32591 |
| 0.1 | 6.88460 | 6.64115 | 6.38460 | 5.82320 | 5.51234 | 5.17580 | 4.80749 |
| 0.2 | 7.28100 | 7.04846 | 6.80459 | 6.27568 | 5.98595 | 5.67522 | 5.39916 |

| n = 0.5 |
|---|---|---|---|---|---|---|---|
| FGM type | \(\alpha\) | \(\Delta T = 0\) | \(\Delta T = 20\) | \(\Delta T = 40\) | \(\Delta T = 80\) | \(\Delta T = 100\) | \(\Delta T = 120\) |
| FGM-I | 0 | 4.51590 | 4.27880 | 4.02262 | 3.43419 | 3.08657 | 2.68473 | 2.19842 |
| 0.1 | 4.58217 | 4.36912 | 4.14070 | 3.62459 | 3.32696 | 2.99211 | 2.60518 |
| 0.2 | 4.66787 | 4.47881 | 4.27764 | 3.83014 | 3.57752 | 3.29397 | 2.98814 |
| FGM-II | 0 | 4.51590 | 4.27880 | 4.02262 | 3.43419 | 3.08657 | 2.68473 | 2.19842 |
| 0.1 | 4.60306 | 4.38030 | 4.14131 | 3.59817 | 3.28272 | 2.92507 | 2.59665 |
| 0.2 | 4.70024 | 4.49155 | 4.26838 | 3.76693 | 3.47998 | 3.15975 | 2.79248 |

| n = 1 |
|---|---|---|---|---|---|---|---|
| FGM type | \(\alpha\) | \(\Delta T = 0\) | \(\Delta T = 20\) | \(\Delta T = 40\) | \(\Delta T = 80\) | \(\Delta T = 100\) | \(\Delta T = 120\) |
| FGM-I | 0 | 3.95833 | 3.72768 | 3.47633 | 2.88841 | 2.53142 | 2.10431 | 1.54950 |
| 0.1 | 3.95094 | 3.74466 | 3.52136 | 3.00870 | 2.70575 | 2.35583 | 1.93314 |
| 0.2 | 3.94060 | 3.75869 | 3.56338 | 3.12155 | 2.86669 | 2.56010 | 2.24933 |
| FGM-II | 0 | 3.95833 | 3.72768 | 3.47633 | 2.88841 | 2.53142 | 2.10431 | 1.54950 |
| 0.1 | 4.00688 | 3.79114 | 3.55734 | 3.01749 | 2.69637 | 2.32232 | 1.86322 |
| 0.2 | 4.05968 | 3.85842 | 3.64125 | 3.14560 | 2.85595 | 2.52553 | 2.13507 |

| n = 2 |
|---|---|---|---|---|---|---|---|
| FGM type | \(\alpha\) | \(\Delta T = 0\) | \(\Delta T = 20\) | \(\Delta T = 40\) | \(\Delta T = 80\) | \(\Delta T = 100\) | \(\Delta T = 120\) |
| FGM-I | 0 | 3.55531 | 3.31306 | 3.08494 | 2.50001 | 2.13548 | 1.68266 | 1.05058 |
| 0.1 | 3.50826 | 3.30846 | 3.09081 | 2.58297 | 2.27629 | 1.91233 | 1.44821 |
| 0.2 | 3.44949 | 3.27413 | 3.08447 | 2.64930 | 2.39340 | 2.09963 | 1.74883 |
| FGM-II | 0 | 3.55531 | 3.31306 | 3.08494 | 2.50001 | 2.13548 | 1.68266 | 1.05058 |
| 0.1 | 3.58170 | 3.37260 | 3.14446 | 3.61022 | 2.82510 | 1.89594 | 1.38919 |
| 0.2 | 3.60999 | 3.41545 | 3.20437 | 2.71627 | 2.42566 | 2.08697 | 1.67090 |

| n = 5 |
|---|---|---|---|---|---|---|---|
| FGM type | \(\alpha\) | \(\Delta T = 0\) | \(\Delta T = 20\) | \(\Delta T = 40\) | \(\Delta T = 80\) | \(\Delta T = 100\) | \(\Delta T = 120\) |
| FGM-I | 0 | 3.23324 | 3.01418 | 2.77210 | 2.18731 | 1.81217 | 1.32264 | 0.424321 |
| 0.1 | 3.16673 | 2.96868 | 2.75554 | 2.25039 | 1.93807 | 1.55512 | 1.02509 |
| 0.2 | 3.07867 | 2.90872 | 2.72367 | 2.29333 | 2.03530 | 1.73218 | 1.35400 |
| FGM-II | 0 | 3.23324 | 3.01418 | 2.77210 | 2.18731 | 1.81217 | 1.32264 | 0.424321 |
| 0.1 | 3.24607 | 3.04239 | 2.81871 | 2.28113 | 1.95627 | 1.54646 | 0.96687 |
| 0.2 | 3.25960 | 3.07073 | 2.86453 | 2.38158 | 2.08843 | 1.73831 | 1.28534 |

1. By increasing the gradient index value, the non-dimensional frequencies are found to decrease.
2. Fundamental frequencies decrease by increasing the four-temperature rising and all two porosity distributions.
3. The responses of the non-dimensional frequencies in the FG porous beams according to geometric parameters, under sinusoidal temperature rise are very similar to that under nonlinear, linear and uniform temperature rise. However, the critical temperature gradient under sinusoidal temperature rise is higher than those under the other temperature rises.
4. The non-dimensional frequency predicted by STR is always greater than those UTR, LTR, NLTR and the uniform temperature rise has more significant effect on the non-dimensional frequencies than the other temperature rise.
5. For FGM-I, at a constant value for changes in temperature, increasing the porosity first causes the increase in fundamental frequencies, however this trend is vice versa for upper values of gradient indexes. This behavior is dependent on power law indexes and temperature changes. For FGM-II, increasing the porosity causes the increase in fundamental frequencies for all values of gradient indexes.
and temperature changes.

6. In FGM I, the porosity has more significant impact on natural frequencies of the beam in comparison of FGM II.

It is concluded that various factors such as porosity parameter, porosity distribution, temperature rising and power-law index have a notable effect on the non-dimensional frequencies of FG beams with porosities. This emphasizes the importance of the inspected porosity volume fraction effect in thermal environments. Therefore, the porosity and thermal effects should be considered in the analysis of vibration behavior of FG structures.

REFERENCES


Figure 6. The variation of the first dimensionless frequency of S-S FGM(I) beam subjected to STR for different porosities and temperature changings ($L/h = 20$).


Figure 7. Comparison of the first non-dimensional frequency of S-S FG (I) beam subjected to UTR and STR for different porosity volume fractions and material graduations. ($\Delta T = 40, L/h = 20$).


Static and Dynamic Stability Analysis of a Rotating Taper Beam Having Elliptical Cross Section Subjected to Pulsating Axial Load With Thermal Gradient

Madhusmita Pradhan, Mrunal Kanti Mishra and Pushparaj Dash
Department of Mechanical Engineering, VSSUT, Burla, India, 768018.

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The static and dynamic stability of a rotating tapered beam having an elliptical cross-section subjected to a pulsating axial load with a thermal gradient is investigated under three different boundary conditions, such as clamped-clamped (C-C), clamped-pinned (C-P), and pinned-pinned (P-P). The governing equations of motion have been derived by using Hamilton’s energy principle. A set of Hill’s equations have been obtained by the application of generalized Galerkin’s method. The effects of taper parameter, hub radius, rotational speed, thermal gradient, and geometric parameter on the static buckling loads and the regions of instability have been studied and the results are presented graphically.

NOMENCLATURE

\[ A(x), A(\xi) \] Area of the generic section of the beam
\[ A_1 \] Cross sectional area \( x = 0 \)
\[ C_0 \] Hub radius
\[ l \] Length of the beam
\[ c_0 \] Dimensionless hub radius, \((C_0/l)\)
\[ b(x), b(\xi) \] Minor diameter of a generic section of the beam
\[ B \] Minor axis diameter at \( x = 0 \)
\[ E \] Young’s modulus
\[ I(x), I(\xi) \] Moment of inertia at a generic section of the beam
\[ I_1 \] Moment of inertia \( x = 0 \)
\[ m(\xi) \] Mass distribution function
\[ P_0 \] Static axial load
\[ P_1 \] Dynamic axial load
\[ p(\tau) \] Dimensionless load
\[ \bar{P}_0 \] Dimensionless static axial load
\[ \bar{P}_1 \] Dimensionless dynamic axial load
\[ S(\xi) \] Moment of inertia distribution function
\[ t \] Time period
\[ w(x,t) \] Transverse deflection of the beam
\[ q(\xi) \] Geometric parameter
\[ \alpha^* \] Minor axis taper parameter
\[ \eta \] Dimensionless transverse deflection, \((=w/l)\)
\[ \xi \] Dimensionless length, \((=x/l)\)
\[ \tau \] Dimensionless time
\[ \rho \] Density of the beam material
\[ \Omega \] Uniform angular velocity of the beam about \( z' \) axis
\[ \Omega_0 \] Rotational speed parameter
\[ \omega \] Excitation frequency
\[ \omega_0 \] Dimensionless fundamental natural frequency
\[ \bar{\omega} \] Non-dimensional excitation frequency
\[ b \] Minor axis diameter at \( a \) Major axis diameter

1. INTRODUCTION

The vibration analysis of rotating beams is of great importance in the design of many engineering examples, such as helicopter rotor blades, turbine blades, turbo engine blades, etc. The effect of thermal gradient is also a vital aspect as the modulus of elasticity for most of the elastic materials are greatly affected by the temperature. Lo and Renbarger obtained the differential equation of motion of a cantilever blade mounted on a rotating disc at a stagger angle.\(^1\) The natural frequencies and mode shapes of a rotating uniform cantilever beam with tip mass were studied by Bhat.\(^2\) The theoretical expression for the work done due to centrifugal effects was derived by Carnegie.\(^3\) The effects of rotational speed, disc radius, and stagger angle of the blade on the frequencies of the lateral vibration were studied by Rao.\(^4\) Liu and Yeh obtained the natural frequencies of a non-uniform rotating beam.\(^5\) Bauer studied the effects of spin, hub radius, and aspect ratio on the vibrational behavior of a rotating beam.\(^6\) The effect of rotational speed and root flexibilities on static buckling loads and the first order simple resonance zones were studied by Abbas by using finite element technique.\(^7\) Ishida et al. investigated the influence of rotational speed on the unstable regions of a system consisting of a disc mounted on an elastic shaft subjected to a pulsating axial load.\(^8\) The explicit stability conditions for a rotating shaft under parametric excitation were derived by
Sri Namachchivaya. Rao derived the formula for the fundamental flexural frequency of tapered cantilever beam. Taylor obtained the power series solution of blade natural frequencies for the case of a uniform beam and a completely tapered beam. Abbas and Thomas and Yokoyama studied the effects of support conditions on the dynamic stability of Timoshenko beams by using finite element method. The stability properties of a periodically loaded non-linear dynamic system considering the damping effects were studied by Svensson. Parametric instability of a non-uniform beam with thermal gradient and elastic end support were studied by Kar and Sujata. The same authors studied the dynamic stability of a rotating beam with various boundary conditions. The dynamic stability of a rotating beam with a constrained damping layer were investigated by Lin and Chen. Rao and Carnegie used the Ritz averaging procedure to solve the differential equation of a rotating blade. The bending frequencies of a rotating cantilever beam were determined by Schilhansl. The results of torsional vibration of a rectangular cross-section cantilever beam were obtained by Vet. Lenci et al. obtained the approximate analytical expressions for the natural frequencies of non-uniform cables and beams by using the asymptotic development method. The effects of thermal gradient on the frequencies of rotating beams were studied by Tomar and Jain. The dynamic stability of a circularly tapered rotating beam were studied by computational method and the results are presented graphically.

2. FORMULATION OF THE PROBLEM

A rotating tapered beam of length \( l \) was set off a distance \( C_0 \) from the axis of rotation and rotating at a uniform angular velocity \( \Omega \) about a vertical \( z' \) axis and was capable of oscillating in the \( x - z \) plane. A pulsating axial force, \( P(t) = P_0 + P_1 \cos \omega t \), was applied at the end of the beam along the point of center of gravity of the cross-section in the axial direction, \( \omega \) being the frequency of the applied load, \( t \) being the time, and \( P_0 \) and \( P_1 \) being the static and dynamic load amplitudes respectively.

The following assumptions were made for deriving the equations of motion:

1. The material of the beam was homogeneous & isotropic in nature.
2. The deflections of the beam were small and the transverse deflection \( w(x,t) \) was the same for all points of a cross-section.
3. Extensional deflection of the beam was neglected.
4. The beam obeyed Euler-Bernoulli beam theory.
5. A steady one-dimensional temperature gradient was assumed to exist along the central length of the beam.
6. Extension and rotary inertia effects were negligible.

Applying Hamilton’s principle gave

\[
\delta \int_{t_1}^{t_2} (T - V + W_p) = 0; \tag{1}
\]

The expressions for potential energy, kinetic energy, and work done were as follows

\[
T = \frac{1}{2} \int_0^l \rho A(x) \left( \frac{\partial w}{\partial t} \right)^2 \, dx + \frac{1}{2} \int_0^l \rho A(x) \Omega^2 w^2 \, dx; \tag{2}
\]

\[
V = \frac{1}{2} \int_0^l E(x) I(x) \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx
+ \frac{1}{2} \int_0^l \rho A(x) \Omega^2 (C_0 + x) \left( \frac{\partial w}{\partial x} \right)^2 \, dx; \tag{3}
\]

\[
W_p = \frac{1}{2} \int_0^l P(t) \left( \frac{\partial w}{\partial x} \right)^2 \, dx; \tag{4}
\]
where \( w(x, t) \) is the transverse deflection of the beam.

Solving Eq. (1), the following equation of motion and boundary conditions are obtained

\[
[E(x)I(x)w_{,xx},x]_{,xx} + \rho A(x)w_{,tt} + \rho \Omega^2 I(x)w_{,xx} - [N(x_1)w_x]_{,x} + P(t)w_{,xx} = 0; \quad (5)
\]

where,

\[
N(x_1) = \frac{1}{2} \rho A(x)\Omega^2 \left[ (C_0 + l)^2 - (C_0 + x')^2 \right]; \quad (6)
\]

The boundary conditions at \( x = C_0 \) and \( x = (C_0 + l) \) are

\[
[E(x)I(x)w_{,xx},x]_{,x} + P(t)w_x = 0; \quad (7a)
\]

\[
[E(x)I(x)w_{,xx},x]_{,t} = 0; \quad (7b)
\]

\[
w_{,tt} = 0. \quad (7c)
\]

In the above expressions

\[
w_{,x} = \frac{\partial w}{\partial x}; \quad (8a)
\]

\[
w_{,xx} = \frac{\partial^2 w}{\partial x^2}; \quad (8b)
\]

\[
w_{,t} = \frac{\partial w}{\partial t}; \quad (8c)
\]

\[
w_{,tt} = \frac{\partial^2 w}{\partial t^2}; \quad (8d)
\]

and the dimensionless parameters are

\[
\xi = \frac{x}{l}; \quad (9a)
\]

\[
\eta = \frac{w}{l}; \quad (9b)
\]

\[
c_0 = \frac{C_0}{l}; \quad (9c)
\]

\[
\tau = ct; \quad (9d)
\]

\[
\left\{ \begin{array}{l}
\gamma^2 = \frac{E(x)I(x)}{\rho A(x)\Omega^2} \\
\frac{\partial w}{\partial x} = \frac{\partial \eta}{\partial \xi} \\
\frac{\partial^2 w}{\partial x^2} = \frac{1}{l^2} \left( \frac{\partial^2 \eta}{\partial \xi^2} \right)^2 \\
\frac{\partial^2 w}{\partial x^2} = \frac{1}{l^2} \left( \frac{\partial^2 \eta}{\partial \xi^2} \right)^2 \\
\frac{\partial w}{\partial \tau} = ct \left( \frac{\partial \eta}{\partial \tau} \right) \\
\frac{\partial w}{\partial \tau} = ct \left( \frac{\partial \eta}{\partial \tau} \right)^2 \\
p(\tau) = \frac{p(t)^2}{E_l I_1}; \\
p(\tau) = p_0 + p_1 \cos \theta \tau; \quad (9m)
\end{array} \right. \quad (9e)
\]

Table 1. Co-ordinate functions.

<table>
<thead>
<tr>
<th>Sl No.</th>
<th>End Arrangement</th>
<th>Co-ordinate function ( i = 1, 2, \ldots, r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P-P</td>
<td>( \eta(\xi) = \sin(n\xi) )</td>
</tr>
<tr>
<td>2</td>
<td>C-P</td>
<td>( \eta(\xi) = 2(i + 2)\xi^{(i+1)} - (4i + 6)\xi^{(i+2)} + 2(i + 1)\xi^{(i+3)} )</td>
</tr>
<tr>
<td>3</td>
<td>C-C</td>
<td>( \eta(\xi) = \xi^{(i+1)} + 2\xi^{(i+2)} + \xi^{(i+3)} )</td>
</tr>
</tbody>
</table>

Non-dimensional equation of motion can be written as

\[
[S(\xi)T(\xi)\eta'''' + m(\xi)\eta'' + [r_5\Omega^2 + p(\tau)] \eta''] - \Omega^2_0[q(\xi)\eta]'' = 0. \quad (10)
\]

Now, non-dimensional boundary conditions are written as

\[
\left\{ \begin{array}{l}
[S(\xi)T(\xi)\eta'''' + p(\tau)\eta']_{\xi = 1} = 0; \quad (11a) \\
[S(\xi)T(\xi)\eta''']_{\xi = 1} = 0; \quad (11b) \\
[S(\xi)T(\xi)\eta''']_{\xi = 1} = 0; \quad (11c) \\
\eta(0, \tau) = 0; \quad (11d) \\
\eta'(0, \tau) = 0. \quad (11e)
\end{array} \right. 
\]

In the above expressions

\[
r_0 = \frac{I(\xi)}{A_1 l^2}; \quad (12a)
\]

\[
\Omega^2_0 = \frac{\rho A(\xi)\Omega^2 l^4}{E_l I_1}; \quad (12b)
\]

\[
\Omega^2_0 q(\xi) = \frac{N(x_1)^2}{E_l I_1}; \quad (12c)
\]

\[
A(\xi) = A_1 \alpha(\xi); \quad (12d)
\]

\[
I(\xi) = I_1 S(\xi); \quad (12e)
\]

\[
b(\xi) = B \{ 1 - \alpha^* \xi \}; \quad (12f)
\]

\[
\alpha^* = 1 - \frac{b}{B}; \quad (12g)
\]

\[
m(\xi) = 1 - \alpha^* \xi; \quad (12h)
\]

\[
S(\xi) = (1 - \alpha^* \xi)^3. \quad (12i)
\]

2.1. Approximate Solution

The approximate solution to the non-dimensional equations of motion was assumed as

\[
\eta(\xi, \tau) = \sum_{r=1}^{N} \eta_r(\xi) f_r(\tau); \quad (13)
\]

where \( f_r(\tau) \) is an unknown function of time and \( \eta_r(\xi) \) is the coordinate function chosen as to satisfy as many of the boundary conditions as possible. It was further assumed that coordinate functions for the various boundary conditions could be approximated by the ones given in Table 1.

Substitution of the series of solutions in the non-dimensional equations of motion and subsequent application of the general
Galerkin method led to the following matrix equation of motion
\[ [M] \{ \ddot{f} \} + [K]\{f\} - \left\{ p_0[H] - p_1 \cos(\theta\tau)[H] \right\} \{f\} = \{0\}; \]
\[ (14) \]
where
\[ \ddot{f} = \frac{\partial^2 f}{\partial \tau^2} \] and \[ \{f\} = \{f_1, \ldots, f_N\}^T. \]

Various matrix elements were given by
\[ \int_0^1 m(\xi) \eta_i(\xi) \eta_j(\xi) d\xi = M_{ij}; \]
\[ (16) \]
\[ \int_0^1 \{ S(\xi)T(\xi) \eta_i'(\xi) \eta_j'(\xi) \} d\xi + \Omega^2_n \{ g(\xi) - r_\theta \} \eta_i^v(\xi) \eta_j^v(\xi) \} d\xi = K_{ij}; \]
\[ (17) \]
\[ \int_0^1 \eta_i'(\xi) \eta_j'(\xi) d\xi = H_{ij}; \]
\[ (18) \]
where \( i, j = 1, 2, \ldots, N. \)

2.2. Static Buckling Load

Substituting \( p_1 = 0 \) and \( \{\ddot{f}\} = \{0\} \) in Eq. (14) led to the eigenvalue problem \( [K]^{-1}[H]\{f\} = \frac{1}{p_0}\{f\} \). The static buckling loads for the first few modes were obtained as the reciprocals of the eigenvalues of \( [K]^{-1}[H] \) using MATLAB.

2.3. Regions of Instability

Let \( [L] \) be the modal matrix of \( [M]^{-1}[K] \). Then by the introduction of the linear coordinate transformation, \( f = [L]\{v\} \), \( \{v\} \) being a new set of generalized coordinates yielded
\[ \{\ddot{v}_n\} + \left\{ \omega_n^2 \right\} \{v\} + p_1 \cos(\tau) [B]\{v\} = \{0\}; \]
\[ (19) \]
where \( \omega_n^2 \) is a spectral matrix corresponding to \( [M]^{-1}K \) and \( [B] = -[L]^{-1}[M]^{-1}[H][L] \).

Equation (19) can be written as
\[ \ddot{v}_n + \omega_n^2 v_n + p_1 \cos(\tau) \sum_{m=1}^N b_{nm} u_m = 0, \]
\[ \quad n = 1, 2, \ldots, N; \]
\[ (20) \]
Equation (20) represents a system of \( N \) coupled Hill’s equations with complex coefficients. Here, \( \omega_n \) and \( b_{nm} \) are complex quantities given by
\[ \omega_n = \omega_n, R + j \omega_n, I; \]
\[ (21a) \]
\[ b_{n,m} = b_{nm, R} + jb_{nm, I}; \]
\[ (21b) \]
The boundaries of the region of instability of simple and combination resonances were obtained using the following conditions by Saito & Otomi.\(^{23}\)

### 2.3.1. Simple Resonance

In this case, the regions of instability when damping is present were given by
\[ \left| \frac{\omega}{2} - \omega_{\mu, R} \right| < \frac{1}{4} \sqrt{\frac{p_1}{\omega_{\mu, R}} \left( b_{\mu, R}^2 + b_{\mu, I}^2 \right)} - 16 \omega_{\mu, I}^2; \]
\[ (22) \]
and for the undamped case
\[ \left| \frac{\omega}{2} - \omega_{\mu, R} \right| < \frac{1}{4} \sqrt{\frac{p_1}{\omega_{\mu, R}} \left( b_{\mu, R}^2 + b_{\mu, I}^2 \right)}; \]
\[ (23) \]
for \( \mu = 1, 2, \ldots, N. \)

### 2.3.2. Combination Resonance of Sum Type

This type of resonance occurs when \( \mu > \nu; \mu, \nu = 1, \ldots, N \) and the regions of instability for the damped case were given by
\[ \left| \frac{\omega}{2} - \frac{1}{2} (\omega_{\nu, R} + \omega_{\mu, R}) \right| < \frac{\omega_{\mu, I} + \omega_{\nu, I}}{\sqrt{\omega_{\mu, R} \omega_{\nu, R}}}; \]
\[ (24) \]
and for the un-damped case
\[ \left| \frac{\omega}{2} - \frac{1}{2} (\omega_{\nu, R} + \omega_{\mu, R}) \right| < \frac{p_1}{4} \sqrt{\frac{b_{\nu, R} b_{\nu, I} b_{\mu, R} b_{\mu, I}}{\omega_{\nu, R} \omega_{\mu, R}}}. \]
\[ (25) \]

### 2.3.3. Combination Resonance of Difference Type

This type of resonance occurs when \( \mu < \nu; \mu, \nu = 1, \ldots, N \) and the regions of instability for the damped case were given by
\[ \left| \frac{\omega}{2} - \frac{1}{2} (\omega_{\nu, R} - \omega_{\mu, R}) \right| < \frac{\omega_{\mu, I} + \omega_{\nu, I}}{\sqrt{\omega_{\mu, R} \omega_{\nu, R}}}; \]
\[ (26) \]
and for the undamped case
\[ \left| \frac{\omega}{2} - \frac{1}{2} (\omega_{\nu, R} - \omega_{\mu, R}) \right| < \frac{p_1}{4} \sqrt{\frac{b_{\nu, R} b_{\nu, I} b_{\mu, R} b_{\mu, I}}{\omega_{\nu, R} \omega_{\mu, R}}}. \]
\[ (27) \]
Dynamic stability analysis of the elliptically tapered rotating beam with axial pulsating under various boundary conditions had been carried out by using Eqs. (23), (25), and (27). From them regions of instability were obtained for various cases.

3. NUMERICAL RESULTS AND DISCUSSION

3.1. Validation of the results

Taking the following non-dimensional parameters of the rotating uniform beam as given in Kar and Sujata, the static buckling loads of the beam without thermal gradient have been determined.\(^{16}\) The hub radius \( c_0 = 1 \) and \( r_g = 0.001 \) have been
Variations of static buckling loads with different parameters are shown in Figs. 2 through 6. As shown in Fig. 2, the static buckling load decreased with an increase in the value of $\alpha^*$. The decrement was non-linear in nature and the rate of decrease was more for the higher modes. The C-C case was the most stable and P-P was the least among the three boundary conditions.

Figure 3 depicts the effect of $C_0$ on the static buckling loads of the system. The static buckling load remained almost independent of the parameter $C_0$. In this case also the C-C case was more stable than the C-P and P-P case.

The effect of $\delta$ on the static buckling loads is shown in Fig. 4. The static buckling loads decreased with an increase in the value of $\delta$ for all the considered cases. The variation was quite linear for the first and second mode. For the third mode the variation was non-linear for the P-P and C-P case.

The variation of static buckling load with an increase in the value of $\Omega_0$ is shown in Fig. 5. The static buckling increased up to a point for $\Omega_0 = 0.75$. After that, it increased linearly with a higher increasing rate. The C-C case was more stable than the C-P and P-P case.

Figure 6 addresses the effect of $q(\xi)$ on the static buckling loads of the system. For the first mode, the static buckling loads increased linearly for all the three boundary conditions. For the second mode under the C-P and C-C case, the static buckling load decreased up to certain value of $q(\xi)$, then increased linearly for higher values of $q(\xi)$. For the P-P case, the static buckling load increased non-linearly with $q(\xi)$. For the third mode, the static buckling load increased with an increase in the value of $q(\xi)$ for all the boundary conditions.

Effects of different parameters on the regions of instability are shown in Figs. 7 through 33.

Figures 7 through 12 address the effects of $\Omega_0$ on the instability regions of the system for $\Omega_0 = 1$ and $\Omega_0 = 5$ with, $C_0 = 1$, $\delta = 0.1$, and $\alpha^* = 0.5$. All the instability zones were shifted towards the right and became narrower for all the three boundary conditions. The combination resonance zone $(\omega_1 + \omega_3)$ appeared for $\Omega_0 = 5$, and it improved the stability as its area was very small for the P-P case. The system stability improved with $\Omega_0$ for the C-P and C-C case, as the instability zones, simple as well as combination zones, were shifted towards higher excitation frequencies and at $\Omega_0 = 5$, the zone near $(\omega_1 + \omega_2)$ for $\Omega_0 = 1$ disappeared.

The effects of $q(\xi)$ on the instability regions are shown in Figs. 13 through 18. With an increase in the value of $q(\xi)$ from 0.2 to 0.5 with $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$, the resonance zones were shifted towards higher excitation frequencies and reduced in their areas. For the P-P case, $(\omega_1 + \omega_3)$ appeared for $q(\xi) = 0.5$, but it improved the stability for increasing value of $q(\xi)$ as the zone area was very negligible. For the C-P and C-C case combination resonance zones, $(\omega_1 + \omega_2)$ and $(\omega_2 + \omega_3)$ appeared for all the values of $q(\xi)$. The C-C case was more stable than the C-P and P-P cases as the excitation frequencies for the C-C case were larger than the C-P and P-P case.

### Table 2. Comparison of static buckling loads

<table>
<thead>
<tr>
<th>End arrangement of the sandwich beam</th>
<th>Static buckling loads</th>
<th>Kar and Sujata from figures 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation speed parameter with $c_0 = 1$, $r_g = 0.001$, $\Omega_0 = 5$ Clamped-clamped</td>
<td>1</td>
<td>45.3</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>77.15</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>140.9</td>
</tr>
<tr>
<td>Rotation speed parameter with $c_0 = 0$, $r_g = 0.0001$, $\Omega_0 = 10$ Clamped-pinned</td>
<td>1</td>
<td>16.4</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>64.25</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>112.4</td>
</tr>
</tbody>
</table>

considered here. The static buckling loads for C-C and C-P end conditions have been determined for the first three modes as given in Table 2. It may be observed that these results are in good agreement with those obtained by Kar and Sujata. This was expected as the equation of motion is the same as that of Kar and Sujata for a rotating uniform beam having a rectangular cross-section in the absence of thermal gradient. This validated the code developed in the present work using MATLAB.

As the static buckling loads obtained using the present analysis were in good agreement with Kar and Sujata, the static buckling loads and regions of instability of the rotating tapered beam having elliptical cross-section subjected to a pulsating axial load with thermal gradient were investigated under three different boundary conditions, C-C, C-P, and P-P.

Numerical results were obtained for various values of the parameters like rotational speed, geometric parameter, taper parameter, hub radius, and thermal gradient. The linearly tapered beam with an elliptical cross-section was assumed to have a minor axis diameter varying according to the relation $b(\xi) = B(1 - \alpha^* \xi)$, where $B$ is the minor diameter of the beam at the end $\xi = 0$, $\alpha^*$ is the minor diameter taper parameter, and $\alpha^* = 1 - \frac{b}{B}$.

Consequently, the mass distribution $m(\xi)$ and the moment of inertia distribution $S(\xi)$ are given by the relations

$$m(\xi) = 1 - \alpha^* \xi; \quad (28a)$$

$$S(\xi) = (1 - \alpha^* \xi)^3; \quad (28b)$$

The temperature above the reference temperature at any point $\xi$ from the end of the beam was assumed to be $\psi = \psi_0(1 - \xi)$. Choosing $\psi = \psi_0$, the temperature at the end $\xi = 1$ as the reference temperature, the variation of modulus of elasticity of the beam is denoted by

$$E(\xi) = E_1[1 - \gamma \psi_1(1 - \xi)], \quad 0 \leq \gamma \psi_1 < 1$$

$$= E_1 T(\xi). \quad (29)$$

where $\gamma$ is the coefficient of thermal expansion of the beam material, $\delta = \gamma \psi_1$ is the thermal gradient parameter, and $T(\xi) = [1 - \delta(1 - \xi)]$. 

Figure 1. System configuration.

Figure 2. Variation of $(P_0)_{crit}$ with $\alpha^*$.  

Figure 3. Variation of $(P_0)_{crit}$ with $c_0$.  

Figure 4. Variation of $(P_0)_{crit}$ with $\delta$.  

Figure 5. Variation of $(P_0)_{crit}$ with $\Omega_0$.  

Figure 6. Variation of $(P_0)_{crit}$ with $q(\xi)$.  

Figure 7. Stability diagrams for $\Omega_0 = 1$, $C_0 = 1$, $\delta = 0.1$ and $\alpha^* = 0.5$.  

P-P Case.
Figure 8. Stability diagrams for $\Omega_0 = 5$, $C_0 = 1$, $\delta = 0.1$ and $\alpha^* = 0.5$.

Figure 9. Stability diagrams for $\Omega_0 = 1$, $C_0 = 1$, $\delta = 0.1$ and $\alpha^* = 0.5$.

Figure 10. Stability diagrams for $\Omega_0 = 5$, $C_0 = 1$, $\delta = 0.1$ and $\alpha^* = 0.5$.

Figure 11. Stability diagrams for $\Omega_0 = 1$, $C_0 = 1$, $\delta = 0.1$ and $\alpha^* = 0.5$.

Figure 12. Stability diagrams for $\Omega_0 = 5$, $C_0 = 1$, $\delta = 0.1$ and $\alpha^* = 0.5$.

Figure 13. Stability diagrams for $q(\xi) = 0.2$, $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$.

Figure 14. Stability diagrams for $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$.

Figure 15. Stability diagrams for $q(\xi) = 0.2$, $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$. 
Figure 16. Stability diagrams for $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$.

Figure 17. Stability diagrams for $q(\xi) = 0.2$, $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$.

Figure 18. Stability diagrams for $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$, $\delta = 0.1$.

Figure 19. Stability diagrams for $\delta = 0.1$, $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$.

Figure 20. Stability diagrams for $\delta = 0.3$, $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$.

Figure 21. Stability diagrams for $\delta = 0.1$, $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$.

Figure 22. Stability diagrams for $\delta = 0.3$, $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$.

Figure 23. Stability diagrams for $\delta = 0.1$, $q(\xi) = 0.5$, $\Omega_0 = 5$, $\alpha^* = 0.5$. 
Figure 24. Stability diagrams for $\delta = 0.3, q(\xi) = 0.5, \Omega_0 = 5, \alpha^* = 0.5$.

Figure 25. Stability diagrams for $C_0 = 1, \Omega_0 = 5, \alpha^* = 0.5, \delta = 0.1$.

Figure 26. Stability diagrams for $C_0 = 1, \Omega_0 = 5, \alpha^* = 0.5, \delta = 0.1$.

Figure 27. Stability diagrams for $C_0 = 1, \Omega_0 = 5, \alpha^* = 0.5, \delta = 0.1$.

Figure 28. Stability diagrams for $\alpha^* = 0.2, C_0 = 1, \Omega_0 = 5, \delta = 0.1$.

Figure 29. Stability diagrams for $\alpha^* = 0.8, C_0 = 1, \Omega_0 = 5, \delta = 0.1$.

Figure 30. Stability diagrams for $\alpha^* = 0.2, C_0 = 1, \Omega_0 = 5, \delta = 0.1$.

Figure 31. Stability diagrams for $\alpha^* = 0.8, C_0 = 1, \Omega_0 = 5, \delta = 0.1$. 
As shown in Figs. 19 through 24, the stability of the system decreased with an increase in the value of $\delta$ from 0.1 to 0.3 with $\alpha^* = 0.5$, $q(\xi) = 0.5$, and $\Omega_0 = 5$, as all the resonance zones were shifted towards lower excitation frequencies and became wider. For the C-P and C-C case, $(\omega_1 + \omega_2)$ disappeared for all the values of $\delta$. Under the C-C case, $(\omega_2 + \omega_3)$ also disappeared for all values of $\delta$. Hence the C-C case was the most stable while the P-P case was the least stable among the three boundary conditions.

Figures 25 through 27 display the effect of $C_0 = 1$ with $\alpha^* = 0.5$, $\Omega_0 = 5$, and $\delta = 0.1$. For all the three considered boundary conditions the instability zones remained unaffected with an increase in the value of $C_0$ from 1 to 3, so the figures are not shown here. From the figures it was clear that the C-C case was more stable than the other two cases.

From Figs. 28 through 33, it is seen that with an increase in the value of $\alpha^*$ from 0.2 to 0.8 with $\Omega_0 = 5$, $\delta = 0.1$, and $C_0 = 1$, the instability regions were shifted towards lower excitation frequencies and became wider for the P-P and C-P case. For the P-P case, the combination resonance zone $(\omega_1 + \omega_2)$ disappeared for $\alpha = 0.8$, and it worsened the stability of the system as the area of other unstable zones was larger in comparison to $\alpha^* = 0.2$. For the C-C condition, the combination resonance of $(\omega_1 + \omega_2)$ and $(\omega_2 + \omega_3)$ disappeared for all the values of $\alpha^*$. Hence the C-C case was more stable.

With an increase in the value of $\Omega_0$, $q(\xi)$, the value of $[k]$ increased. This improved the stability of the system. Similarly with an increase in the value of $\alpha^*$, $\delta$, the value of $[k]$ decreased, which worsened the stability of the system. Thus, the results showing the effects of various parameters on the static buckling loads and regions of instability were justified.

4. CONCLUSIONS

In this work, a computational analysis of the static and dynamic stability of a rotating taper beam having an elliptical cross-section subjected to pulsating axial load with thermal gradient under three different boundary conditions is considered. The programming has been developed by using MATLAB and comparisons are made with the results of earlier researchers to test the validity of the analysis.\(^{16}\)

An increase in rotational speed and geometric parameters increase the static buckling load for all the three modes under the three considered boundary conditions. A higher thermal gradient and taper parameter have a detrimental effect on the static stability of the system. The static buckling loads are almost independent of the hub radius for all the considered cases.

It is seen that the dynamic stability of the system improves with an increase of rotational speed and geometric parameter. However, an increase in thermal gradient and taper parameter makes the beam more sensitive to periodic forces by shifting the instability regions towards lower excitation frequencies as well as widening of the instability zones. The stability of the system remains unaffected with an increase in the value of hub radius. For all the cases, the C-C condition is the most stable and the P-P case is the least among the three boundary conditions. Combination resonances of sum type occur for all the three boundary conditions.

REFERENCES


Comparisons of Paraboloidal Shells and Sinusoidal-Shaped Shells in Natural Frequencies

Yeong-Bin Yang
Department of Civil Engineering, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan. School of Civil Engineering, Chongqing University, 83 Shabei Street, Shapingba District, Chongqing, China, 400045.

Jae-Hoon Kang
Chung-Ang University, 221 Heuksuk-Dong, Dongjak-Ku, Seoul 156-756, South Korea.

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Natural frequencies and mode shapes are obtained for a sinusoidal-shaped shell of revolution by using the Ritz method from a three-dimensional (3-D) analysis instead of a mathematically two-dimensional (2-D) thin shell theory or high order thick shell theory. The present analysis uses circular cylindrical coordinates instead of 3-D shell coordinates, which have been used in traditional shell analyses. Convergence studies can analyze the first five frequencies to four-digit exactitude. Results are given for a variety of shallow and deep sinusoidal-shaped shells with different boundary conditions. The sinusoidal-shaped shells are very similar to paraboloidal shells in shape. The frequencies of the sinusoidal-shaped shells from the present 3-D method are compared with those from 2-D thin shell theories for paraboloidal shells. The present 3-D method is applicable to very thick as well as thin shells.

1. INTRODUCTION

Vibrations of sinusoidal-shaped curved beams were investigated. However, studies involving sinusoidal-shaped shells of revolution have not been reported. The sine-shaped shells of revolution with various curvatures are obtained through the ratio of amplitude and period of a sine curve. Such shells are used in architectural, civil, and aeronautical engineering. Paraboloidal shells are very similar to sinusoidal shells in shape. A vast published literature exists for the free vibrations of shells. The monograph of Leissa summarized approximately 1000 relevant publications world-wide from the 1960s. Almost all of these publications dealt with shells (e.g. circular cylindrical, conical, spherical). Among them were eight references considering paraboloidal shells. Some additional investigations of the static and dynamic characteristics of paraboloidal shells have also been uncovered. However, these studies were either experimental or were based on thin shell theories, which are mathematically two-dimensional (2-D). That is, for thin shells, one assumes the Kirchhoff hypothesis that the shell middle surface remains normal during deformations (vibratory, in this case) and unstretched in length. The conventional 2-D shell theory is only applicable to thin shells. A high order thick shell theory, which considers the effects of shear deformation and rotary inertia, could be used and would be useful for the low frequency modes of moderately thick shells. However, such a theory would also be 2-D. Vibration analyses of paraboloidal shells have been studied by a three-dimensional (3-D) Ritz method. Recently, experiments on the vibration control of a piezoelectric laminated paraboloidal shell were made. The current study provides a 3-D analysis of the vibrations of sinusoidal-shaped shells of revolution by using the Ritz method. The present analysis uses circular cylindrical coordinates instead of related 3-D shell coordinates, which are normal and tangent to the shell mid-surface. 3-D shell coordinates are usually used in traditional shell analyses. It takes much more time to compute the energy functionals based on the 3-D shell coordinates compared to the circular cylindrical coordinates. Convergence studies are demonstrated for the first five frequencies to four-digit exactitude. Results are given for a variety of shallow and deep sinusoidal-shaped shells with varying boundary conditions. The frequencies from the present 3-D method are compared with those from 2-D thin shell theories for paraboloidal shells. The 3-D method is only limited to small strains. No other constraints are placed on the displacements. This is an obvious difference between the 3-D analysis and the classical 2-D thin shell theories, which make very limiting assumptions about the displacement variation through the shell thickness. Therefore, the present analysis is applicable to very thick shells as well as thin shells. Also, even for shells with varying curvature, the present method is valid.

2. METHOD OF ANALYSIS

A representative cross-section of a sinusoidal-shaped shell of revolution is depicted in Fig. 1. The period and amplitude of the outer cosine curve $z_i$ were of $4a$ and $b$, respectively. The shell thickness was denoted as $H$ at the bottom ($z = 0$) and at the top ($r = 0$) of the shell. Thus, the period and amplitude of the inner cosine curve $z_o$ were of $4(a - H)$ and $b - H$, and the middle cosine curve $z_m$ of $(a - H/2)$ and $b - H/2$, respectively as shown in Fig. 1. The equations of the curves became

$$z_i = (b - H) \cos \left( \frac{\pi}{2(a - H)}r \right), \quad z_o = b \cos \left( \frac{\pi}{2a}r \right);$$
$$z_m = (b - H/2) \cos \left( \frac{\pi}{2(a - H/2)}r \right).$$

The circular cylindrical coordinate system $(r, z, \theta)$, also shown in the figure, was used in the analysis, and $\theta$ is the circumference-
where the strains \( \varepsilon_{ij} \) were expressed in terms of the three displacements by Eq. (2). The kinetic energy \( (V) \) was the integral over the domain \( (A) \):

\[
V = \frac{1}{2} \int \int \int_A \left[ (\sigma_{rr} \varepsilon_{rr} + \sigma_{zz} \varepsilon_{zz} + \sigma_{\theta\theta} \varepsilon_{\theta\theta})^2 + 2G(\varepsilon_{rr}^2 + \varepsilon_{zz}^2 + \varepsilon_{\theta\theta}^2) + 2(\varepsilon_{rz}^2 + \varepsilon_{\theta r}^2 + \varepsilon_{\theta z}^2) \right] r dr dz d\theta;
\]

where the strains \( \varepsilon_{ij} \) were an arbitrary phase functionals. The Ritz method uses the maximum strain energy (V) and the maximum kinetic energy (\( T_{\text{max}} \)) functionals. The functionals were expressed in terms of the non-dimensional coordinates \( \Psi \) and \( \zeta \) as follows:

\[
V_{\text{max}} = \frac{bG}{2} \left[ \int_0^1 \int_0^\zeta_0 I_V \psi d\zeta d\psi - \int_0^{1-H^*} \int_0^\zeta_0 I_V \psi d\zeta d\psi \right];
\]

\[
T_{\text{max}} = \frac{\rho \omega^2 a^2 b}{2} \left[ \int_0^1 \int_0^\zeta_0 I_T \psi d\zeta d\psi - \int_0^{1-H^*} \int_0^\zeta_0 I_T \psi d\zeta d\psi \right];
\]

where \( U_r, U_z, \) and \( U_\theta \) were displacement functions of \( \Psi \) and \( \zeta \), \( \omega \) was a natural frequency, and \( \alpha \) was an arbitrary phase angle. The circumferential wave number was taken to be an integer \( (n = 0, 1, 2, 3, \ldots, \infty) \), to ensure periodicity in \( \theta \).

The Ritz method uses the maximum strain energy \( (V_{\text{max}}) \) and the maximum kinetic energy \( (T_{\text{max}}) \) functionals. The functionals were expressed in terms of the non-dimensional coordinates \( \Psi \) and \( \zeta \) as follows:
where

\[ I_{V} = \left[ \frac{\lambda (\kappa_1 + \kappa_2 + \kappa_3)^2}{G} + 2(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) + \kappa_2^2 \right] \Gamma + \left( \kappa_1^2 + \kappa_2^2 \right) I_2; \] (18)

\[ I_{T} \equiv \left( U_r^2 + U_z^2 \right) \Gamma_1 + U_\theta^2 \Gamma_2; \] (19)

and

\[ \kappa_1 \equiv U_r + nU_\theta/\psi; \quad \kappa_2 \equiv U_r/\psi; \quad \kappa_3 \equiv U_z/\psi; \quad \kappa_4 \equiv nU_\theta/\psi - \theta_\psi/\psi; \quad \kappa_5 \equiv (nU_r + U_\theta)/\psi - \theta_\psi/\psi; \] (20)

and \( \Gamma_1 \) and \( \Gamma_2 \) were defined by

\[ \Gamma_1 \equiv \int_0^{2\pi} \cos^2 \theta \, d\theta = \begin{cases} 2\pi & \text{if } n = 0 \\ \pi & \text{if } n \geq 1 \end{cases}; \]

\[ \Gamma_2 \equiv \int_0^{2\pi} \sin^2 \theta \, d\theta = \begin{cases} 0 & \text{if } n = 0 \\ \pi & \text{if } n \geq 1 \end{cases}. \] (21)

From Eq. (2) it is seen that the non-dimensional constant \( \lambda/G \) in Eq. (19) involved only \( \nu \) as follows

\[ \frac{\lambda}{G} = \frac{2\nu}{1 - 2\nu}. \] (22)

The displacement functions \( U_r, U_z, \) and \( U_\theta \) in Eqs. (2) were further assumed as algebraic polynomials

\[ U_r(\psi, \zeta) = \eta_r(\psi, \zeta) \sum_{i=0}^{j} \sum_{j=0}^{J} A_{ij} \psi^i \zeta^j; \]

\[ U_z(\psi, \zeta) = \eta_z(\psi, \zeta) \sum_{k=0}^{K} \sum_{L=0}^{l} B_{kl} \psi^k \zeta^l; \]

\[ U_\theta(\psi, \zeta) = \eta_\theta(\psi, \zeta) \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn} \psi^m \zeta^n; \] (23)

where \( i, j, k, l, m, \) and \( n \) were integers; \( I, J, K, L, M, \) and \( N \) were the highest degrees taken in the polynomial terms; \( A_{ij}, B_{kl}, \) and \( C_{mn} \) were arbitrary coefficients to be determined, and \( \eta_r, \eta_z, \eta_\theta(\psi, \zeta) \) were functions based on geometric boundary conditions. For example:

1. completely free: \( \eta_r = \eta_z = \eta_\theta = 1; \)
2. the bottom edge (\( \zeta = 0 \)) fixed: \( \eta_r = \eta_z = \eta_\theta = \zeta. \)

The eigenvalue problem was formulated by minimizing the free vibration frequencies with respect to the arbitrary coefficients \( A_{ij}, B_{kl}, \) and \( C_{mn}. \) This corresponded to the following equations.\(^{23}\)

\[ \frac{\partial}{\partial A_{ij}} (V_{\max} - \omega^2 T_{\max}) = 0; \]

\[ (i = 0, 1, 2, \ldots, I; \, j = 0, 1, 2, \ldots, J); \]

\[ \frac{\partial}{\partial B_{kl}} (V_{\max} - \omega^2 T_{\max}) = 0; \]

\[ (k = 0, 1, 2, \ldots, K; \, l = 0, 1, 2, \ldots, L); \]

\[ \frac{\partial}{\partial C_{mn}} (V_{\max} - \omega^2 T_{\max}) = 0; \]

\[ (m = 0, 1, 2, \ldots, M; \, n = 0, 1, 2, \ldots, N); \] (24)

where \( T_{\max} = \omega^2 T_{\max}. \) The minimizing equations (2) yielded a set of \( (I+1)(J+1)+(K+1)(L+1)+(M+1)(N+1) \) linear, homogeneous, algebraic equations (or Ritz system) in the unknowns \( A_{ij}, B_{kl}, \) and \( C_{mn}. \) The equations were written in the form

\[ (\mathbf{K} - \Omega \mathbf{M}) \mathbf{x} = \mathbf{0}; \] (25)

where \( \mathbf{K} \) and \( \mathbf{M} \) were stiffness and mass matrices that resulted from the maximum strain energy \( (V_{\max}) \) and the maximum kinetic energy \( (T_{\max}) \), respectively. The variable \( \Omega \) was an eigenvalue of the vibrating system, expressed as the square of non-dimensional frequency, \( \Omega = \omega^2 a^2 \rho/G, \) and the vector \( \mathbf{x} \) took the form

\[ \mathbf{x} = (A_{00}, A_{01}, ..., A_{IJ}; B_{00}, B_{01}, ..., B_{KL}; C_{00}, C_{01}, ..., C_{MN})^T. \] (26)

In the present problem, the Ritz system had the following form:

\[ \begin{bmatrix} K_{ij,ij} & K_{ij,kl} & K_{ij,\tilde{m}n} \\ K_{kl,ij} & K_{kl,kl} & K_{kl,\tilde{m}n} \\ K_{mn,ij} & K_{mn,kl} & K_{mn,\tilde{m}n} \end{bmatrix} \begin{bmatrix} A_{ij} \\ B_{kl} \\ C_{\tilde{m}n} \end{bmatrix} = 0; \] (27)

where

\[ K_{ij,ij} = \Gamma_1 \left[ \left( \frac{\lambda}{G} + 2 \right) \left( \frac{a}{b} \right)^2 \hat{P}_{ij,\zeta} \hat{P}_{ij,\zeta} \right] + \left( \hat{P}_{ij,\psi} \hat{P}_{ij,\psi} \right) + n^2 \Gamma_2 \left( \hat{P}_{ij,\psi} \hat{P}_{ij,\psi} \right); \]

\[ K_{kl,kl} = \Gamma_1 \left[ \left( \frac{\lambda}{G} + 2 \right) \left( \frac{a}{b} \right)^2 \hat{P}_{kl,\psi} \hat{P}_{kl,\psi} \right] \]

\[ + \left( \hat{P}_{kl,\psi} \hat{P}_{kl,\psi} \right) + \left( \frac{a}{b} \right)^2 \left( \hat{P}_{kl,\zeta} \hat{P}_{kl,\zeta} \right); \]

\[ K_{mn,\tilde{m}n} = n^2 \Gamma_1 \left( \frac{\lambda}{G} + 2 \right) \left( \hat{P}_{mn,\psi} \hat{P}_{mn,\psi} \right) \]

\[ + \left( \hat{P}_{mn,\psi} \hat{P}_{mn,\psi} \right); \]

\[ K_{ij,\tilde{m}n} = n \Gamma_1 \left( \frac{\lambda}{G} + 2 \right) \left( \hat{P}_{ij,\psi} \hat{P}_{\tilde{m}n,\psi} \right) \]

\[ + \left( \hat{P}_{ij,\psi} \hat{P}_{\tilde{m}n,\psi} \right); \]

\[ K_{kl,\tilde{m}n} = n \Gamma_1 \left( \frac{\lambda}{G} + 2 \right) \left( \hat{P}_{kl,\psi} \hat{P}_{\tilde{m}n,\psi} \right) \]

\[ + \left( \hat{P}_{kl,\psi} \hat{P}_{\tilde{m}n,\psi} \right); \]

\[ K_{mn,\tilde{m}n} = n \Gamma_1 \left( \frac{\lambda}{G} + 2 \right) \left( \hat{P}_{mn,\psi} \hat{P}_{\tilde{m}n,\psi} \right) \]

\[ + \left( \hat{P}_{mn,\psi} \hat{P}_{\tilde{m}n,\psi} \right); \]

\[ \hat{P}_{mn,\psi} \hat{P}_{\tilde{m}n,\psi} \].
3. CONVERGENCE STUDIES

Tables 1-3 showed convergence studies for a sinusoidal shell of revolution fixed at the bottom edge (\(z = 0\)) for \(b/a = 1\) and \(H/a = 1/10\). The tables listed the first five non-dimensional frequencies in \(\omega_a \sqrt{\rho/G}\) for \(\nu = 0.3\), for torsional modes (\(n = 0^T\)) in Table 1, axisymmetric modes (\(n = 0^A\)) in Table 2, and bending modes (\(n = 2\)) in Table 3. The bending modes (\(n = 2\)) had two circumferential waves in their mode shapes.

To make the study of convergence less complicated, equal numbers of polynomial terms were taken in both the \(r\) (or \(\Psi\)) coordinate (i.e., \(l = K = M\)) and \(z\) (or \(\zeta\)) coordinate (i.e., \(J = L = N\)). Computational optimizations were obtained for some configurations and some mode shapes by using unequal numbers of polynomial terms. The symbols TZ and TR in the tables indicate the total numbers of polynomial terms used in the \(z\) (or \(\zeta\)) and \(r\) (or \(\Psi\)) directions, respectively. Note that the frequency determinant order \(\text{DET}\) was related to TZ and TR as follows:

\[
\text{DET} = \begin{cases} 
Tz \times Tz & \text{for torsional modes (} n = 0^T\) \\
2 \times Tz \times Tz & \text{for axisymmetric modes (} n = 0^A\) \\
3 \times Tz \times Tz & \text{for general modes (} n = 1, 2, 3, \ldots\) 
\end{cases}
\]  

(31)

Tables 1-3 showed the monotonic convergence of all five frequencies as \(\text{TZ} = J + 1, L + 1, \text{and} N + 1\) in Eqs. (2) were increased, as well as \(\text{TR} = I + 1, K + 1, \text{and} M + 1\) in Eqs. (2). For example, Table 3 showed that the first non-dimensional frequency in \(\omega_a \sqrt{\rho/G}\) for \(n = 2\) converged to four digits 1.342 when \((\text{TZ}, \text{TR})=(7,10)\) terms were used, which resulted in \(\text{DET} = 3 \times (7 \times 10) = 210\). Frequencies in underlined, bold-faced type in Tables 1-3 were the most accurate values (to four significant figures) achieved with the smallest determinant sizes.

4. COMPARISON WITH PARABOLOIDAL SHELLS BY 2-D SHELL THEORY

Studies on vibrations of sinusoidal shells of revolution have not been reported. Paraboloidal shells of revolution are very similar to sinusoidal shells in shape. Lin and Lee analyzed free vibrations of paraboloidal shells applying a 2-D inextensional shell theory, which depended on the assumption that the length of line elements remained invariant under the deformation of the shell.26 Based on Love’s equation, Lin and Lee obtained the natural frequencies of paraboloidal shells,25 \((\omega)\) for completely free boundaries as follows:

\[
\omega^2 = \frac{n^2(n^2 - 1)^2 GH^2}{96\rho^4(1 - \nu) \rho} \times 
\int_0^{\pi/2} \tan^{2n-3} \varphi \cdot \sec^3 \varphi \cdot (\cos^2 \varphi + \sec^2 \varphi + 2 - 4\nu) \, d\varphi
\]

\[
\int_0^{\pi/2} \tan^{2n+1} \varphi \cdot \sec^3 \varphi \cdot [2n + (n^2 + 1) \sec^2 \varphi] \, d\varphi
\]

(32)

where \(\hat{a}\) was the focal distance of the mid-surface of a paraboloidal shell. The variable \(\varphi\) was a meridian coordinate that represented the angle between the normal to the mid-surface and the axis of revolution (\(z\)-axis). \(\varphi_0\) represented \(\varphi\) at the bottom face of the shell. Comparing the sinusoidal-shaped shell in Fig. 1 with a paraboloidal shell with thickness \(H\), the coordinates of the top and bottom of the mid-surface of the paraboloidal shell were \((r, z) = (0, b - H/2)\) and \((r, z) = ( \pm a \mp H/2, 0)\), respectively. Thus, the equation of the mid-surface of the paraboloidal shell became

\[
z_p = -\frac{(b - H/2)}{(a - H/2)^2} r^2 + (b - H/2);
\]

(33)

and the angle resulted in

\[
\tan^{-1}\left[\frac{2(b - H/2)}{a - H/2}\right] = \tan^{-1}\left[\frac{2(k - H^*/2)}{1 - H^*/2}\right]
\]

(34)

and the focal distance yielded

\[
\hat{a} = \frac{(a - H/2)^2}{4(b - H/2)} = \frac{a(1 - H^*/2)^2}{4(k - H^*/2)^2}.
\]  

(35)
Table 3. Convergence of frequencies $\omega a/\sqrt{\rho/G}$ of a sinusoidal-shaped shell fixed at the bottom edge ($z = 0$) for the five lowest bending modes ($n = 2$) for $b/a = 1$ and $H/a = 1/10$ ($\nu = 0.3$).

<table>
<thead>
<tr>
<th>TZ</th>
<th>TR</th>
<th>DET</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6</td>
<td>108</td>
<td>1.347</td>
<td>2.440</td>
<td>3.638</td>
<td>4.410</td>
<td>5.176</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>126</td>
<td>1.345</td>
<td>2.434</td>
<td>3.618</td>
<td>4.408</td>
<td>5.083</td>
</tr>
</tbody>
</table>

Table 4. Comparisons of the first ($s = 1$) non-dimensional frequencies $\omega a/\sqrt{\rho/G}$ for each $n (= 2, 3, 4, 5, 6)$ from the 3-D Ritz method (3DR) and 2-D shell theory (2DS) of completely free paraboloidal shells $H/R = 1/2$ and $\nu = 0.3$.

<table>
<thead>
<tr>
<th>$h_b/R$</th>
<th>Methods</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/90</td>
<td>3DR</td>
<td>0.2583</td>
<td>0.6327</td>
<td>0.1140</td>
<td>0.1774</td>
<td>0.2526</td>
</tr>
<tr>
<td></td>
<td>(Difference %)</td>
<td>$(-1.6%)$</td>
<td>$(-2.1%)$</td>
<td>$(2.9%)$</td>
<td>$(3.7%)$</td>
<td>$(4.8%)$</td>
</tr>
<tr>
<td>1/30</td>
<td>3DR</td>
<td>0.1864</td>
<td>0.3325</td>
<td>0.5127</td>
<td>0.7255</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(Difference %)</td>
<td>$(0.8%)$</td>
<td>$(4.0%)$</td>
<td>$(5.9%)$</td>
<td>$(7.7%)$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2 shows comparisons of the middle surfaces of paraboloidal shell $z_p$ and the sinusoidal-shaped shell $z_m$ in Eqs. (2) for $b/a = 1$ and 2.

Comparisons of the present 3-D Ritz method (3DR) with the 2-D shell theory (2DS) of Lin and Lee (1960) are made in Tables 4 and 5 for the first ($s = 1$) non-dimensional frequencies in $\omega R/\sqrt{\rho/G}$ for each $n (= 2, 3, 4, 5, 6)$ of completely free paraboloidal shells for $H/R = 1/2$ (in Table 4) and 1 (in Table 5), and $\nu = 0.3$. The percent difference in frequencies obtained by the two analyses was given by

$$\text{Difference (\%) = } \frac{2\text{DS} - 3\text{DR}}{3\text{DR}} \times 100. \quad (36)$$

It was observed that the 3-D Ritz method yielded lower frequencies than the 2-D thin shell results in all the frequencies irrespective of thickness parameter ($h_b/R$), curvature ($H/R$), and circumferential wave number ($n$), as expected. The Ritz method guaranteed the upper bound convergence of the frequencies in terms of functions sets that are mathematically complete, such as algebraic polynomials. An accurate 3-D analysis should yield lower frequencies than those 2-D thin shell theory, mainly because shear deformation and rotary.

Inertia effects were accounted for in a 3-D analysis, but not in 2-D, thin shell theory. It is particularly interesting that the inextensional theory of Lin and Lee includes bending stiffness, but neglects membrane-type stretching effects, and therefore the 2DS frequencies in Tables 4 and 5 were close to the accurate 3DR results for the thin shell ($h_b/R = 1/90$). This confirmed that the middle surface of the shell can deform inextensionally for these modes ($n = 2, 3, \ldots$) when the shell is completely free. It was observed in Tables 4 and 5 that the frequency differences become larger as shell thickness ($h_b/R$) increases. It is interesting to note that, for the fundamental modes occurring at $n = 2$, the differences for $H/R = 1/2$ (shallower shell) were smaller than ones for $H/R = 1$ (deeper shell), and vice versa for the higher modes ($n = 2, 3, 4, 5$) with two exceptions for ($n, h_b/R = (6, 1/90)$ and $(3, 1/30)$).

5. NUMERICAL RESULTS AND DISCUSSION

Tables 6-7 present the non-dimensional frequencies $\omega a/\sqrt{\rho/G}$ of sinusoidal-shaped shells of revolution for $H/a = 0.1$. Table 6 is for the sinusoidal-shaped shells with completely free boundary condition while Table 7 is for the shells with fixed boundary condition at the bottom. Poisson’s ratio ($\nu$) is taken to be 0.3. Each table shows the frequencies for the shells with five values of $b/a = 1/3, 1/2, 1, 2,$ and $3$. Thirty frequencies are given for each shell configuration, which arise from six circumferential wave numbers ($n = 0^2, 0^4, 1, 2, 3, 4$) and the first five modes ($s = 1, 2, 3, 4, 5$) for each $n$, where the superscripts $T$ and $A$ indicated torsional and axisymmetric modes, respectively.

The two-dimensional mode shapes of the sinusoidal-shaped shells of revolution at arbitrary $z$ ($z = z_0$) and $0 \leq \theta \leq 2\pi$ irrespective of boundary conditions are given in Fig. 3 for each value of $n$ except for the torsional ones ($n = 0^2$). The mode shapes have $2n$ nodal points ($\mu = 0$) for each $n$. The numbers in parentheses identify the first five frequencies for each shell configuration. For example, in the case of $b/a = 1/3$ in the first column of Table 6, the first five frequencies are modes for ($n, s = (2, 1), (3, 1), (0^4, 1), (1, 1),$ and $(4, 1)$ in this order. The zero frequencies of rigid body modes occurring at the sinusoidal-shaped shells having completely free boundary condition are omitted from the table.

It is interesting to note in Tables 6-7 that the fundamental (lowest) frequencies are all for modes having two ($n = 2$) circumferential waves in their modes for completely free boundary conditions irrespective of the values of $b/a$, and one ($n = 1$) circumferential waves or axisymmetric modes ($n = 0^4$) for fixed ones. It is seen in Tables 6-7 that the torsional ($n = 0^2$) and axisymmetric modes ($n = 0^4$) are more important for the fixed boundary conditions than to the completely free ones.
Table 6. Non-dimensional frequencies $\omega a/\sqrt{\rho/G}$ of completely free, sinusoidal-shaped shells of revolution for $H/a = 1/10$ ($\nu = 0.3$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s$</th>
<th>1/3</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>5.030</td>
<td>4.792</td>
<td>3.559</td>
<td>2.039</td>
<td>1.398</td>
</tr>
<tr>
<td>2</td>
<td>7.815</td>
<td>7.319</td>
<td>5.661</td>
<td>3.478</td>
<td>2.426</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10.56</td>
<td>9.844</td>
<td>7.654</td>
<td>4.852</td>
<td>3.426</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15.84</td>
<td>14.87</td>
<td>11.67</td>
<td>8.017</td>
<td>6.116</td>
<td></td>
</tr>
</tbody>
</table>

Table 7. Non-dimensional frequencies $\omega a/\sqrt{\rho/G}$ sinusoidal-shaped shells of revolution fixed at the bottom of the shell for $H/a = 1/10$ ($\nu = 0.3$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s$</th>
<th>1/3</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>4.325</td>
<td>3.786</td>
<td>2.351</td>
<td>1.253(4)</td>
<td>0.8470(3)</td>
</tr>
<tr>
<td>2</td>
<td>6.871</td>
<td>6.165</td>
<td>4.636</td>
<td>2.770</td>
<td>1.918</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9.537</td>
<td>8.637</td>
<td>6.661</td>
<td>4.165</td>
<td>2.922</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14.92</td>
<td>13.65</td>
<td>10.61</td>
<td>6.911</td>
<td>5.004</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2. Comparisons of the middle surfaces of the sinusoidal shell $z_n$, and the paraboloidal shell $z_p$ for $b/a = 1$ and 2.

Figure 3. Mode shapes for each $n$ at $z = z_o$.

That is, the torsional and axisymmetric modes are among the lowest frequencies of the shells.

6. CONCLUSIONS

Extensive and accurate frequency data determined by the 3-D Ritz analysis have been presented for sinusoidal-shaped shells of revolution based on circular cylindrical coordinates instead of 3-D shell coordinates. The comparisons of natural frequencies for the sinusoidal shells of revolution by the present 3-D analysis with those for paraboloidal shells by 2-D shell theory, which are very similar to the sinusoidal shells.
in shape, show a good agreement. The convergence studies demonstrated that the Ritz method guarantees upper bound convergence of the frequencies. Therefore, the data in Tables 6-7 may be used as benchmark results against those obtained by other 3-D methods, such as finite elements and finite differences and may be compared to determine the accuracy of the latter. The present 3-D analysis is applicable to very thick shells as well as thin shells.

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Characteristics of In-Plane Waves in Composite Plates

K. Renji, S. Josephine Kelvina Florence and Sameer Deshpande

ISRO Satellite center, ISRO Vimanapura Post, Bangalore, India 560017.

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The high frequency dynamic excitations generate both in-plane as well as bending waves in structures. In aerospace applications, many of these structures are made of composite materials. There are two types of in-plane motions, longitudinal and in-plane shear. Although these motions are uncoupled in isotropic materials, composite structures show coupled behaviour. The works reported on in-plane waves in composite structures assume that two in-plane motions are uncoupled as in isotropic plates. In this work, characteristics of the in-plane waves in composite laminated plates are investigated. Expressions for wavenumber, phase speed and group speed are derived. It is seen that in composite plates the two in-plane waves are coupled in longitudinal and shear propagations and are non-dispersive. The phase speeds of in-plane waves in composite plates can be much different from those determined using the expressions for isotropic plates where the waves are uncoupled. To validate the expressions derived the phase speeds of in-plane waves in a typical composite panel are determined experimentally. It is seen that the experimentally obtained phase speeds match well with the theoretical results.

1. INTRODUCTION

Broadband high frequency acoustic excitation is one of the loads for the design of spacecraft elements. These excitations generate both in-plane as well as bending waves in structures. Characteristics of these waves should be well understood in order to study the response behaviour of structures in the high frequency region. The wavenumber, phase speed and group speed are the most important parameters that govern the wave motion.

The characteristics of bending waves in various forms of structural elements are well discussed. Many of the spacecraft panels are made of honeycomb sandwich construction with composite face sheets. Closed form expressions for the bending wave characteristics are derived for such panels considering even the transverse shear deformation. Though the structure–borne sound caused by acoustic excitation is dominated by bending waves, the in-plane waves become important at locations far away from the excitation as well as in structural health monitoring. Most of the work on the wave characteristics carried out in structural health monitoring relate to bending waves. Very few works are reported on the characteristics of in-plane waves in composite plates.

Datta et al. studied the wave propagation in laminated composite plates, but were limited to bending waves. Due to the fact that they used a numerical method, no expressions were derived. Some studies used spectral elements for modelling the in-plane wave propagation and hence no expressions were derived for their characteristics. Though there are a few more studies reported on the characteristics of the in-plane waves, no expression is derived for these characteristics. These characteristics must instead be obtained through numerical methods. The works reported in also do not present any expression for the wave characteristics, but instead use numerical techniques to determine the characteristics at every frequency. In a few other references, although the transmission of in-plane waves are discussed, no expressions for computing the speeds of these waves are derived.

There are two types of in-plane waves, in-plane longitudinal and in-plane shear. These planes are uncoupled in isotropic materials. In composite plates, which will be shown later, they are coupled. There are several works dealing with the speed of in-plane waves in composite panels. All of them assume that these two in-plane waves are uncoupled. Thus, the results do not include the coupled in-plane shear and longitudinal behaviour. Instead, the results give the speeds as though they are uncoupled, as in isotropic plates.

In the present work, expressions for the characteristics of in-plane waves in a composite laminate are derived. It takes into account the coupled motion of both the in-plane waves. Sensitivities of these characteristics to various parameters of the composite plate are discussed. Experimental results are obtained for a typical composite plate. The experimentally determined phase speeds are compared with the phase speeds determined using the expression derived in this work.

2. DIFFERENTIAL EQUATIONS OF WAVE MOTION

Consider a plate as shown in Fig. 1.

The coordinate axes are in-plane (x and y) and normal (z). The displacements along the in-plane directions are represented by u and v and the normal displacement is represented by w. Corresponding mid-plane displacements are \( u_0, v_0 \) and \( w_0 \). The equilibrium equations for the free vibration of thin plates, neglecting rotary inertia, are as follows:

\[
\frac{\partial^2 N_{xx}}{\partial x^2} + 2 \frac{\partial^2 N_{xy}}{\partial x \partial y} + \frac{\partial^2 N_{yy}}{\partial y^2} = \rho_m \frac{\partial^2 w_0}{\partial t^2}; \quad (1)
\]

\[
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = \rho_m \frac{\partial^2 w_0}{\partial t^2}; \quad (2)
\]

\[
\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = \rho_m \frac{\partial^2 w_0}{\partial t^2}; \quad (3)
\]
where \( \rho_m \) represented the mass per unit area, \( N_{xx}, N_{yy} \) and \( N_{xy} \) are the stress resultants (per unit width) and \( M_{xx}, M_{yy} \) and \( M_{xy} \) are the moment resultants (per unit width).

Assuming Classical Lamination theory, based on Kirchoff’s theory, in which the displacements in the in-plane directions vary linearly through the plate thickness while the normal displacement is independent of the thickness,\(^{21}\) the displacement field is determined by:

\[
\begin{align*}
  u &= u^0 - z \frac{\partial w}{\partial x}, \\
  v &= v^0 - z \frac{\partial w}{\partial y}, \\
  w &= w_0.
\end{align*}
\]

Here, the transverse shear effects are neglected and the corresponding shear strains are zero.

The relationships of in-plane strains, curvature and twist with the displacements are called the generalized strain-displacement or kinematic equations. The strains are expressed in terms of the mid-plane strains \( \epsilon_{xx}^0, \epsilon_{yy}^0 \& \gamma_{xy}^0 \) and the curvature and the twist \( \kappa_{xx}, \kappa_{yy} \& \kappa_{xy} \).

\[
\begin{align*}
  \epsilon_{xx} &= \epsilon_{xx}^0 + z\kappa_{xx}; \\
  \epsilon_{yy} &= \epsilon_{yy}^0 + z\kappa_{yy}; \\
  \gamma_{xy} &= \gamma_{xy}^0 + z\kappa_{xy}.
\end{align*}
\]

As the stresses in a laminate vary from layer to layer, it was convenient to deal with a simpler but equivalent system of forces and moments on a laminate cross-section, commonly called as the stress resultants and moment resultants.\(^{21}\) These six quantities formed a system that is equivalent to the stress system on the laminate, but it is applied at the geometrical mid-plane.

As the laminate is in a plane-stress state, normal stress and the transverse shear stresses were neglected. Only \( \sigma_{xx}, \sigma_{yy} \) and \( \tau_{xy} \) were considered. In order to find a system of forces and moments acting at the geometric mid-plane that was equivalent to the effect of these stresses, three stress resultants were defined which are equal to the sum or integral of these stresses in the thickness direction. These forces and moments were defined as:

\[
\begin{align*}
  \left[ \begin{array}{c}
  N_{xx} \\
  N_{yy} \\
  N_{xy}
\end{array} \right] &= \sum_{k=1}^{h_k} \int_{h_k-1}^{h_k} \left[ \begin{array}{c}
  \sigma_{xx} \\
  \sigma_{yy} \\
  \tau_{xy}
\end{array} \right] \, dz; \\
  \left[ \begin{array}{c}
  M_{xx} \\
  M_{yy} \\
  M_{xy}
\end{array} \right] &= \sum_{k=1}^{h_k} \int_{h_k-1}^{h_k} \left[ \begin{array}{c}
  \sigma_{xx} \\
  \sigma_{yy} \\
  \tau_{xy}
\end{array} \right] \, dz.
\end{align*}
\]

For a thin plate, using the kinematic equations as described before and combining Eqs. (10) & (11) we determined that:

\[
\begin{align*}
  \left[ \begin{array}{c}
  N_{xx} \\
  N_{yy} \\
  N_{xy}
\end{array} \right] &= \left[ \begin{array}{cccccc}
  A_{11} & A_{12} & A_{16} & A_{16} & B_{11} & B_{12} & B_{16} \\
  A_{12} & A_{22} & A_{26} & A_{26} & B_{21} & B_{22} & B_{26} \\
  A_{16} & A_{26} & A_{66} & A_{66} & B_{11} & B_{16} & B_{66}
\end{array} \right] \left[ \begin{array}{c}
  \epsilon_{xx}^0 \\
  \epsilon_{yy}^0 \\
  \gamma_{xy}^0
\end{array} \right], \quad \kappa_{xx}, \kappa_{yy}, \kappa_{xy}.
\end{align*}
\]

Here,

\[
A_{ij} = \sum_{k=1}^{h_k} (Q_{ij})_k (h_k - h_{k-1}), \quad \text{called as extensional stiffness terms};
\]

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{h_k} (Q_{ij})_k (h_k^2 - h_{k-1}^2), \quad \text{called as coupling stiffness terms};
\]

\[
D_{ij} = \frac{1}{4} \sum_{k=1}^{h_k} (Q_{ij})_k (h_k^3 - h_{k-1}^3), \quad \text{called as bending stiffness terms}.
\]

In the present work, symmetric laminate was considered, therefore \( B_{ij} = 0 \). It was assumed that the laminate had negligible values of \( A_{16}, A_{26}, D_{16} \) and \( D_{26} \). Hence the above relation became,

\[
\begin{align*}
  \left[ \begin{array}{c}
  N_{xx} \\
  N_{yy} \\
  N_{xy}
\end{array} \right] &= \left[ \begin{array}{cccccc}
  A_{11} & A_{12} & 0 & 0 & 0 & 0 \\
  A_{12} & A_{22} & 0 & 0 & 0 & 0 \\
  0 & 0 & A_{66} & 0 & 0 & 0 \\
  0 & 0 & 0 & D_{11} & D_{12} & 0 \\
  0 & 0 & 0 & D_{12} & D_{22} & 0 \\
  0 & 0 & 0 & 0 & 0 & D_{66}
\end{array} \right] \left[ \begin{array}{c}
  \epsilon_{xx}^0 \\
  \epsilon_{yy}^0 \\
  \gamma_{xy}^0 \end{array} \right].
\end{align*}
\]

Using Eq. (13) in the first two equations of motion given by Eqs. (1) & (2) (corresponding to the in-plane direction) the equilibrium equations were cast in terms of displacements.

\[
A_{11} \frac{\partial^2 w_0}{\partial x^2} + A_{66} \frac{\partial^2 w_0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} = \rho_m \frac{\partial^2 w_0}{\partial t^2}; \quad (14)
\]

\[
A_{66} \frac{\partial^2 w_0}{\partial x^2} + A_{22} \frac{\partial^2 v_0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 w_0}{\partial x \partial y} = \rho_m \frac{\partial^2 v_0}{\partial t^2}. \quad (15)
\]

3. EQUATIONS FOR WAVEMOTIONS

Using the above governing differential equations (Eqs. (14) and (15)), the characteristics of the waves were obtained.

The solution to the wave equation was as follows:

\[
w_0 = U_0 e^{i(\omega - k_x x - k_y y)} \quad \text{and} \quad v_0 = V_0 e^{i(\omega - k_x x - k_y y)}; \quad (16)
\]

where \( k_x \) and \( k_y \) represent the wavenumbers in the \( x \) and \( y \) directions respectively and \( \omega \) represents the radian frequency.
Upon the substitution of Eq. (16) into Eqs. (14) and (15), we produced equations for wave motions.

\[
\begin{bmatrix}
A_{11}k_x^2 + A_{66}k_y^2 - \rho_m\omega^2 \\
(A_{12} + A_{66})k_xk_y
\end{bmatrix}
\begin{bmatrix}
U_0 \\
V_0
\end{bmatrix} = 0. \tag{17}
\]

4. Wavenumber and Wave Speed

The above determinant being zero leads to the equations relating the wavenumbers. From the expressions for wavenumber, the phase speeds and group speeds of the waves are determined.

4.1. Wavenumber

Determinant of Eq. (17) is given below.

\[
A_{11}A_{66}k_x^4 + A_{22}A_{66}k_y^4 + k_x^2k_y^2(A_{11}A_{22} + A_{66}^2 - A_{12} - 2A_{12}A_{66}) - \rho_m\omega^2(A_{11}k_x^2 + A_{66}k_y^2 + k_x^2k_y^2) + \rho_m\omega^4 = 0. \tag{18}
\]

By defining \( \alpha = \frac{A_{11}}{A_{11}^2} \); \( \beta = \frac{A_{66}}{A_{11}} \); \( \delta = \frac{A_{12}}{A_{11}} \), the determinant became:

\[
A_{11}A_{66}(k_x^2 + \delta k_y^2) + A_{11}^2(\delta - \alpha^2 - 2\alpha\beta)k_xk_y^2 - \rho_m\omega^2A_{11}(k_x^2 + \beta k_y^2 + \delta k_x^2) + \rho_m\omega^4 = 0. \tag{19}
\]

For a panel having \( A_{11} = A_{22} \),

\[
A_{11}A_{66}(k_x^2 + k_y^2) + A_{11}(1 - \alpha^2 - 2\alpha\beta)k_x^2k_y^2 - \rho_m\omega^2A_{11}(k_x^2 + \beta k_y^2 + k_x^2) + \rho_m\omega^4 = 0. \tag{20}
\]

Since \( k_x^2 + k_y^2 = k^2 \) and \( k_x^4 + k_y^4 = k^4 - 2k_x^2k_y^2 \), we determined that:

\[
A_{11}A_{66}k^4 + A_{11}^2((1 - \alpha(\alpha + 2\beta)) - 2\beta)k_x^2k_y^2 - \rho_m\omega^2A_{11}k^2(1 + \beta) + \rho_m\omega^4 = 0. \tag{21}
\]

Since \( k_x = k \cos \theta; k_y = k \sin \theta \) in the wavenumber domain,

\[
\rho_m\omega^4 - \rho_m\omega^2A_{11}(1 + \beta)k^2 + A_{11}A_{66}k^4 + A_{11}^2k^4(1 - \alpha^2 - 2\alpha\beta - 2\beta) \sin^2 \theta \cos^2 \theta = 0. \tag{22}
\]

By defining:

\[
e = \{1 - \alpha^2 - 2\alpha\beta - 2\beta\} \sin^2 \theta \cos^2 \theta; \tag{23}
\]

the above equation can be simplified as:

\[
\rho_m\omega^4 - \rho_m\omega^2A_{11}(1 + \beta)k^2 + A_{11}A_{66}k^4 + A_{11}^2k^4e = 0. \tag{24}
\]

Eq. (24) is the dispersion relation which is a function of \( \omega \) and wavenumber \( k \), from which the expression for the wavenumber can be derived as:

\[
k^2 = \frac{\rho_m\omega^2}{2A_{11}} \{ (1 + \beta) \pm \sqrt{(1 - \beta^2)^2 - 4\epsilon} \}. \tag{25}
\]

4.2. Phase Speed

Using the definition of the wavenumber, that is \( k = \frac{\omega}{c} \), in Eq. (24), we determined that:

\[
\begin{align*}
\rho_m\omega^4 & - \rho_m\omega^2A_{11}(1 + \beta) \left( \frac{\omega}{c} \right)^2 + \\
& + A_{11}A_{66} \left( \frac{\omega}{c} \right)^4 + A_{11}^2(\frac{\omega}{c})^4 \epsilon = 0;
\end{align*} \tag{26}
\]

which, on simplification, became:

\[
c^4\rho_m - \rho_m A_{11}(1 + \beta)c^2 + (A_{11}A_{66} + A_{11}^2\epsilon) = 0. \tag{27}
\]

This can be further simplified as:

\[
c^4\rho_m - \rho_m A_{11} - c^2(1 + \beta) + A_{11}(\beta + \epsilon) = 0. \tag{28}
\]

Eq. (28) is quadratic in \( c^2 \), the solution of which gives the velocity of the in-plane wave as:

\[
c^2 = \frac{(1 + \beta) \pm \sqrt{(1 + \beta)^2 - 4(\frac{\rho_m}{A_{11}}) \frac{A_{11}}{\rho_m}(\beta + \epsilon)}}{2(\frac{\rho_m}{A_{11}})}; \tag{29}
\]

\[
c^2 = \frac{A_{11}}{2\rho_m}[(1 + \beta) \pm \sqrt{(1 - \beta)^2 - 4\epsilon}]. \tag{30}
\]

Eq. (30) gives the phase speed of the in-plane wave in a composite plate. It is to be noted that the results showed the occurrence of two waves as in isotropic plates and they depended on the orientation of the wave which is represented by the parameter \( \epsilon \).

4.3. Group Speed

The gradient \( \frac{\partial \omega}{\partial k} \) is the group velocity, denoted by \( c_g \). From Eq. (24) we determined that:

\[
\rho_m\omega^4 - k^2(1 + \beta)\omega^2 + \frac{A_{11}}{\rho_m}k^4(\beta + \epsilon) = 0. \tag{31}
\]

The above expression is quadratic in \( \omega^2 \) and the solution was:

\[
\omega^2 = \frac{k^2(1 + \beta) \pm \sqrt{k^4(1 + \beta)^2 - 4(\frac{\rho_m}{A_{11}}) \frac{A_{11}}{\rho_m}k^4(\beta + \epsilon)}}{2\frac{\rho_m}{A_{11}}}. \tag{32}
\]

Upon applying certain algebraic operations, we determined that:

\[
\omega = k \sqrt{\frac{A_{11}}{2\rho_m} \{ (1 + \beta) \pm \sqrt{(1 - \beta)^2 - 4\epsilon} \}}; \tag{33}
\]

\[
c_g^2 = \frac{A_{11}}{2\rho_m} \{ (1 + \beta) \pm \sqrt{(1 - \beta)^2 - 4\epsilon} \}. \tag{34}
\]

Eq. (34) represents the group speed of the in-plane wave in a composite plate. It can be seen that the group speed is the same as the phase speed, which was as expected since the group speed was independent of frequency.
4.4. Assumptions

The assumptions used in arriving at the above expressions are summarized below. Some of these assumptions are part of any two-dimensional structural analysis but given here for completeness.

1. The plate is thin, i.e., thickness of the plate was much less compared to the other dimensions.
3. $\epsilon_z = 0$, i.e., the displacement w was independent of z.
4. The transverse planes that were normal to the undeformed layers remained plane and normal after deformation (CLPT is used). This means the shear deformations were negligible.
5. Material was linearly elastic.
6. The laminate was symmetric, therefore $B_{ij} = 0$.
7. The laminate had negligible values of $A_{16}, A_{26}$. This was satisfied in most of the practical cases. If the laminate is balanced, these parameters vanished.
8. The parameters $D_{16}, D_{26}$ of the laminate were negligible.
9. The laminate had $A_{11} = A_{22}$, otherwise the expressions for the characteristics were more complex.
10. Mass distribution was uniform, i.e. mass per unit area was constant.
11. Rotary inertia was neglected.

5. SIMPLIFIED EXPRESSION

Eq. (30) gives the phase speed of the in-plane waves in a composite plate. It is to be noted that the speed depends on the orientation of the wave which is represented by the parameter $\epsilon$. This expression can be simplified in certain cases.

5.1. Simplification of the Expression

Consider the equation for the phase speed, $c^2 = \frac{A_{11}}{2\rho_m} (1 + \beta) \pm \sqrt{1 - \beta^2}$. Here, $\epsilon$ is given by the expression $\{1 - \alpha^2 - 2\alpha\beta - 2\beta^2\} \sin^2 \theta \cos^2 \theta$ which served as a function of $\theta$. The parameter establishes the relation between the wavenumber components along the two directions. For a particular value of wavenumber, it is possible for one to have various combinations of wavenumber components such that the resultant wavenumber is the same. Since there was no specific directional preference, i.e. equal probability of occurrence for the wave components, an average value of $\sin^2 \theta \cos^2 \theta$ was used. The average value of $\sin^2 \theta \cos^2 \theta$ can be shown to be equal to 1/8 using $\epsilon = \{1 - \alpha^2 - 2\alpha\beta\} \frac{1}{2} - \frac{1}{4}$. Now the term inside the square root of the expression for the phase speed became $1 + \beta^2 - \beta - \{1 - \alpha^2 - 2\alpha\beta\} \frac{1}{2}$.

Using the above term, the phase speed of the in-plane wave can be simplified as:

$$c^2 = \frac{A_{11}}{2\rho_m} \left[ (1 + \beta) \pm \sqrt{1 + \beta^2 - \beta - \{1 - \alpha^2 - 2\alpha\beta\} \frac{1}{2}} \right]. \quad (35)$$

For small values of $\beta$, $\beta^2$ was neglected and hence:

$$c^2 = \frac{A_{11}}{2\rho_m} \left[ (1 + \beta) \pm \sqrt{1 - \beta^2 - \beta - \{1 - \alpha^2 - 2\alpha\beta\} \frac{1}{2}} \right]. \quad (36)$$

The square-root appearing in Eq. (36) can be eliminated if $|\beta + \frac{1 - \alpha^2 - 2\alpha\beta}{2}| < 1$, and the expression for the phase speed results in:

$$c^2 = \frac{A_{11}}{2\rho_m} \left[ (1 + \beta) \pm \left(1 - \frac{\beta^2}{2} - \left\{1 - \alpha^2 - 2\alpha\beta\right\} \frac{1}{4}\right)\right]. \quad (37)$$

Considering the positive root of Eq. (37):

$$c^2 = \frac{A_{11}}{\rho_m} \left[ 1 + 0.25\beta - \left\{1 - \alpha^2 - 2\alpha\beta\right\} \frac{1}{8}\right]. \quad (38)$$

Considering the negative root of Eq. (37):

$$c^2 = \frac{A_{11}}{\rho_m} \left[ 0.75\beta + \left\{1 - \alpha^2 - 2\alpha\beta\right\} \frac{1}{8}\right]. \quad (39)$$

Thus, we found two waves with phase speeds that are determined by Eqs. (38) and (39). The significance of these two wave speeds will be clear when we discuss the results for the isotropic plates. The additional assumptions used for the simplified expression were the equal probability of occurrence for the waves. The value of $\beta$ was so small that $\beta^2$ could be neglected and the value $|\beta + \frac{1 - \alpha^2 - 2\alpha\beta}{2}| < 1$, which on simplification, became $|2\beta - \alpha^2 - 2\alpha\beta| < 1$.

5.2. Typical Results

Phase speed is obtained for a typical honeycomb sandwich plate with orthotropic face sheets. The material properties are given below for reference. The face sheet material is two layers of bi-directional CFRP ($0^\circ/90^\circ$) M18/43090. The elastic modulus of the material (each layer) is 1.47 E11 N/m$^2$ and the shear modulus is 4.09 E9 N/m$^2$ with a Poisson’s ratio of 0.03. The honeycomb core has a thickness of 12 mm. The mass per unit area of the panel is 0.92 kg/m$^2$. The panel has $A_{11} = 4.71 \times 10^7$ N/m; $A_{12} = 1.41 \times 10^7$ N/m; $A_{66} = 1.28 \times 10^8$ N/m; $\alpha = 0.03$, $\beta = 0.027$, $\epsilon = 0.117$ and $1 - \alpha^2 - 2\alpha\beta = 0.125$. Using these parameters, the phase speeds are 6650 m/s and 2941 m/s. The phase speeds obtained using the simplified expression are 6737 m/s and 2731 m/s and they are very close to those obtained using the general expression.

Thus, if $\beta^2$ is negligible and $|2\beta - \alpha^2 - 2\alpha\beta| < 1$, Eqs. (38) and (39) can give the values of phase speeds. Otherwise, the general expression given by Eq. (35) can be used in determining the phase speeds.

5.3. Isotropic Plates

In the case of isotropic material,

$$\alpha = \frac{A_{12}}{A_{11}} = \mu, \quad \beta = \frac{A_{66}}{A_{11}} = \frac{1 - \mu}{2}, \quad 1 - \alpha^2 - 2\alpha\beta = 1 - \mu. $$

$\phi$
Considering positive root given by Eq. (38) we get:

\[ c^2 = \frac{A_{11}}{\rho m}. \]  

(40)

Considering negative root given by Eq. (39) we get:

\[ c^2 = \frac{A_{11}}{\rho m} \left[ 1 - \frac{\mu}{2} \right]. \]  

(41)

Thus, we get two phase speeds. One speed is related to the in-plane longitudinal and the other is related to in-plane shear. The two in-plane waves present in the composite plate correspond to the in-plane longitudinal and in-plane shear. In an isotropic plate they are decoupled and this decoupling results in two independent waves. However, the in-plane waves are coupled in a composite plate.

6. WAVE CHARACTERISTICS

Two in-plane waves exist in a composite plate, which are coupled in longitudinal and shear. These waves are non-dispersive. Both the phase and group speeds are the same. The phase speeds depend on two parameters, namely \( \beta \) and \( \alpha \). The parameter \( \beta \) represents the shear properties and \( \alpha \) represents Poisson’s effect. Relationships of phase speeds with the above parameters are discussed here.

In Fig. 2, phase speed is shown against various values of \( \beta \) for a value of 0.3 for \( \alpha \). As discussed earlier, phase speed will have two values. One of the waves has a higher phase speed, denoted in the figure as wave 1, compared to the other, denoted as wave 2. As in most of the composite structures the values of \( \beta \) are much lower and the results are provided for various values of \( \beta \) up to a value of 0.4. For an isotropic material \( \beta = \frac{1-\mu}{2} \) and \( \alpha = \mu \). For a Poisson’s ratio of 0.3, \( \beta = 0.35 \) and \( \alpha = 0.3 \).

For convenience, the phase speed is normalized with respect to \( \frac{A_{11}}{\rho m} \). This factor \( c^2/(A_{11}/\rho m) \), termed here as normalized phase speed, is taken as the ordinate in the plots.

Therefore the figures give \( \left[ 1 + 0.25 \beta - \left( \frac{1 - \alpha^2 - 2 \alpha \beta}{8} \right) \right] \) and \( \left[ 0.75 \beta + \left( \frac{1 - \alpha^2 - 2 \alpha \beta}{8} \right) \right] \) for various values of \( \beta \). The term \( \frac{A_{11}}{\rho m} \) is the phase speed of the longitudinal wave if the material was isotropic.

6.1. Influence of Parameter \( \beta \)

Consider wave 1. For an isotropic plate, the phase speed converges to that of the longitudinal wave. In this case, the wave is termed here as the longitudinal dominant wave. It can be seen that the phase speeds of such waves increase with parameter \( \beta \). Typical values of the parameters of a composite panel are \( \beta = 0.03 \) and \( \alpha = 0.03 \). Fig. 3 gives the phase speed for \( \alpha = 0.03 \) for various values of \( \beta \). It can be seen that \( c^2/(A_{11}/\rho m) \) can be about 0.5 times that of the equivalent isotropic plate.

Consider wave 2. For an isotropic plate the phase speed converges to that of shear wave therefore termed here as shear dominant wave. It can be seen that the phase speed increases with parameter \( \beta \). For an isotropic plate \( \beta = 0.35 \) and \( \alpha = 0.3 \), and hence \( c^2/(A_{11}/\rho m) \) is equal to 0.35. For a typical composite panel \( (\beta = 0.03 \text{ and } \alpha = 0.03) \), it can be seen that \( c^2/(A_{11}/\rho m) \) can be about 0.5 times that of the equivalent isotropic plate.

6.2. Influence of Parameter \( \alpha \)

The influence of parameter \( \alpha \) is now explored. Variations of phase speeds with \( \beta \) are plotted for various values of \( \alpha \). Fig. 4 gives these results for the longitudinal dominant wave and Fig. 5 gives these results for the shear dominant wave. As \( \alpha \) increases, the phase speeds of longitudinal waves decrease while the phase speeds of shear dominant waves increase. Nevertheless, the influence of \( \alpha \) is negligible, meaning that the phase speeds do not significantly change with \( \alpha \).

6.3. Use of Simplified Expression

The above results are obtained using the simplified expressions for the phase speeds. The characteristics are now ob-

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**Figure 2.** Variation of phase speed with \( \beta \) for \( \alpha = 0.3 \).

**Figure 3.** Variation of phase speed with \( \beta \) for \( \alpha = 0.03 \).

**Figure 4.** Phase speeds of longitudinal dominant waves for various values of \( \alpha \).
6.4. Summary of the Results

The phase speeds of composite panels can be much lower than those computed using the expression for the isotropic plate, thus justifying the need for using the expressions derived in this work for determining the phase speeds of in-plane waves in composite plates. The impact is very significant for the shear dominant waves than the longitudinal dominant waves. Influence of the parameter $\beta$ is quite significant though the influence of the parameter $\alpha$ is relatively negligible.

7. EXPERIMENTAL RESULTS

Experiments are conducted on a typical composite panel to validate the expressions derived here for the phase speeds of the in-plane waves. The panel is excited using a modal hammer. Accelerations are measured at salient points and at the point of impact. The time lag of the acceleration response from the acceleration at the excitation point is used to determine the phase speed. Experimentally obtained phase speeds are compared with the phase speeds estimated by the expressions derived in this work.

7.1. Details of the Panel

A honeycomb sandwich panel with composite face sheet was used for the experiment. The panel had dimensions $1440 \text{ mm} \times 480 \text{ mm} \times 15 \text{ mm}$. The face sheet had two layers of CFRP (+45/-45). The CFRP material was M18 / BD43090. The elastic modulus of the material (each layer) was $1.47 \text{ E11 N/m}^2$ and the shear modulus was $4.09 \text{ E9 N/m}^2$ with a Poisson’s ratio of 0.03. The honeycomb core had a thickness of 15 mm. The mass per unit area of the panel was $1.21 \text{ kg/m}^2$. The panel has $A_{11} = 4.71 \times 10^7 \text{ N/m}$, $A_{22} = 4.71 \times 10^7 \text{ N/m}$, $A_{12} = 1.41 \times 10^6 \text{ N/m}$, $A_{66} = 1.28 \times 10^6 \text{ N/m}$, $\alpha = 0.03$, $\beta = 0.027$. The above values were determined with respect to the principal material directions.

7.2. Test Setup

The panel was kept on a foam and isolated from the ground as shown in Fig. 8. The panel was impacted at the center of an edge with a modal hammer of Kistler make (Model no: - Kistler 9722A500). Using the Impact hammer, an impact was generated at location 1 along $Y$ direction. A Teflon tip (Serial no 9904A), with additional dead weight, was used for impact. Acceleration responses were measured at three locations as shown in Fig. 9. Response was measured at location 1. Portable data acquisition system of the LMS was used for the data acquisition. All the channels were connected in IEPE mode directly to the data acquisition systems.

7.3. Data Acquisition Parameters

To obtain the wave speed properly, the data acquisition parameters had to be selected properly. For a speed of 10000 m/s, the in-plane wave was expected to reach the other edge in $45 \mu s$. A resolution of about $10 \mu s$, wherein there will be 4 lines in the duration of the arrival time, was expected to give good results. Therefore, the data was acquired with a sampling rate of 102.4 kHz. This resulted in a bandwidth of 51.2 kHz. The number of spectral lines selected was 64. This resulted in a frequency resolution of 800 Hz. The above settings provided the acquisition for a duration of 0.00125 s = 1/(800). The data had a high resolution of 0.00001 s (10 $\mu$s). Several trials were carried out in arriving at these parameters. Use of 20.48 kHz as a bandwidth with a resolution of 24.4 $\mu$s was not sufficient to capture the arrival of the wave. Usage of higher sampling rate, e.g. a bandwidth of 102.4 kHz caused several issues with the measured acceleration response like resonance of the accelerometers etc.
7.4. Test Results

The acceleration responses measured at the impact location (1Y) and at location 2Y are shown in Fig. 10 and impact location (1Y) and at location 4Y are shown in Fig. 11.

The wave speeds are determined from the delay in the disturbance reaching the edge from the initiation of the disturbance. The time or arrival of the in-plane wave at location 2 gives a speed of 5861 m/s. Based on the position of location 2 this wave is expected to be a longitudinal dominant wave. The time or arrival of the in-plane wave at location 4 gives a speed of 3806 m/s. Based on the position of location 4 this wave is expected to be an in-plane shear dominant wave. Thus, the experimental results reveal existence of two in-plane waves and show that these in-plane waves are coupled.

7.5. Comparison with Theoretical Estimation

The phase speeds of the in-plane waves in the above panel are theoretically estimated using the expressions derived in this work (Eq. (30)).

The speeds thus determined are 5772 m/s (against the experimentally obtained speed of 5861 m/s) and 2561 m/s (against the experimentally obtained speed of 3806 m/s). The phase speed of the longitudinal dominant wave is obtained by using the expression derived here. In this case, the phase speed of the longitudinal dominant wave is in good agreement with the experimentally determined phase speed. However, the experimentally determined phase speed of the in-plane shear dominant wave is quite higher than the estimated phase speed. This is investigated further.
It is important to analyse the way in which the waves are generated in the experimental set up. It can be seen that the impact is given at location 1 along direction $Y$. This impact set a longitudinal dominant wave in both face sheets as there is a metallic embedment, called an insert, at the impact point connecting both face sheets. Since the longitudinal dominant wave is coupled to in-plane shear motion, the in-plane shear waves are now generated from the progressing longitudinal dominant waves. In this case, the longitudinal dominant waves are already set in the face sheets. Since the core is very flexible, the in-plane shear waves that are generated in the face sheets travel independently. Considering the above logic, the in-plane shear waves are expected to be set independently in both the face sheets. Therefore, considering the properties of one face sheet alone (the axial stiffness becomes half) and the core does not move along with the face sheets (mass per unit area is 0.3656 kg/m$^2$) we determined that the phase speed of the in-plane shear is 3296 m/s (against the experimentally obtained speed of 3806 m/s). The phase speed of the in-plane shear is about 0.87 of the experimentally obtained phase speed.

Thus, it can be seen that the phase speeds estimated using the expression derived here are in good agreement with the experimentally obtained phase speeds.

It is interesting to compare the experimentally obtained results with those computed using the existing expressions. As discussed earlier (section 6.1), the speeds computed using the existing expression can cause large errors in the speed of the shear dominant in-plane wave. The speed of this wave, which is determined by the existing expression, is 1323 m/s (against the experimentally obtained speed of 3806 m/s). There is a significant difference between the experimentally determined speed and those determined using the existing expressions. The speed of this wave, which is computed using the expression derived here, is 3296 m/s. The experimental results signify the coupled motion of both the in-plane waves and also the need for the expression derived in this work.
8. SUMMARY AND CONCLUSIONS

Expressions for the phase speed and group speed of in-plane waves in a composite laminated plate are derived. Two in-plane waves exist in a plate and they are non-dispersive. In an isotropic plate they are uncoupled and form independent in-plane longitudinal and in-plane shear waves. In a composite plate these waves are coupled. The phase speeds in a typical composite panel are determined experimentally. The experimental results show the existence of the two in-plane waves with different phase speeds. The phase speeds determined using the expressions derived here match very well with the experimental results.

The phase speeds of in-plane waves in composite plates can be much lower than those computed using the expression for isotropic plates in which the two waves are uncoupled. The additional parameters that govern the phase speed are the in-plane shear stiffness and Poisson’s ratio related parameters. The influence of the in-plane shear stiffness parameter on the phase speed is quite significant, but the influence of Poisson’s ratio related parameters is relatively less. The impact is considerable in the case of the shear dominant waves.

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Determination of Roller Bearing Inner Race Defect Width Based on Vibration Signal

Mohamed El Morsy
Helwan University – Cairo - Egypt.

Gabriela Achtenová
Czech Technical University in Prague.

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The present article’s intent is to measure and identify the roller bearing inner race defect width and its corresponding characteristic frequency based on filtered time-domain vibration signal. In case localized fault occurs in a bearing, the rolling elements encounter some slippage as the rolling elements enter and leave the bearing load zone. As a consequence, the incidence of the impacts never reproduce exactly at the same position from one cycle to another. Moreover, when the position of the defect is moving with respect to the load distribution zone of the bearing, the series of impulses are modulated in amplitude in time-domain and the conforming Bearing Characteristic Frequencies (BCFs) arise in frequency domain. In order to verify the ability of time-domain in measuring the fault of rolling bearing, an artificial fault is introduced in the vehicle gearbox bearing: an orthogonal placed groove on the inner race with the initial width of 0.6mm approximately. The faulted bearing is a roller bearing quantification of the characteristic features relevant to the inner race bearing defect. It is located on the gearbox input shaft—on the clutch side. To jettison the frequency associated with interferential vibrations, the vibration signal is filtered with a band-pass filter based on an optimal daughter Morlet wavelet function whose parameters are optimized based on maximum Kurtosis (Kurt.). The residual signal is performed for the measurement of defect width. The proposed technique is used to analyse the experimental signal of vehicle gearbox rolling bearing. The experimental test stand is equipped with two dynamometer machines; the input dynamometer serves as an internal combustion engine, the output dynamometer introduces the load on the flange of the output joint shaft. The Kurtosis and Pulse Indicator (PI) are selected as the evaluation parameters of the de-noising effect. The results show the reliability of the proposed approach for identification and quantification of the characteristic features relevant to the inner race bearing defect.

1. INTRODUCTION

The rolling bearing is one of the most important components in rotary machines. Although a significant amount of work has been done on bearing condition monitoring, the estimation and identification of defect size in bearing elements is still a challenge. Usage vibration measured signal is quite extensive in the field of condition monitoring and diagnostics of rotating machines. The presence of a variety of noise and a wide spectrum of bearing defect signals possess difficulty in the recognition of bearing condition based on the time-domain of vibration signal. These necessities need a modern and reliable approach for identification and quantification of the characteristic features relevant to the bearing conditions.1-3

Utilizing the advantage of both the acoustic and vibration sensing towards a low-cost solution to complex problems, a signal processing scheme has been proposed.4 A new approach for bearing fault diagnosis has been proposed based on probabilistic principal component analysis and cyclic bi-spectrum with optimal cycle frequency; the effectiveness of this method has been demonstrated by numerical simulation and experimental investigation of a rolling bearing with an outer race fault.5 The scheme has implemented for identification and measurement of inner race defect of a cylindrical roller bearing. The defect identification has been carried out by envelope spectrum of intrinsic mode functions (IMFs) generated by Ensemble Empirical Mode Decomposition (EEMD) of acoustic signal. After identification of the defect, evaluation of its

NOMENCLATURE

| BCFs | Bearing Characteristic Frequencies |
| Kurt. | Kurtosis |
| PI | Pulse Indicator |
| CWT | Continuous Wavelet Transform |
| TMI | Time Marginal Integration |
| IMFs | Intrinsic mode functions |
| EEMD | Ensemble Empirical Mode Decomposition |
| RMS | Root mean square |
| WT | Wavelet transform |
| STFT | Short Time Fourier Transform |
| b | Time translation parameter |
| a | Scale parameter |
| BPFI | Ball pass frequency for inner race |
| BPFO | Ball pass frequency for outer race |
| BSF | Ball spin frequency |
| \( L_{IR} \) | Inner race defect width |
| \( \Delta t \) | Burst duration |
| \( D_I \) | Inner race diameter |
| \( F_s \) | Shaft speed |
| \( f_c \) | Fundamental train frequency |
| \( f \) | Relative rotating frequency between inner and outer races |
| \( \alpha \) | Contact angle between the race and the roller |
| \( z \) | Number of rollers |
| \( d \) | Roller diameter |
| \( D \) | Bearing’s pitch diameter |
size has been made by applying Continuous Wavelet Transform (CWT) using adaptive wavelet to the vibration signal to produce a 2D map of CWT coefficient. CWT operation acts as band pass filtering and produces high coefficient at a scale analogous to frequency of burst. Finally, the Time Marginal Integration (TMI) of CWT coefficient has been carried out for the measurement/evaluation of defect width. The EEMD has been employed to decompose the nonlinear and non-stationary acoustic signal due to race defect into a number of IMFs. To enhance the feature of defect in the vibration signal, CWT has been carried out using adaptive wavelet.6–8 The TMI of CWT coefficient has been carried out for the evaluation of defect width. The TMI is analogous to the instantaneous power of the signal and reveals how the power of the signal changes with time. The TMI of CWT coefficient over scale and time location has been expressed.9

Bearing defects are representing a high percentage of all induction machine faults, and their detections have been desired to avoid damages of industrial processes and reduce maintenance cost.10 Though rotor defects appear less significant than bearing defects in terms of computable manner, most of the bearing failures have been caused by shaft misalignment, rotor eccentricity, and other rotor related defects.10,11 These defects produce some symptoms such as unbalanced air-gap voltages and line currents, increased torque pulsations, decreased average torque, amplified losses in efficiency, and excessive heating.12

In vibration condition monitoring and fault diagnostic, the use of time and frequency domain features are the common techniques. The monitoring process has been improved continuously over the last years because new signal pre-processing techniques have been introduced.13–15 Due to the high number of possible features and different ways of feature generation, a kind of framework for the feature selection process is required.

Vibration-based signal analysis in the time and frequency domain has been a major technique for bearing fault diagnosis. Several statistical parameters in the time domain and the frequency domain, such as (root mean square) RMS, Kurt., and Skewness, have been shown to be capable of machine fault detection.16,17 In,18 nine features in the time domain and seven features in the frequency domain were used for bearing fault detection.18 An adaptive wavelet filter was applied, based on the Morlet wavelet function, on the rotational vibration data measured from a single-stage gearbox with artificially induced cracks in the pinion.19 The wavelet transform (WT) was capable of decomposing a signal into different frequencies with different resolutions, (i.e., it provides a time-scale (frequency) representation of the measured signal).20

The present research work is focused on identification and quantification of the characteristic features related to the inner race bearing defect based on filtered time-domain vibration signal. An optimal Morlet wavelet function is applied to the raw vibration signal as a band-pass filter, whose factors are optimized based on maximum Kurt.

2. DESCRIPTION OF THE TEST SET-UP

2.1. Experimental Set-Up

The experimental works were done on an open loop test rig consisting of two three-phase dynamometric machines. The gearbox to be tested, used as the internal combustion engine, was used as a dynamometer KS 56–4 and torque flanges for torque measurement. On the input, the first one on the output side was an eddy current water-cooled dynamometer 2VD 110 / 6 as a brake as shown in Figure 1. The test-rig has been described in detail.21 The power flowed through one output shaft only.

2.2. Investigated Gearbox

The tested gearbox used for our experimental work was the model most frequently used in modern small to mid-sized passenger vehicles with transversely mounted powertrain and front wheel drive. The internal arrangement of the tested gearbox and the faulty bearing are illustrated in Figure 2.

Figure 1. Diagram of the open loop test bed used for investigation of fault in an automotive gearbox.

Figure 2. Diagram of the investigated five-speed automotive gearbox, the highlighted bearing is the faulty element.
3. SIGNAL PROCESSING

3.1. Time-frequency Analysis

The time-frequency analyses used time-frequency distributions, which represented the energy of waveform signals in two-dimensional functions of both time and frequency, to better reveal fault patterns for more accurate diagnostics. The Short Time Fourier Transform (STFT) or spectrogram (the power of STFT) and Wigner-Ville distribution were the most popular time-frequency distributions.\textsuperscript{22, 23} Another transform approach for time-frequency analysis was the wavelet transform.

3.2. Morlet Wavelet Filter

The mechanical waveform signals were managed to obtain feature vectors. The CWT was applied to obtain the wavelet coefficients of the measured signals. The statistical parameters of the wavelet transform coefficients were extracted, which constitute the feature vectors.

The Morlet wavelet (Mother Morlet function) is one of the most popular non-orthogonal wavelets and it is defined as

\[
\psi(t) = \exp\left(\frac{\beta^2 \cdot t^2}{2}\right) \cos(\pi t). \tag{1}
\]

However, in practice, it was not easy to provide a proper threshold for wavelet de-noising. For this case, an adaptive (daughter) wavelet function was applied instead of traditional (Mother) wavelet function.\textsuperscript{3}

3.2.1. Adaptive Morlet Wavelet Filter

A daughter Morlet wavelet was obtained by time translation and scale dilation from the mother wavelet the following formula:

\[
\psi_{a,b}(t) = \psi \left(\frac{t-b}{a}\right) = \exp\left[\frac{-\beta^2(T-B)^2}{2a^2}\right] \cos\left[\frac{\pi(t-b)}{a}\right]. \tag{2}
\]

To recognize the immersed impulses by de-noising, the location and the shape of the frequency band conforming to the impulses was determined first. Scale ($\alpha$) and parameter ($\beta$) controlled the location and the shape of the daughter Morlet wavelet function respectively. Several investigators have reported on how to select a mother wavelet that adapts the best to the signal to be isolated. The method of ($\beta$) selection in Morlet wavelet was based on maximum Kurt.\textsuperscript{6, 8} Details on how to select ($\beta$) and ($\alpha$) in a Morlet wavelet based on Kurt. to make the mother wavelet match the signal to be isolated have been provided.

3.3. Bearing Characteristic Frequencies (BCFs)

For a bearing with a stationary outer race, the above defect characteristic frequencies were obtained as follows:\textsuperscript{24}

characteristic frequency of the outer-race

\[
f_o(inHz) = \frac{zf}{2} \left(1 - \frac{d}{D} \cos \alpha\right); \tag{3}
\]

characteristic frequency of the inner-race

\[
f_i(inHz) = \frac{zf}{2} \left(1 + \frac{d}{D} \cos \alpha\right). \tag{4}
\]

Roller (BPFR) or the Ball Spin Frequency (BSF)

\[
f_r(inHz) = \frac{D}{d} f \left[1 - \left(\frac{d}{D} \cos \alpha\right)^2\right]; \tag{5}
\]

and characteristic frequency of the cage

\[
f_c(inHz) = \frac{f}{2} \left[1 - \frac{d}{D} \cos \alpha\right]. \tag{6}
\]

This equation was based on the assumption of a pure rolling motion (no slipping) for the rotating elements. However, in reality some sliding motion may occur, which caused slight deviation of the bearing characteristic frequencies (BCF) locations. Therefore, this equation should be regarded as approximations only. Table 1 illustrates the fault BCF at different input shaft speeds.

3.4. Fault Signature Theories

Vibration signature handling from a bearing with localized fault may be mainly performed for two objectives: fault diagnosis and size quantifying. Fault diagnosis focuses on detecting the cyclo-stationary characteristics associated with the passage of the rolling element over the considered fault. The fault location could be identified by matching the measured impact frequency, using the well-established envelope analysis technique with one of fault characteristic frequencies: the ball pass frequency for inner race (BPFI) for a fault, the ball pass frequency for outer race (BPFO) for a fault, or the BSF for a fault on the rolling element.\textsuperscript{22}

For fault size quantification, the time domain signal features associated with the passage of the roller elements over the spalled region should be further analyzed. Several studies have been conducted to describe vibrational features that are correlated with fault size.\textsuperscript{23–25} These studies agree on the

| Table 1. Fault bearing characteristic frequencies (BCF) at different input shaft speeds. |
|----------------------------------|------------------|------------------|------------------|
| (BCFs, in Hz)                   | 2500 rpm         | 2000 rpm         | 1500 rpm         |
| $f_o$                           | 347.85           | 278.28           | 208.71           |
| $f_i$                           | 443.81           | 355.05           | 266.29           |
| $f_r$                           | 338.70           | 270.96           | 203.22           |
| $f_c$                           | 18.31            | 14.65            | 10.98            |
fact that two main parts of the vibration response should be distinguished: the entry and the leaving. The fault size could be measured by the separation (in samples) between the roller element entry point into the fault zone and the impact point.

The earlier theory by Dowling explains the spacing between two high-frequency impulses as a symptom to the entry and exit points as shown in Fig. 4. Dowling’s interpretations could not be observed for other data. An in-depth study by Epps and McCallion shows different behaviors for the entry and the exit. The entry into the fault part comprised gradual destress mainly dominated by low-frequency content. The exit part started when the rolling element instantaneously departed from the fault zone and caused a sudden change in the direction and high-frequency impulse as shown in the right plot in Fig. 4.

The low-frequency content of the weak entry event made it very hard to identify from background noise. The envelope analysis, which is the most common vibration-based diagnosis technique, detected the impulse response resulting from the high-frequency impact events and thus the possibility of identifying the actual fault size or severity level diminished.

Recently, two methods have been discussed to enhance entry partly through intensive signal processing schemes, namely joint treatment and separate treatment. The joint treatment utilized pre-whitening and wavelet analysis to highlight both entry and exit events. On the other hand, separate treatment used a threshold to isolate respectively low-frequency and high-frequency events into two separate envelops with statistical evaluation.

The inner race defect width ($L_{IR}$) can be calculated by making use of burst duration ($\Delta t$) determined from TMI graph, inner race diameter ($D_I$), shaft speed ($F_s$) and fundamental train frequency ($f_c$). The mathematical expression of inner race defect width is expressed as

$$L_{IR} = \pi \times \Delta t \times D_I \times (F_s - f_c).$$

### 3.5. Vibration Measurements

One non-destructive technique, vibration acceleration generation, was employed to record the gearbox during operation. A Bruel & Kjaer portable and front-end type 3050-B-040 4 channel input Module 50 kHz analyser was used. The speed was measured using a tachometer type MM360, a one axial (Delta Tron type 4507 B 004 with measuring range $+/-700$ ms-2) accelerometer was used for recording vibration acceleration signals, both mounted upon the gearbox case as shown in Fig. 1. The vibration signal in vertical and radial terms is presented in this article. The selected sampling frequency was 50 kHz, and signals of 0.128 sec duration were recorded. Recordings were carried out at different operation conditions as shown in Table 2.

### 4. RESULTS AND DISCUSSION

Table 2 illustrates the vibration signals for each operation in gearbox for bearing fault diagnosis. The vibration signals condition with 0.128 sec duration, which were measured from vehicle were recorded with sampling frequency 50 kHz to cover maximum bearing harmonics. The length of the segments was chosen carefully to contain enough information to capture localized features of the signal such that the computation time was a minimum.

Figures 5a and 5b show the raw waveform measured signals and its frequency domain of faulty bearing at input speed 2000 rpm and input load 50 Nm. In Fig. 5a the waveform signal displays high peaks repeated approximately at certain times that represent the bearing impacts, which was repeated approximately every 0.03 sec (rotating frequency 33.3 Hz). Also, around each bearing impact there were few peaks representing the periodic of bearing assembly resonance, in case of real vibration measurement, the noise to signal ratio was significant as clarified in Fig. 5a. The CWT was applied as filter to reduce the noise ratio in the measurement signal. The Morlet function was selected among the CWT functions to achieve this task. To
Figure 6. Optimization of a Morlet function parameter based on maximum Kurtosis.

Figure 7. Continuous wavelet transforms using optimal Morlet function based on max. Kurtosis.

increase from the ability of a Morlet filter and to identify the immersed impulses by de-noising, the location and the shape of the frequency band corresponding to the impulses must be strong-minded first. The adaptive Morlet wavelet function parameters were optimized based on maximum Kurt. as shown in Fig. 6. The optimal values of the measured signal at input shaft speed 2000 rpm and input shaft load 50 Nm are \( a = 2 \) and Beta \( \beta = 0.1 \) while at this point Kurt. values reached its maximum.

Figure 7 shows the effect of the Morlet function on the prior raw measured signal (as shown in Fig. 7a by using the optimal value \( \beta = 0.1 \). Figure 7b illustrates the wavelet map with significant distortions at constant sampling times corresponding to the rotating frequency of the input shaft. The residual signal was extracted by a cross section in the wavelet map at optimal scale value \( a = 2 \), the residual signal was displayed as function of sampling time. To be simpler, the residual signal redraws as function of time as shown in Fig. 8a, reading the repeated peaks values around the bearing impacts. Bearing assembly resonance (inner race frequency = 353.36 Hz) was extracted as shown in zooms around the bearing assembly resonance and entry/exit vibrational events at bearing spall edges in Figs. 8b and 8c.

Referring to the faulty bearing characteristics and details of Fig. 8c, the \( D_I \) of the specified bearing was 29 mm. The \( \Delta t \) was calculated by averaging taken from 8 bursts was 0.00035 sec. At shaft speed of 33.33 Hz and \( f_c \) of 14.65 Hz, the defect width evaluated using the proposed scheme was 0.6 ± 0.05 mm according to Eq. (7).

In this section, to verify the accuracy and the effectiveness of the proposed method, the input shaft speeds were changed up and down (2500 rpm and 1500 rpm) around the tested speed (2000 rpm) at the same constant load 50Nm as illustrated in Table 3.

Figures 9 and 10 show the time and frequency response of measured signals at input shaft speed 2500 rpm, 1500 rpm, and constant input shaft load 50 Nm respectively. The bearing impacts were repeated about every 0.0024 sec (rotating speed 41.67 Hz) and 0.04 sec (rotating speed 25 Hz) at each speed (2500 rpm and 1500 rpm) as shown in Figs.fig9a and 10a respectively. Furthermore, bearing harmonics were displayed in frequency response at high frequency range as shown in Figs. 9b and 10b.
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(a) Time domain response of faulty bearing.

(b) Frequency domain response of faulty bearing.

Figure 9. Vibration measurement signals of faulty bearing (inner race) at input shaft speed 2500 rpm, input shaft load 50 Nm.

(a) Raw measured signal divided into number of samples = 16383.

(b) Wavelet map at Scale = 1:20 and Beta ($\beta$) = 0.6.

(c) Residual signal at optimized values $a = 2$ and Beta ($\beta$) = 0.6.

Figure 11. Continuous wavelet transforms using optimal Morlet function based on max. Kurtosis at input shaft speeds at 1500 rpm.

In the following figures, the time response of measured raw signals at input shaft speed 2500 rpm and 1500 rpm by using the optimal Morlet function with optimized values, ($a$) = 2, ($\beta$) = 0.6 and ($a$) = 2, ($\beta$) = 0.7 respectively, were filtered to reduce the noise effect as shown in Fig. 13.

Figures 11 and 12 show the effect of the Morlet function on the prior raw measured signal (as shown in Figs. 9a and 10a) by using the optimal value ($\beta$) = 0.6 and 0.7 respectively. Figures 11b and 12b illustrate the wavelet map with significant distortions at constant sampling time corresponding to the rotating frequency of input shaft. The residual signal was extracted by a cross section in the wavelet map at optimal scale value ($a$) = 2.

The residual signals redrew as function of time as shown in Figs. 13a and 14a and Figs 13b and 14b, by reading the repeated peaks around the bearing impacts, bearing assembly resonances (inner race frequencies in each case is 442.5 Hz and 265.9 Hz) were found as shown in Figs. 13b and 14b.

Figures 13c and 14c show enlarged around entry/exit vibrational events at bearing spall edges at 2500 rpm and 1500 rpm and constant load 50 Nm respectively. By applying Eq. (7) for both tested speeds (2500 and 1500 rpm respectively), the inner race diameter ($D_I$) of the specified bearing was 29 mm. The burst duration ($\Delta t$) was calculated by averages taken from 8 bursts was 0.00024 sec and 0.00046 sec. At shaft speed of 41.67 Hz and 25 Hz and $f_c$ of 14.65 and 10.98 Hz, the defect width evaluated using the proposed scheme was 0.6 ± 0.055 mm.

The results in the present work show the reliability of the approach for identification and quantification of the characteristic features relevant to the inner race bearing defect. A conclusion can be drawn that time domain vibration signal can be used uniquely as a bearing fault diagnosis tool independent on all spectral forms such as traditional frequency domain, power spectrum, and envelope analysis. Table 3 illustrates the sum-
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(a) Raw measured signal divided into number of samples = 16383.

(b) Wavelet map at Scale = 1:20 and Beta ($\beta$) = 0.7.

(c) Residual signal at optimized values $a = 2$ and Beta ($\beta$) = 0.7.

Figure 12. Continuous wavelet transforms using optimal Morlet function based on max. Kurtosis at input shaft speeds at 1500 rpm.

Table 3. The calculated and extracted inner race of faulty bearing at different shaft rotational speed.

<table>
<thead>
<tr>
<th>Input shaft speed (rpm)</th>
<th>Calculated (Hz)</th>
<th>Period extracted (s)</th>
<th>Period extracted (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>443.81</td>
<td>0.00226</td>
<td>442.50</td>
</tr>
<tr>
<td>2000</td>
<td>355.05</td>
<td>0.00283</td>
<td>353.36</td>
</tr>
<tr>
<td>1500</td>
<td>266.29</td>
<td>0.00376</td>
<td>265.96</td>
</tr>
</tbody>
</table>

Conclusions

This paper presents an optimal Morlet wavelet filter, whose parameters are optimized based on maximum Kurtosis, for the determination of roller bearing inner race defect based on time-domain vibration signal. The optimal Morlet wavelet filter was used to eliminate the interfering in the measured vibration signal resulting from other sources and extract adequately the impulsive feature of a defective element. The proposed method can be conducted in an almost automatic way, with a minimum degree of user intervention and reliable at different operation conditions. Kurt. and PI are introduced to evaluate the improvement in the raw signal due to the de-noising effect.

A simple method to process the fused data of vibration signal has been proposed to resolve complicated issues, such as identification of inner race defect as well as estimation of defect size.

The optimal Morlet function is used as a filter with a bandpass filter to decrease the residual in-band noise and highlight the periodic impulsive feature.
6. ACKNOWLEDGEMENTS

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REFERENCES


Numerous dynamic models of spur gears, helical gears, bevel gears, and face gears can be found in various studies. However, studies that focus on the dynamic model of a double helical gear pair are quite limited. The author proposed a model of a double helical gear pair by only considering the axial vibration. The author did not consider the friction and multiple backlashes in the proposed model. The friction force of the tooth surface and backlash are important factors that can cause complex non-linear phenomena in gear pairs. Therefore, a dynamic model of a double helical gear pair that takes into consideration the axial vibration, friction and multiple backlashes is proposed. Firstly, based on the tooth contact analysis (TCA) of a double helical gear pair, the path of contact and meshing time from engagement to disengagement are obtained. The formula for determining the sliding friction coefficient is introduced. Based on TCA and the dynamic meshing force provided by the subsequent dynamics model of double helical gear pair, the sliding friction coefficient of the tooth surface is calculated. Secondly, the stiffness excitation, gear-into impact excitation and error excitation (including the axial displacement caused by the errors of manufacture and installation under low speed) are calculated according to the existing research results. Following this, a dynamic model of a double helical gear pair that takes into consideration the axial vibration, friction and multiple backlashes is both built and solved. Finally, an example is presented to verify the corresponding results.

**NOMENCLATURE**

\[ j = 1, 2 \]  
\[ i = p, q \]  
\[ n = x, y, z \]  
\[ \zeta = h, p, q \]  
\[ l = 1, 2, \ldots, 12 \]  
\[ y_{ij} \] translational vibration displacement of two ends helical gear center point along the y-direction (Fig. 4)  
\[ z_{ij} \] translational vibration displacement of two ends helical gear center point along the z-direction (Fig. 4)  
\[ x_{ij} \] translational vibration displacement of two ends helical gear center point along the x-direction (Fig. 4)  
\[ \theta_{ij} \] angular vibration displacement of two ends helical gear center point (Fig. 4)  
\[ m \] equivalent mass of double helical gear pair  
\[ b_l \] half backlash  
\[ k_j \] torsional stiffness of two ends gear pair  
\[ c_j \] torsional damping of two ends gear pair  
\[ k_{p1x}, k_{p1y}, k_{q1x}, k_{q1y}, c_{p1x}, c_{p1y}, c_{q1x}, c_{q1y}, k_{p2x}, k_{p2y}, k_{q2x}, k_{q2y}, k_{q2z} \] Support stiffness  
\[ k_{pq}, k_{q2z} \] Support damping  
\[ c_{pq} \] meshing damping along the tangential direction  
\[ k_{pq} \] meshing stiffness along the tangential direction  
\[ c_{pj} \] meshing damping along the axial direction  
\[ k_{pj} \] meshing stiffness along the axial direction  
\[ c_{pj} \] meshing error along the tangential direction  
\[ k_{pj} \] meshing error along the axial direction  
\[ k_{px} \] torsional stiffness along X-direction of pinion shaft (Fig. 4)  
\[ c_{px} \] torsional damping along X-direction of pinion shaft  
\[ k_{qy} \] torsional stiffness along the Y-direction of pinion shaft (Fig. 4)  
\[ c_{qy} \] torsional damping along the Y-direction of pinion shaft  
\[ k_{py} \] torsional stiffness along the Y-direction of gear shaft  
\[ c_{py} \] torsional damping along the Y-direction of gear shaft  
\[ k_{qz} \] tensile (compression) stiffness along the Z-direction of pinion shaft (Fig. 4)
Numerous analytical models on gear dynamics, including spur, helical, bevel, face gears etc., can be found in various studies. However, studies that focus on a double helical gear pair are quite limited. Jauregui proposed a single-degree dynamic model. Sondkar built a dynamic model of a double helical planetary gear set. Kahraman proposed a linear, time-invariant dynamic model of double helical gear pair systems including shafts and bearing supports. In the above models, the axial vibration caused by axial displacement under low speed and tooth surface friction is not considered. Although Wang proposed a dynamic model of a double helical gear pair in consideration of axial vibration, the friction force and multiple backlashes are not considered.

The friction force of the tooth surface, backlash and time-varying meshing stiffness are important factors that cause complex non-linear phenomena in gear pairs. For double helical gear pairs, the axial displacement under low speeds should also be considered. Combining TCA and loaded tooth contact analysis (LTCA), the axial displacement caused by the errors of manufacture and installation under low speeds can be obtained, and then the error excitation, stiffness excitation and impact excitation can be calculated. Therefore, the specific objectives of this study are as follows:

1. Based on TCA of double helical gear pair, the path of contact and meshing time from engagement to disengagement are obtained. The formula for determining the sliding friction coefficient is introduced. Based on TCA and the dynamic meshing force provided by the subsequent dynamics model of a double helical gear pair, the sliding friction coefficient on tooth surface is calculated out.

2. The dynamic model of a double helical gear pair that takes the axial vibration, friction and multiple backlashes into consideration is built up and solved. Finally, an example is demonstrated.

A flow diagram for the dynamic model of a double helical gear pair is shown in Fig. 1. It should be pointed out that the related research about stiffness excitation, gear-into impact excitation and error excitation can be found in references. The TCA and LTCA of a double helical gear pair can be found in references. The related research of TCA and LTCA can be also found in references.
2.3. The Calculations of Distance Between Friction Force and Gear Center

According to Fig. 3, the calculation of distance between friction force and gear center was represented as

\[
\begin{align*}
\delta_1 &= \sqrt{a^2 - (r_{pb} + r q)^2} - \sqrt{r_{pa}^2 - r_{qpb}^2 + r_{pb} w_p k} - \lambda \\
\delta_2 &= \sqrt{a^2 - (r_{pb} + r q)^2} - \sqrt{r_{pa}^2 - r_{qpb}^2 + r_{pb} w_p k} - \lambda \\
\delta_3 &= \sqrt{a^2 - (r_{pb} + r q)^2} - \lambda \\
\delta_4 &= y_p + \theta_q r_q - y_p - \theta_p r_p - c_y
\end{align*}
\] (3)

Where, \( \lambda \) was the relative displacement between meshing points of two teeth surfaces along the meshing line. \( y_p, y_q \) were the translational vibration displacements of two ends of helical gear center points along the meshing line. \( \theta_q, \theta_p \) were the angular vibration displacements of two ends of helical gear center points, \( c_y \) was the meshing error along the meshing line. In Fig. 3, \( B \) was the width of the teeth, \( \beta \) was the helical angle.
3.2. Descriptions of the Backlash

The existence of backlash generated impact between engaged teeth, which influenced the stability of gear transmission. The clearance function \( f \) was expressed as

\[
f_{c,jn}(y) = \begin{cases} 
  y - b_l & y > b_l \\
  0 & y \leq |b_l| \\
  y + b_l & y < -b_l 
\end{cases} 
\]  

(5)

For the left gear pair, taking the interaction between the left and right end gear pair into consideration, the dynamic meshing force along the tangential direction and the axial direction were respectively represented as

\[
F_{p1y} = k_{p1}f_{h1y}(\bar{y}_{p1} - \bar{y}_{q1} - c_{y1}) + c_{y1}(\bar{y}_{p1} - \bar{y}_{q1} - \bar{x}_{y1}) = \cos \beta [k_1f_{h1y}(\bar{y}_{p1} - \bar{y}_{q1} - c_{y1}) + c_1(\bar{y}_{p1} - \bar{y}_{q1} - \bar{x}_{y1})]; 
\]

(6)

\[
F_{z1} = k_{z1}(\bar{z}_{p1} - \bar{z}_{q1} - c_{z1}) + c_{z1}(\bar{z}_{p1} - \bar{z}_{q1} - \bar{x}_{z1}) = \sin \beta [k_1(\bar{z}_{p1} - \bar{z}_{q1} - c_{z1}) + c_1(\bar{z}_{p1} - \bar{z}_{q1} - \bar{x}_{z1})] - k_{z1}\bar{z}_e - c_{z1}\bar{z}_e 
\]

(7)

In this case, the relation between the vibration displacement of center point \( q_1, p_1, q_2, \) and \( p_2 \) and generalized displacement of driving gear and driven gear were represented as \( \bar{y}_{p1}, \bar{y}_{q1}, \bar{z}_{p1}, \) and \( \bar{z}_{q1}. \)

For the right gear pair, considering the interaction between the left and right end gear pair, the dynamic meshing force along the tangential direction and the axial direction were respectively represented as

\[
F_{p2y} = k_{p2}f_{h2y}(\bar{y}_{p2} - \bar{y}_{q2} - c_{y2}) + c_{y2}(\bar{y}_{p2} - \bar{y}_{q2} - \bar{x}_{y2}) = \cos \beta [k_2f_{h2y}(\bar{y}_{p2} - \bar{y}_{q2} - c_{y2}) + c_2(\bar{y}_{p2} - \bar{y}_{q2} - \bar{x}_{y2})]; 
\]

(8)

\[
F_{z2} = k_{z2}(\bar{z}_{p2} - \bar{z}_{q2} - c_{z2}) + c_{z2}(\bar{z}_{p2} - \bar{z}_{q2} - \bar{x}_{z2}) = \sin \beta [k_2(\bar{z}_{p2} - \bar{z}_{q2} - c_{z2}) + c_2(\bar{z}_{p2} - \bar{z}_{q2} - \bar{x}_{z2})] + k_{z2}\bar{z}_e + c_{z2}\bar{z}_e 
\]

(9)

The formula (5)–formula (9) came from reference. According to Fig. 4, the dynamic equations of this system were established by the Newton second law.

\[
m_p\ddot{x}_{p1} + c_{p1x}\dot{x}_{p1} + k_{p1x}f_{p1x}(x_{p1}) + c_{px}(x_{p1} - \dot{x}_{p1}) + k_{px}(x_{p1} - x_{p2}) = \chi_{p1}F_{p1y}; 
\]

(10)

\[
m_p\ddot{x}_{p2} + c_{p2x}\dot{x}_{p2} + k_{p2x}f_{p2x}(x_{p2}) + c_{px}(x_{p2} - \dot{x}_{p2}) + k_{px}(x_{p2} - x_{p1}) = -\chi_{p2}F_{p2y}; 
\]

(11)

\[
m_p\ddot{y}_{p1} + c_{p1y}\dot{y}_{p1} + k_{p1y}f_{p1y}(y_{p1}) + c_{py}(y_{p1} - \dot{y}_{p1}) + k_{py}(y_{p1} - y_{p2}) = -F_{p1y}; 
\]

(12)

\[
I_{p1}\ddot{\theta}_{p1} + F_{p1y}R_{p1} + R_{p1}\left[k_{p1}(\theta_{p1} - \theta_{p2}) + c_{p1}(\dot{\theta}_{p1} - \dot{\theta}_{p2})\right] - s_1\chi_{p1}F_{p1y} = -T_{p1}; 
\]

(13)

\[
I_{q1}\ddot{\theta}_{q1} - F_{q1y}R_{q1} - R_{q1}\left[k_{q1}(\theta_{q1} - \theta_{p2}) + c_{p1}(\dot{\theta}_{q1} - \dot{\theta}_{p2})\right] + s_2\chi_{p1}F_{p1y} = T_{q1}; 
\]

(17)

\[
I_{p2}\ddot{\theta}_{p2} + F_{p2y}R_{p2} + R_{p2}\left[k_{p2}(\theta_{q2} - \theta_{p1}) + c_{p1}(\dot{\theta}_{q2} - \dot{\theta}_{p1})\right] - s_2\chi_{p2}F_{p2y} = -T_{p2}; 
\]

(21)

The relative displacement \( \lambda_j \) between the meshing points of two teeth surfaces along the meshing line was expressed as

\[
\lambda_j = y_{qj} - y_{pj} - \epsilon_{yj}.
\]

(26)

The phase separation of the rotation angle between two ends gear pair \( \gamma_i \) was represented as

\[
\gamma_i = \theta_{j1} - \theta_{j2};
\]

(27)

\( \lambda_j \) and \( \lambda_i \) were taken as new free degrees. The transformation equations of Eq. (A1)–Eq. (A20) are shown in Appendix 1.

4. ILLUSTRATIVE EXAMPLE

The dynamic model of a double helical gear pair was considered as an example. The parameters of a double helical gear pair are given in Table 1. In the example, through preliminary calculation, the displacement in x-direction, y-direction, z-direction and meshing line direction was very small, therefore, all support backlashes were set to 0 and the two ends’ meshing backlashes were set to 0.004 mm.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Pinion</th>
<th>Gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tooth number</td>
<td>31</td>
<td>102</td>
</tr>
<tr>
<td>Normal modulus /( \text{mm} )</td>
<td>4.5</td>
<td></td>
</tr>
<tr>
<td>Normal pressure angle /( \degree )</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Helical angle /( \degree )</td>
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<td></td>
</tr>
<tr>
<td>Face width /( \text{mm} )</td>
<td>90 x 2</td>
<td></td>
</tr>
<tr>
<td>Helical direction</td>
<td>Right left</td>
<td>Left right</td>
</tr>
<tr>
<td>Run-out groove width /( \text{mm} )</td>
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<td></td>
</tr>
<tr>
<td>Rotation speed of pinion /( \text{r/min} )</td>
<td>2881</td>
<td></td>
</tr>
<tr>
<td>Output torque /( \text{N·m} )</td>
<td>2000</td>
<td></td>
</tr>
</tbody>
</table>

Based on the results of TCA and LTCA, the meshing synthetic stiffness, meshing stiffness of single-pair teeth, the axial displacement under low rotation speeds and the gear-into impact excitation were calculated. The calculation results can be found in reference.\(^{13}\)

4.1. The Related Calculations of Friction Force on Tooth Surface

Based on TCA of double helical gear pair, Fig. 5 shows the path of contact on pinion tooth surface. In the example, the meshing time of a pair of teeth from engagement to disengagement is 0.0035271 s.

For the left gear pair, From mesh position 1 to mesh position 5, the distance between friction force and gear center were represented as

\[
s_1 = \sqrt{a^2 - (r_{pb} + r_{qb})^2} - \sqrt{r_q^2 - r_{qb}^2 + r_{pb}w_{pt} - \lambda} \]

\[
s_2 = \sqrt{a^2 - (r_{pb} + r_{qb})^2} - s_1 \]

\[
\lambda = y_q + \theta_q\dot{r}_q - y_p - \theta_p r_p - \epsilon_y
\]

(28)

From mesh position 5 to mesh position 12, the distance between friction force and gear center were represented as

\[
s_1 = \sqrt{a^2 - r_b^2 - \lambda} \]

\[
s_2 = \sqrt{a^2 - (r_{pb} + r_{qb})^2} - s_1 \]

\[
\lambda = y_q + \theta_q\dot{r}_q - y_p - \theta_p r_p - \epsilon_y
\]

(29)

From mesh position 12 to mesh position 15, the distance be-
tween friction force and gear center were represented as

\[
\begin{align*}
    s_1 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - \sqrt{\sqrt{r_{q_a}^2 - r_{q_b}^2} + r_{ph}w_p t - \lambda}, \\
    s_2 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - s_1, \\
    \lambda &= y_q + \theta_y r_q - y_p - \theta_p r_p - c_y \\
\end{align*}
\]

(30)

Where, \( t_1 \) was the time from position 5 to position 12. For the right gear pair, From mesh position 16 to mesh position 20, the distance between friction force and gear center were represented as

\[
\begin{align*}
    s_1 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - \sqrt{\sqrt{r_{q_a}^2 - r_{q_b}^2} + r_{ph}w_p t - \lambda}, \\
    s_2 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - s_1, \\
    \lambda &= y_q + \theta_y r_q - y_p - \theta_p r_p - c_y \\
\end{align*}
\]

(31)

From mesh position 20 to mesh position 27, the distance between friction force and gear center were represented as

\[
\begin{align*}
    s_1 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - \sqrt{\sqrt{r_{q_a}^2 - r_{q_b}^2} + r_{ph}w_p t - \lambda}, \\
    s_2 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - s_1, \\
    \lambda &= y_q + \theta_y r_q - y_p - \theta_p r_p - c_y \\
\end{align*}
\]

(32)

From mesh position 27 to mesh position 30, the distance between friction force and gear center were represented as

\[
\begin{align*}
    s_1 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - \sqrt{\sqrt{r_{q_a}^2 - r_{q_b}^2} + r_{ph}w_p(t - t_1) - \lambda}, \\
    s_2 &= \sqrt{a^2 - (r_{ph} + r_{qh})^2} - s_1, \\
    \lambda &= y_q + \theta_y r_q - y_p - \theta_p r_p - c_y \\
\end{align*}
\]

(33)

Where, \( t_1 \) was the time from position 20 to position 27.

According to Eq. (1)–Eq. (2), the values of sliding friction coefficient on the left tooth surface are shown in Fig. 6. There were similar results on the right tooth surface. In this example, the meshing time of a pair of teeth from engagement to disengagement was 0.0035271 s. Combined Fig. 5, the meshing time near the pitch point was also shown in Fig. 6. It can be found that the value of \( \mu \) near the pitch point was almost zero, which fits the results mentioned by reference.\(^{19}\)

4.2. The Solution of Dynamic Model of Double Helical Gear Pair

The equations are solved by the method of the fourth-order Runge-Kutta algorithm with varying step lengths. The gear-mesh frequency is 1489 Hz. Taking the left end gear pair as an example, we get the vibratory responses of displacement, speed, and acceleration in time-domain spectrum shown in Fig. 7, respectively. The phase diagram, poincare mapping graph and corresponding frequency responses transformed by FFT are shown in Fig. 8. From Fig. 7 and Fig. 8, the example shows that the bending vibrations (including x-direction, y-direction) are much smaller than axial vibration and torsional vibration. The example confirms that the axial vibration should not be neglected. Lastly, the displacements in each direction of x, y, z and meshing line in the example is very small. Therefore, all support backlashes and meshing backlashes should avoid being too large.

5. CONCLUSIONS

Based on the TCA of a double helical gear pair, the contact path and meshing time from engagement to disengagement is obtained. The sliding friction coefficient of the tooth surface is calculated by using the existing formula for determining the sliding friction coefficient, the related results of TCA and the dynamic meshing force provided by the subsequent dynamics model of a double helical gear pair. The dynamic model of a double helical gear pair that takes the axial vibration, friction and multiple backlashes into consideration is built up by Newton’s second law and solved by the method of the fourth-order Runge-Kutta algorithm with variable step lengths. A pair of double helical gear pairs are taken as an example. The distances between the friction force, gear center, and sliding fric-
Figure 7. The vibratory responses of displacement, speed, and acceleration in time-domain spectrum.

Figure 8. The phase diagram, poincare mapping graph and corresponding frequency responses.
tion coefficient is calculated, respectively. The effect of the sliding friction coefficient is highly consistent with the results mentioned by reference. Based on these results, the dynamic model of a double helical gear pair is calculated. The corresponding results are verified by the example of the left end gear pair. The dynamic model of a double helical gear pair laid the foundation for the dynamic design of the next step.

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The transformation dynamic equations of double helical gear pair The transformation equations of Eq. (6) is represented as

\[ F_{p1y} = \cos \beta \left[ k_1 f_{h1y}(\lambda_1) + c_1 \lambda_1 \right] . \]  

(A1)

The transformation equations of Eq. (7) is represented as

\[ F_{z1} = \sin \beta \left\{ k_1 \left[ z_{p1} - z_{q1} - \lambda_1 \tan \beta - 2\varepsilon_{z1} \right] + c_1 \left[ \dot{z}_{p1} - \dot{z}_{q1} - \lambda_1 \tan \beta - 2\dot{\varepsilon}_{z1} \right] - k_1 \dot{z}_e - c_1 \dot{z}_e \right\} ; \]  

(A2)

The transformation equations of Eq. (8) is represented as

\[ F_{p2y} = \cos \beta \left[ k_2 f_{h2y}(\lambda_2) + c_2 \lambda_2 \right] . \]  

(A3)

The transformation equations of Eq. (9) is represented as

\[ F_{z2} = \sin \beta \left\{ k_2 \left[ z_{p2} - z_{q2} - \lambda_2 \tan \beta - 2\varepsilon_{z2} \right] + c_2 \left[ \dot{z}_{p2} - \dot{z}_{q2} - \lambda_2 \tan \beta - 2\dot{\varepsilon}_{z2} \right] + k_2 \dot{z}_e + c_2 \dot{z}_e \right\} . \]  

(A4)

The transformation equations of Eq. (10)–Eq. (25) is represented as

\[ m_y \ddot{z}_{q1} + c_{q1} \ddot{z}_{q1} + k_{q1} f_{q1z}(z_{q1}) + c_{qz}(\dot{z}_{q1} - \dot{z}_q) + k_{qz}(\ddot{z}_{q1} + \ddot{z}_q) = -F_{z1} ; \]  

(A10)

\[ m_1 (\ddot{\lambda}_1 - \ddot{\lambda}_p + \ddot{\dot{\lambda}}_{q1} + \ddot{\dot{\lambda}}_{q2}) + \frac{m_1}{m_p} [F_{p1y}(1 - s_1 \chi_1 \mu_l)] + (k_{p1 \gamma_p} + c_{p1 \gamma_p}) - \frac{m_1}{m_q} [F_{p1y}(1 + s_2 \chi_1 \mu_l) - (k_{q1 \gamma_q} + c_{q1 \gamma_q})] - \frac{m_1}{m_p} \frac{T_{p} \cdot \dot{R}_p}{\mu_l} = 0 ; \]  

(A11)

\[ m_y \ddot{p}_x + c_{px} \ddot{p}_x + k_{px} f_{pxz}(x_{p2}) + c_{px}(\dot{x}_{p1} - \dot{x}_p) + k_{px}(\ddot{x}_{p1} + \ddot{x}_p) = \chi_2 \mu_2 F_{p2y} ; \]  

(A12)

\[ m_y \ddot{y}_p + c_{py} \ddot{y}_p + k_{py} f_{pyz}(y_{p2}) + c_{py}(\dot{y}_{p1} - \dot{y}_p) + k_{py}(\ddot{y}_{p1} + \ddot{y}_p) = \chi_2 \mu_2 F_{p2y} ; \]  

(A13)

\[ m_y \ddot{q}_y + c_{qy} \ddot{q}_y + k_{qy} f_{qyz}(q_{p2}) + c_{qy}(\dot{q}_{p1} - \dot{q}_p) + k_{qy}(\ddot{q}_{p1} + \ddot{q}_p) = F_{p2y} ; \]  

(A14)

\[ m_y \ddot{p}_z + c_{pz} \ddot{p}_z + k_{pz} f_{pzz}(z_{p2}) + c_{pz}(\dot{z}_{p1} - \dot{z}_p) + k_{pz}(\ddot{z}_{p1} + \ddot{z}_p) = -F_{z2} ; \]  

(A15)

\[ m_2 \ddot{\lambda}_2 - \ddot{\dot{\lambda}}_{p2} + \ddot{\dot{\lambda}}_{q2} + \ddot{\dot{\lambda}}_{q2} + \ddot{\dot{\lambda}}_{q2} = F_{z2} ; \]  

(A16)

\[ m_2 \ddot{\lambda}_2 - \ddot{\dot{\lambda}}_{p2} + \ddot{\dot{\lambda}}_{q2} + \ddot{\dot{\lambda}}_{q2} + \ddot{\dot{\lambda}}_{q2} = F_{z2} ; \]  

(A17)

\[ m_2 \ddot{\lambda}_2 - \ddot{\dot{\lambda}}_{p2} + \ddot{\dot{\lambda}}_{q2} + \ddot{\dot{\lambda}}_{q2} + \ddot{\dot{\lambda}}_{q2} = F_{z2} ; \]  

(A18)

\[ I_{\gamma_p} \gamma_p = -T_{p} \gamma_{p} - (F_{p1y} - F_{p2y}) R_p \]  

(A19)

\[ I_{\gamma_q} \gamma_q = -T_{p} \gamma_{p} - (F_{p1y} - F_{p2y}) R_q \]  

(A20)
Thermal Vibrations of a Graphene Sheet Embedded in Viscoelastic Medium based on Nonlocal Shear Deformation Theory

Ashraf M. Zenkour  
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.  
Department of Mathematics, Kafrelsheikh University, Kafrelsheikh 33516, Egypt.  
A. H. Al-Subhi  
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia.  

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The nonlocal first-order shear deformation plate theory is used to present the thermal vibration of a single-layered graphene sheet (SLGS) resting on a viscoelastic foundation. The viscous damping term is added to the elastic foundation to get a three-parameter visco-Pasternak medium. The nonlocal shear deformation theory is applied to obtain the equations of motion of the simply-supported SLGSs. The effects of the nonlocal parameter as well as the length of the SLGS, mode numbers, three-parameters of the foundation, and the thermal parameter are discussed carefully for the vibration problem. The validation of the present frequencies is discussed with excellent comparison to the existing literature. For future comparisons, additional thermal vibration results of SLGSs are investigated to take into consideration the effects of thermal, nonlocal, and visco-Pasternak mediums.

1. INTRODUCTION

The classical continuum mechanics approaches are less computationally expensive when treating the analyses of micro- and/or nano-structures, compared with other approaches. The classical continuum elasticity is scale-independent in different problems and so, it needs to be modified to account for the small-scale effect. The higher-order continuum theories become effective and reliable approaches to model nanostructures mathematically, which can accurately capture the small-scale effect of nanostructures. Recently, non-local elasticity, 1–4 couple stress, 5–7 and strain gradient elasticity 8, 9 have been employed as higher-order continuum theories. These theories are reasonably considered as a useful role to show size effects in the analysis of nanoscale structures.

The vibration analysis of nanoplates using the nonlocal theory of elasticity has been a main subject of research works in recent years. Single-layered graphene sheets (SLGS), double-layered graphene sheets (DLGS), or multi-layered graphene sheets (MLGS) have high resistance and unique properties. So, all of them are used in the manufacturing of many devices such as oscillators, clocks, and sensor devices. Sakhaee-Pour et al. used a molecular structural mechanics method to present the vibration analysis of SLGSs. 10 Pradhan and Phadikar studied the nonlocal vibration of SL and DL nano-plates based upon the classical plate theory (CPT) and the first-order shear deformation plate theory (FSPT). 11 Ansari et al. investigated the vibrational behaviour of SLGS based on FSPT, and differential equations that were solved by generalized differential quadrature method for various boundary conditions. 12 The vibration analysis of orthotropic SLGS using the CPT was carried out by Pradhan and Kumar and the governing equations of motion were solved by the differential quadrature method. 13 Satish et al. presented thermal vibration analyses of orthotropic nanoplates based on nonlocal continuum mechanics for small-scale effects. 14 Shen et al. used the Galerkin method to present the vibration of a SLGS-based nano-mechanical sensor via the nonlocal Kirchhoff plate theory. 15

Most of the nanostructures were resting on or embedded in an elastic foundation medium. Ansari et al. studied vibration of a MLGS using the FSPT, according to a Winkler-type foundation. 16, 17 Murmu and Adhikari investigated the nonlocal vibration of bonded double nanoplate systems and the governing equations of motion, in terms of displacements, were solved by a new analytical method, according to a Winkler-type foundation. 18 Wang et al. used the nonlocal theory to derive the nonlinear governing equations for double-layered nanoplates subjected to four different boundary conditions, according to a Winkler-type foundation. 19 Behfar and Naghdabadi presented nanoscale vibration analysis of a multi-layered graphene sheet embedded in an elastic medium. 20 Chien et al. investigated the nonlinear vibration of laminated plates resting on a nonlinear elastic medium. 21 Liew et al. proposed a continuum-based plate model to study the vibration behaviour of MLGSs that were embedded in an elastic matrix. 22 Pradhan and Murmu employed the nonlocal plate theory and used DQM for the vibration analysis of nano-SLGSs embedded in an elastic medium. 23 Pradhan and Kumar presented vibration analysis of orthotropic SLGS embedded in a Pasternak elastic medium. 24 The transverse vibration of orthotropic DLGSs em-
bedded in an elastic medium under thermal gradient has been studied by Ghorbanpour Arani et al., using the nonlocal elasticity orthotropic plate theory.25

Moreover, additional nanostructures were resting on or embedded in a three-parameter viscoelastic foundation which was simulated as a visco-Pasternak foundation. Ghorbanpour Arani used the nonlocal elasticity theory of orthotropic plate to present the vibration analysis of DLGSs embedded in a viscoelastic foundation.26 Mohammadi et al. studied the vibration behaviour of circular and annular graphene sheets embedded in a visco-Pasternak foundation, coupled with a temperature change and under an in-plane pre-load.27 Pouresmaeeli et al. proposed the nonlocal orthotropic Kirchhoff plate theory to investigate the vibration analysis of nanoplates embedded in a viscoelastic medium.28 Goodarzi et al. presented the free vibration behaviour of a rectangular graphene sheet under a shear in-plane load and embedded in a visco-Pasternak foundation.29 Nonlocal elasticity theory has been implemented to study the vibration analysis of orthotropic SLGSs subjected to a shear in-plane load. Karličić et al. presented the complex eigenvalues of a viscoelastic orthotropic multi-nanoplate system.30 Hashemi et al. presented the exact solution for free vibration of coupled double viscoelastic GSs embedded in a visco-Pasternak medium.31

The thermal natural vibration frequency of a SLGS resting on a visco-Pasternak foundation is investigated in the present article.32-35 The nonlocal elasticity via the first-order shear deformation theory is presented. The differential governing equations were derived and their solutions are analytically presented for a simply-supported SLGS. The effects of nonlocal index, temperature increment, and the foundation medium on the natural vibration frequencies are illustrated. A comparison with the literature is presented and benchmark results are plotted for sensing the effect of all used parameters. For future comparisons, additional results are presented to investigate and report the thermal, nonlocal, and visco-Pasternak’s parameters.

2. BASIC EQUATIONS

Let us consider a SLGS embedded in a viscoelastic medium and subjected to uniform biaxial temperature changes. The SLGS was of length \( a \), width \( b \), and uniform thickness \( h \), as shown in Fig. 1. The SLGS was made of a homogeneous isotropic and linearly elastic material with Young’s modulus \( E \), Poisson’s ratio \( \nu \), shear modulus \( G \), and material density \( \rho \).

The standard local constitutive equations according to nonlocal theory of elasticity were given by

\[
\tau_{jk}(x') = \lambda \varepsilon_{mm}(x') \delta_{jk} + 2\mu \varepsilon_{jk}(x');
\]

in which \( \tau_{jk}(x') \) and \( \varepsilon_{jk}(x') \) represented the classical local stress and strain tensors at \( x' \). The small Chauzy strain-displacement relations were represented by the usual notation

\[
\varepsilon_{jk}(x') = \frac{1}{2} \left( \frac{\partial u_j(x')}{\partial x_k} + \frac{\partial u_k(x')}{\partial x_j} \right);
\]

where \( u_j(x') \) was the displacement vector at a reference point \( x' \) in the body.1-4 The pseudo-local constitutive equation of gradient type was expressed as

\[
[1 - (ae_0)^2 \nabla^2] \sigma_{jk} = \tau_{jk};
\]

where \( \sigma_{jk} \) represented the nonlocal stress tensor. The parameter \( a \) denoted an internal characteristic length and \( e_0 \) denoted a material constant determined experimentally.

The displacement field \( u_j \) of the FSPT was written as

\[
\begin{align*}
\mathbf{u}_1(x, y, z, t) &= u(x, y, t) + z \psi(x, y, t); \\
\mathbf{u}_2(x, y, z, t) &= v(x, y, t) + z \phi(x, y, t); \\
\mathbf{u}_3(x, y, z, t) &= w(x, y, t);
\end{align*}
\]
in which \( u_1, u_2, \) and \( u_3 \) were the displacements in the \( x, y, \) and \( z \) directions, \( u, v, \) and \( w \) were the mid-plane displacements. Here, \( u \) and \( v \) were the in-plane displacements, while \( w \) denoted transverse displacement (deflection), and \( \psi \) and \( \phi \) were rotational displacement about the \( y- \) and \( x- \) axes, respectively. The displacements of classical plate theory were given by setting \( \psi = -\frac{\partial w}{\partial x} \) and \( \phi = -\frac{\partial u}{\partial y} \).

The strain-displacement equations of elasticity were given by

\[
\begin{align*}
|\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz} \\
\varepsilon_{zz}
\end{bmatrix} | &= \begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \\
\frac{\partial u}{\partial x} - \frac{\partial w}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}
\end{bmatrix} + z \begin{bmatrix}
\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} \\
\frac{\partial \omega}{\partial y} - \frac{\partial \omega}{\partial x} \\
\frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial x} \\
\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} \\
\frac{\partial \omega}{\partial y} - \frac{\partial \omega}{\partial x} \\
\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y}
\end{bmatrix},
\end{align*}
\]

(5)

The constitutive equations of an isotropic SLGS according to the nonlocal elasticity was expressed as

\[
\begin{align*}
\begin{cases}
\sigma_{xx} - \xi \nabla^2 \sigma_{xx} = E \left( \frac{1}{1 - \nu^2} \right) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}; \\
\sigma_{yy} = \xi \nabla^2 \sigma_{yy} = 0; \\
\sigma_{yz} = \xi \nabla^2 \sigma_{yz} = G \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};
\end{cases}
\end{align*}
\]

(6)

where \( \xi = (a_0)^2 \) represented the nonlocal index.

The dynamic version of principle of virtual displacements may be applied to obtain the differential equations of motion. That is,

\[
\begin{align*}
\int_{t_2}^{t_2} \int_{-h/2}^{h/2} & \left\{ \rho \frac{\partial^2 u}{\partial t^2} \delta u_1 + \sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + \\
& \sigma_{xy} \delta \gamma_{xy} + \sigma_{yz} \delta \gamma_{yz} + \sigma_{xz} \delta \gamma_{xz} \right\} d\Omega dz - \\
\int_{\Omega} & \left\{ q - R_f \frac{\partial w}{\partial x} S_x \frac{\partial \sigma}{\partial x} + \frac{\partial w}{\partial y} S_y \frac{\partial \sigma}{\partial y} \right\} dw d\Omega \right\} dt = 0;
\end{align*}
\]

(7)

where \( q \) was the transverse distributed load, \( S_x = S_y = S \) denoted membrane force caused by uniform biaxial temperature changes prior to buckling. It was interesting to let the temperature change, as given by

\[
S = E \left( \frac{1}{1 - \nu} \right) \int_{-h/2}^{h/2} \Theta(z) dz.
\]

(8)

Also, \( R_f \) denoted the three-parameter visco-Pasternak medium. That was

\[
R_f = \left( K_1 - K_2 \nabla^2 + c_0 \frac{\partial}{\partial t} \right) w;
\]

(9)

where \( \nabla^2 \) was the Laplacian, \( K_1 \) represented the linear Winkler’s modulus, \( K_2 \) denoted the Pasternak’s (shear) foundation modulus, and \( c_0 \) was the damping coefficient of the visco-Pasternak medium. The visco-Winkler medium was given by setting \( K_2 = 0 \). The elastic medium was given by neglecting the viscosity damping term (i.e., \( c_0 = 0 \)).

So, the governing equations were derived from the above functional by integrating the displacement gradients in \( \varepsilon_{ij} \) by parts. The extremum conditions of the obtained functional gave the following equilibrium equations:

\[
\begin{align*}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \psi}{\partial t^2}; \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi}{\partial t^2}; \\
\frac{\partial Q_z}{\partial x} + \frac{\partial Q_y}{\partial y} + q - R_f - S \nabla^2 w &= I_0 \frac{\partial^2 w}{\partial t^2}; \\
\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= I_1 \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial^2 \psi}{\partial t^2}; \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y &= I_1 \frac{\partial^2 v}{\partial t^2} + I_2 \frac{\partial^2 \phi}{\partial t^2};
\end{align*}
\]

(10)

where \( N_{xx}, N_{yy}, N_{yy}, \) and \( N_{xy} \) denoted basic components of stress resultants; \( M_{xx}, M_{xy}, \) and \( M_{yy} \) denoted basic components of stress couples; and \( Q_x \) and \( Q_y \) represented shear stress resultants. They were obtained by integrating Eq. (6) over the thickness of the plate as

\[
\begin{align*}
\int_{-h/2}^{h/2} & \left\{ \sigma_{xx} \right\} dz; \\
\int_{-h/2}^{h/2} & \left\{ \sigma_{yy} \right\} dz; \\
\int_{-h/2}^{h/2} & \left\{ \sigma_{xy} \right\} dz;
\end{align*}
\]

(11)

where \( k \) was the transverse shear correction factor. In addition, \( I_j \) denoted mass moments of inertia and defined by

\[
\begin{align*}
I_0 = \int_{-h/2}^{h/2} & \left\{ 1 \right\} dz; \\
I_1 = \int_{-h/2}^{h/2} & \left\{ z \right\} dz.
\end{align*}
\]

(12)

According to the stress-strain relationships that appeared in Eq. (6) and the stress results definition that appeared in Eq. (11) with the aid of Cauchy’s relations, Eq. (5), the stress resultants were expressed, in terms of the displacements, as

\[
\begin{align*}
\begin{bmatrix} N_{xx} \\
N_{yy} \end{bmatrix} &= \xi \nabla^2 \begin{bmatrix} N_{xx} \\
N_{yy} \end{bmatrix} = E h \left[ \begin{bmatrix} 1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \end{bmatrix} \right]; \\
\begin{bmatrix} M_{xx} \\
M_{yy} \end{bmatrix} &= \xi \nabla^2 \begin{bmatrix} M_{xx} \\
M_{yy} \end{bmatrix} = D \left[ \begin{bmatrix} 1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \psi}{\partial x} \\
\frac{\partial \psi}{\partial y} \\
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \end{bmatrix} \right];
\end{align*}
\]

(13)

\[
\begin{align*}
\begin{bmatrix} Q_x \\
Q_y \end{bmatrix} &= k G h \left( \psi + \frac{\partial w}{\partial y} \right); \\
\begin{bmatrix} Q_x \\
Q_y \end{bmatrix} &= k G h \left( \psi + \frac{\partial w}{\partial y} \right);
\end{align*}
\]

(15)
where $D = \frac{E h^3}{12(1-\nu^2)}$ represented the bending rigidity of the SLGS.

The substitution of Eqs. (13)–(15) into Eqs. (10) yielded the following nonlocal partial differential equations of motion in terms of displacements ($q = 0$),

$$D \left( \frac{\partial^2 u}{\partial x^2} + \frac{1 + \nu}{2} \frac{\partial^2 v}{\partial x \partial y} + \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial y^2} \right) = \frac{h^2}{12} (1 - \xi \nabla^2) \left( I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 w}{\partial t^2} \right);$$

$$D \left( \frac{1 - \nu}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1 + \nu}{2} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) = \frac{h^2}{12} (1 - \xi \nabla^2) \left( I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \psi}{\partial t^2} \right);$$

$$kGh \left( \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} + \nabla^2 w \right) = (1 - \xi \nabla^2) \left( K_1 - K_2 \nabla^2 + c_{\partial \psi \partial} - S \nabla^2 \right) w = I_0 (1 - \xi \nabla^2) \frac{\partial \psi}{\partial x};$$

$$D \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{1 + \nu}{2} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{1 - \nu}{2} \frac{\partial^2 \psi}{\partial y^2} \right) - kGh (\psi + \frac{\partial w}{\partial x}) = (1 - \xi \nabla^2) \left( I_0 \frac{\partial^2 \psi}{\partial t^2} + I_2 \frac{\partial^2 \psi}{\partial t^2} \right);$$

$$D \left( \frac{1 - \nu}{2} \frac{\partial^2 \phi}{\partial x^2} + \frac{1 + \nu}{2} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2} \right) - kGh (\phi + \frac{\partial w}{\partial y}) = (1 - \xi \nabla^2) \left( I_0 \frac{\partial^2 \phi}{\partial t^2} + I_2 \frac{\partial^2 \phi}{\partial t^2} \right).$$

3. SOLUTION OF NONLOCAL VIBRATION FREQUENCIES

The determination of natural frequencies is of fundamental importance in design of many nano-structures. The assumed form of displacement components was expressed as

$$\begin{bmatrix} u(x, t) \, \psi(x, t) \\ v(x, t) \, \phi(x, t) \end{bmatrix} = \begin{bmatrix} h u^* \, \psi^* \cos(\lambda_m x) \sin(\mu_n y) \\ h v^* \, \sin(\lambda_m x) \cos(\mu_n y) \\ h u^* \, \phi^* \cos(\lambda_m x) \sin(\mu_n y) \\ h v^* \, \sin(\lambda_m x) \cos(\mu_n y) \end{bmatrix} e^{i\omega t};$$

where $\omega$ represented the natural frequency for the SLGS; $u^*, \psi^*$, $v^*$, $\phi^*$ were arbitrary constant parameters; $\lambda_m = m\pi/a$ and $\mu_n = n\pi/b$ in which $m$ and $n$ were the wave numbers; and $i = \sqrt{-1}$.

The substitution of the solution given above into equations of motion, for constant density, yielded:

$$D \left( \frac{\lambda_m^2}{2} + \frac{1 - \nu}{2} \mu_n^2 \right) u^* + \frac{1 + \nu}{2} D \lambda_m \mu_n v^* = \frac{\rho h^3}{12} \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] u^*;$$

$$\frac{1 + \nu}{2} \lambda_m^2 \mu_n u^* + \frac{1 - \nu}{2} \mu_n^2 \lambda_m v^* = \frac{\rho h^3}{12} \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] v^*;$$

$$kGh (\lambda_m^2 + \mu_n^2) w^* + \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] \left[ K_1 + (\lambda_m^2 + \mu_n^2)(K_2 + S) + i\omega c_z \right] w^* + kG\mu_n \psi^* + kG\mu_n \phi^* = \rho h^3 \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] w^*;$$

$$\frac{1 + \nu}{2} \lambda_m^2 \mu_n \psi^* + \frac{1 - \nu}{2} \mu_n^2 \lambda_m \phi^* = \frac{\rho h^3}{12} \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] \psi^*;$$

$$\left( \frac{1 - \nu}{2} \lambda_m^2 + \mu_n^2 \right) \psi^* + kG \mu_n \phi^* + \left[ \frac{1 + \nu}{2} \lambda_m^2 \mu_n + \frac{1 - \nu}{2} \mu_n^2 \lambda_m \right] kG \phi^* = \frac{\rho h^3}{12} \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] \phi^*. $$

In this present work, we studied the effect of nonlocal parameter on natural frequencies with and without the effect of thermal field. In the case of thermal vibration, we supposed that $\Theta(z) = \alpha \Delta T$ in which $\Delta T$ denoted the increment temperature and $\alpha$ represented the coefficient of thermal expansion for the SLGS. The above equations of motion were written in terms of displacement parameters $w^*$, $\psi^*$, and $\phi^*$ only as

$$\{ |P| + i\omega |J| - \omega^2 |R| \} \{ X \} = \{ 0 \};$$

where $\{ X \} = \{ w^*, \psi^*, \phi^* \}^T$ was the solution vector. The non-vanishing elements of the symmetric matrices $|P|$, $|J|$ and $|R|$ were expressed as:

$$P_{11} = kGh^3 (\lambda_m^2 + \mu_n^2) + h^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] \left[ K_1 + (\lambda_m^2 + \mu_n^2)(K_2 + S) \right];$$

$$P_{12} = kGh^2 \lambda_m \psi^*;$$

$$P_{13} = kGh^2 \mu_n \phi^*;$$

$$P_{22} = D \left( \lambda_m^2 + \frac{1 - \nu}{2} \mu_n^2 \right) + kGh;$$

$$P_{23} = \frac{1 + \nu}{2} D \lambda_m \mu_n;$$

$$P_{33} = D \left( \frac{1 - \nu}{2} \lambda_m^2 + \mu_n^2 \right) + kGh;$$

$$J_{11} = h^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right] c_z;$$

$$R_{11} = \rho h^3 \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right];$$

$$R_{22} = R_{33} = \frac{\rho h^3}{12} \omega^2 \left[ 1 + \xi (\lambda_m^2 + \mu_n^2) \right].$$

The frequency equation for the SLGS was given by setting $||P| + i\omega |J| - \omega^2 |R|| = 0$ to get

$$\sum_{j=0}^{6} A_j (i\omega)^j = 0;$$

$$\omega = \sqrt{\frac{\sum_{j=0}^{6} A_j (i\omega)^j}{\sum_{j=0}^{6} A_j (i\omega)^j}}.$$
in which

\[
A_0 = P_{22} \left( P_{11} P_{33} - P_{13}^2 \right) - P_{12} \left( P_{12} P_{33} - P_{13} P_{23} \right) - P_{23} \left( P_{11} P_{23} - P_{12} P_{33} \right) ;
\]

\[
A_1 = J_{11} \left( P_{22} P_{33} - P_{23}^2 \right) ;
\]

\[
A_2 = R_{11} \left( P_{22} P_{33} - P_{23}^2 \right) + R_{22} \left( P_{11} P_{33} - P_{13}^2 \right) + R_{22} \left( P_{11} P_{22} - P_{12}^2 \right) ;
\]

\[
A_3 = J_{11} \left( P_{22} + P_{33} \right) R_{22} ;
\]

\[
A_4 = \left[ P_{11} R_{22} + \left( P_{22} + P_{33} \right) R_{11} \right] R_{22} ;
\]

\[
A_5 = J_{11} R_{22}^2 ;
\]

\[
A_6 = R_{11} R_{22}^2 .
\]

\[ (30) \]

4. NUMERICAL RESULTS AND DISCUSSIONS

In this section, the properties of graphene sheet are considered as: modulus of elasticity \( E = 1.02 \) TPa, Poisson’s ratio \( \nu = 0.16 \), thermal expansion coefficient \( \alpha = 1.1 \times 10^{-6} \) (1/K), and material density \( \rho = 2250 \) kg/m\(^3\). \(^{11}\) Here, the fundamental vibration frequency was firstly compared with the corresponding ones in the literature.\(^{11,13,15,24}\) For this purpose, the frequency ratio \( (\hat{\omega} = \omega^{NL}/\omega^L) \) was considered where \( \omega^{NL} \) represented the frequency calculated using nonlocal theory while \( \omega^L \) represented the frequency calculated using the local theory. The comparison of the vibration frequency ratio of the nonlocal square SLGS is reported in Table 1 with \( a = b = 10 \) nm and \( h = 0.34 \) nm. In this example, the effects of visco-Pasternak medium and temperature changes were ignored. An excellent agreement appeared and the present ratio was the same as those presented in the literature.

Now, let us consider additional examples to put into evidence the effect of length \( \alpha \), temperature increment \( \Delta T = 10^7 \tau \) K in which \( \tau \) was a thermal parameter, the foundation parameters \( K_1 \) and \( K_2 \), and the viscous damping coefficient \( c_t \) on thermal vibration of the present SLGS. It is to be noted that, we got the local thermal vibration of the SLGSs by setting \( \xi = 0 \) in the preceding equations. The fundamental frequency was obtained by setting \( m = n = 1 \), while the natural one was given by setting \( m = n = 2 \).

Figures 2 and 3 show the fundamental and natural frequency ratios \( \hat{\omega} \) vs nonlocal index \( \xi \) and the length \( a \) of the SLGS without viscoelastic foundation \( (K_1 = K_2 = c_t = 0) \) and neglecting the thermal effect \( (\tau = 0) \). The frequency ratio \( \hat{\omega} \) increased as \( a \) increased and \( \xi \) decreased. The maximum frequency ratio occurred when \( \xi = 0 \). The fundamental frequency ratio was larger than the corresponding natural one.

Figures 4–7 show the fundamental and natural frequency ratios \( \hat{\omega} \) of a square SLGS vs nonlocal index \( \xi \) and Winkler’s parameter \( K_1 \). The medium was depending only on the Winkler parameter in Figs. 4 and 5. The thermal effect was also neglected \( (\tau = 0) \). The frequency ratio \( \hat{\omega} \) increased as \( \xi \) decreased. The maximum frequency ratio occurred when \( \xi = 0 \) and for higher values of \( K_1 \). The frequency ratio suddenly increased with the increase of the Winkler’s parameter \( K_1 \). Also, for constant values of \( K_1 \), the frequency ratio was increasing as \( K_2 \) increased, as shown in Figs. 6 and 7. In the last two figures, the fundamental frequency ratio was smaller than the corresponding natural one.

The plots of fundamental and natural frequency ratios \( \hat{\omega} \) of a square SLGS vs the elastic foundation parameters \( K_1 \) and \( K_2 \)
appear in Figs. 8 and 9. The frequency ratio was independent of \( K_2 \) for the higher values of \( K_1 \). The fundamental frequency ratio was also independent of \( K_2 \) when \( K_1 = 0 \) while the natural frequency ratio may be increasing, as \( K_2 \) increased when \( K_1 = 0 \). Also, the fundamental frequency ratio was smaller than the corresponding natural one.

The effects of thermal parameter \( \tau \) and nonlocal index \( \xi \) on the fundamental and natural frequency ratios \( \hat{\omega} \) of a square SLGS are discussed in Figs. 10 and 11. The SLGS was embedded in an elastic medium with \( K_1 = K_2 = 10 \) nN. The frequency ratio increased as \( \tau \) increased. The frequency ratio vanished for higher values of \( \xi \) when the thermal effect was neglected. The fundamental frequency ratio was larger than the corresponding natural one.

To discuss the effect of viscous damping parameter we obtained the frequency parameter \( \varpi = a \sqrt{\frac{1}{\rho_0}} \omega \) in terms of new parameters as

\[
\Delta T = 10^4 \tau; \quad \bar{c}_t = \frac{a}{\sqrt{\rho G h}} c_t; \quad K_1 = \frac{h \kappa_1}{a}; \quad K_2 = \frac{h^2 \kappa_2}{a^2}.
\]

(31)

Figures 12 and 13 show the fundamental and natural frequency parameter \( \varpi \) of a square SLGS vs thermal parameter \( \tau \) and damping coefficient \( \bar{c}_t \). The other elastic foundation parameters were set as \( \kappa_1 = 2 \) and \( \kappa_2 = 1 \). The frequency parameter \( \varpi \) increased as the thermal parameter \( \tau \) increased and damping coefficient \( \bar{c}_t \) decreased. The frequency parameter \( \varpi \) vanished for the small values of \( \tau \) and the higher values of \( \bar{c}_t \).
5. CONCLUSIONS

The nonlocal thermal vibration frequencies of a single-layered graphene sheet embedded in three-parameter visco-Pasternak’s medium are investigated. The local-to-nonlocal vibration frequency ratios compare well with the corresponding results in the literature. The fundamental and natural frequency ratios are presented to serve as the benchmarks for future comparisons. Some novel phenomena can be observed from the discussion of the results. The results are very sensitive to the variation of nonlocal index. The inclusion of three-parameter visco-Pasternak foundation is very pronounced. The frequency ratios are obviated through three-dimensional (3-D) plots to discuss and investigate the effect of all parameters. The maximum frequency ratio occurs when neglecting the nonlocal index. The frequency ratio increases with the increase of the thermal parameter. The frequency ratio may vanish for the higher values of the nonlocal index, especially when neglecting thermal effect.

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Figure 12. Fundamental frequency of a square SLGS vs thermal parameter $\tau$ and damping coefficient $c_t$ ($a = 10$ nm, $h = 0.34$ nm, $\kappa_1 = 2$, $\kappa_2 = 1$, $\eta = 0.5$).

Figure 13. Natural frequency of a square SLGS vs thermal parameter $\tau$ and damping coefficient $c_t$ ($a = 10$ nm, $h = 0.34$ nm, $\kappa_1 = 2$, $\kappa_2 = 1$, $\eta = 0.5$).


Analytical Solution for Free Vibration of Annular Mindlin Plate with a Circumferential Open Crack

Eshagh Derakhshan
Civil Engineering Department, Isfahan University of Technology, Isfahan, Iran.

Mahboobeh Fakhrzarei
Civil Engineering Department, Yazd University, Yazd, Iran.

Shahram Derakhshan
Mechanical Engineering Department, Iran University of Science and Technology, Tehran, Iran.

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Mindlin plate theory is employed to obtain the free vibration response of an annular moderately thick plate with a circumferential open crack with fixed-free boundary conditions. To model the crack, a set of continuously distributed rotational springs are employed at the crack location. The corresponding spring stiffness value is a function of the crack depth and is given as a closed-form function. To obtain the vibration behaviour, the eigenvalue problem is solved to obtain the natural frequencies and mode shapes. The current method is verified by comparing the results with those obtained from finite element analysis. Through a parametric study, the effects of the crack depth and its radial location on the natural frequencies and mode shapes are investigated. The results show that for a constant crack depth, the reduction in natural frequency is a strong function of the radial location of the crack.

1. INTRODUCTION

Plates with various shapes are being widely used in different industries, such as the marine and automotive industries. Annular plates, one of the most applicable shapes of plates, have received great attention from many researchers, and their dynamic response has been studied thoroughly. Free vibration analysis is a reasonable method to study the dynamic behaviour of industrial shapes. Civanek O. investigated the free vibration analysis of annular plates based on the first-order shear deformation theory using the method of discrete singular convolution. Mercan, et al. studied the free vibration response of circular cylindrical shells with functionally graded material. They graded the material properties based on the volume fraction power law indexes.

It is clear that emerging faults in plates, such as cracks, are inescapable under working loads, which can alter their mechanical response. It has been well documented that cracks in the structures change the natural frequencies and mode shapes.

The vibration problem of cracked rectangular plates has been well established. Hosseini-Hashemi et al. investigated the effects of the crack depth and its location on the natural frequencies of a vibrating rectangular Mindlin plate. Huang et al. extracted solutions for the free vibration of side-cracked rectangular functionally graded material (FGM) thick plates by employing the Reddy third-order shear deformation plate theory.

There are, however, much fewer publications on the vibration behaviour of cracked annular plates; most of them devoted to considering radial or arc cracks. Si et al. investigated the free vibration of a baffled circular plate with radial side cracks in contact with water and showed an increase in the distinctions between the dry and wet mode shapes when the cracks appeared. Shi conducted an analysis on an annular sector bilayer plate with an arc-shaped interface under axial shear loading and showed the existence of a coupled effect between the physical and geometrical parameters on the interfacial fracture behaviour.

Circumferential cracks in annular plates may occur either within the manufacturing processes or under the application of variable loadings, which has been rarely studied. Demir et al. investigated the effects of circumferential cracks on the vibration behaviour of annular plates. They showed that the natural frequencies may increase by increasing the number of cracks. Anifantis et al. studied the natural frequencies of the cracked annular thin plates. However, the study did not consider the effects of either shear force or the crack radial location. Bahaloo et al. conducted an analysis on the transverse vibration of a functionally graded rotating annular disk with a circumferential crack. The results of their study indicated that radial distance and the depth of the crack affected the critical speed of the disk.

Bose and Mohanty investigated the large amplitude axisymmetric vibration of a circular plate having a radial crack, where they considered simply supported and clamped boundary conditions. Among the conclusions they reported was that the natural frequency of a cracked plate approaches that of an un-cracked plate for a particular crack position independent of the crack depth.

Lack of a thorough study in the literature to investigate a cracked annular plate when the shear effects in the plate are important was the incentive for the current study. In this study, the effect of crack radial location and its depth on the natural frequencies and mode shapes of an annular Mindlin plate with a circumferential crack with fix-free boundary condition are obtained.

2. MODELLING THE CIRCUMFERENTIAL CRACK

The rotational stiffness for an open edge crack in a rectangular plate, $K_r$, with the assumption of the validity of Linear
Elastic Fracture Mechanics (LEFM) conditions is given by Eq. (1):\[ K_i = \frac{E h^2}{36 (1 - v^2)} \frac{1}{\alpha_b}; \] where, \( E \) is Young’s modulus, \( h \) is the thickness of the plate, \( v \) is Poisson’s ratio, and \( \alpha_b \) is expressed in Eq. (2):

\[ \alpha_b = \eta^2 \sum_{i=0}^{12} C_i \eta^i; \]

where, \( \eta \) is the ratio of crack depth to plate thickness, and the values of \( C_i \) are listed in Table 1.\[ \]

For an annular plate, Finite Element Analysis (FEA) was conducted in Abaqus FEA by applying the bending moment on the axisymmetric model, and the rotational stiffness was conducted in Abaqus FEA by applying the bending moment to the slope change in the crack section. Bending moment and slope in the crack section were obtained using Eqs. (3) and (4), respectively:

\[ M = \int_{h-c}^{h} z \sigma_r \, dz; \]

\[ \Delta \theta = \frac{u_1 - u_2}{h_c}; \]

where, \( h_c \) is the crack depth and \( \sigma_r \) is the radial stress in the crack location. The factors \( u_1 \) and \( u_2 \) indicate radial displacement in the upper surface of the plate in two sides of the crack.

The schematic of the annular plate under study is presented in Fig. 1(a). The dimensions \( a, b, R, \) and \( h \) are plate inner radius, crack radial location, plate outer radius, and plate thickness, respectively. A section of the model used in Abaqus FEA with singular elements at the crack location is depicted in Fig. 1(b). CPS6 elements were employed in a singular area around the crack and CPS8R elements were used in the rest of the model. Mesh sensitivity analysis was conducted to reach the mesh-independent results. Mesh sensitivity analysis was conducted to reach the mesh-independent results.

It was found that the crack depth was only a function of the crack depth and it was independent of the crack location. Furthermore, it was found that with a simple mapping from the polar coordinates to Cartesian coordinates, a modified form of Eq. (1) may be used to obtain the crack stiffness in the annular plate as:

\[ K_i = \frac{E h^2}{36 (1 - v^2)} \frac{\pi (R - a)}{\alpha_b}. \]

### 3. EQUATIONS OF MOTION FOR MINDLIN PLATE

An annular Mindlin plate of radius \( R \) with a crack at radial location \( b \) was considered. The plate material properties are given in Table 2. The outer edge of the plate was free to vibrate while the inner edge was fixed. The whole plate was considered as two annular sub-plates connected at the crack location by rotational springs that represented the circumferential crack.

#### 3.1. Equations of Motion for Each Sub-Plate

In order to increase the accuracy of the analysis and to capture the higher mode shapes of vibration, the thick plate theory of Mindlin and Deresiewicz was adopted in this study.\[ \]

According to this theory, the rotations \( \psi_i \) and \( \theta_i \) of the plate can be expressed based on the three potential functions \( \Theta_1, \Theta_2, \) and \( \Theta_3, \) as presented in Eq. (6):

\[ \psi_{ri} = (\sigma_{1i} - 1) \frac{\partial \Theta_{1i}}{\partial x} + (\sigma_{2i} - 1) \frac{\partial \Theta_{2i}}{\partial \theta} + \frac{1}{\chi} \frac{\partial \Theta_{3i}}{\partial \theta}; \]

\[ \psi_{bi} = (\sigma_{1i} - 1) \frac{\partial \Theta_{1i}}{\partial \theta} + (\sigma_{2i} - 1) \frac{\partial \Theta_{2i}}{\partial \theta} - \frac{1}{\chi} \frac{\partial \Theta_{3i}}{\partial x}; \]

\[ \bar{w}_i = \Theta_{1i} + \Theta_{2i}; \]

where,

\[ \sigma_{1i}^2, \sigma_{2i}^2 = \frac{2}{\delta_{Ri}^2 (1 - v^2)} \left[ \frac{\tau_{1i}^2}{12} + \frac{\tau_{2i}^2}{6 (1 - v)^2} \right] \pm \sqrt{\left( \frac{\tau_{1i}^2}{12} - \frac{\tau_{2i}^2}{6 (1 - v)^2} \right)^2 + \frac{4}{\lambda_i^2}}; \]

\[ \text{Table 1. Values for coefficient } C_i. \]

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<th>( C_i )</th>
<th>( i )</th>
<th>( C_i )</th>
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Figure 1. (a) schematic partial cut view model of the plate, (b) schematic of the axisymmetric FE model in Abaqus.
Table 2. Plate material properties.

<table>
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<th>Parameter</th>
<th>Value</th>
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<tr>
<td>Young’s Modulus, E</td>
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<tr>
<td>Correction Factor, $κ^2$</td>
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<tr>
<td>Density, $ρ$</td>
<td>7772 Kg/m³</td>
</tr>
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</table>

$$\delta_{ji}^2 = \frac{2}{(1-ν)} \left[ \frac{τ^2_i λ_j^2}{12} - \frac{6(1-ν)κ^2}{τ^2_i} \right] ; \quad (7c)$$

$$\vec{w}_i = \vec{w}_i \frac{R_i}{R}, \quad τ_i = \frac{h_i}{R}, \quad λ_i = \omega R^2 \sqrt{\frac{ρ h_i}{D_i}} ; \quad (7d)$$

In the above equations, $r$ and $θ$ are the radial and circumferential coordinates of the polar coordinate system employed in this analysis, respectively. $ω$ is the natural angular frequency of the whole system. For sub-plate $i$, $D_i = E h^3 /[12(1-ν^2)]$ is the flexural rigidity, $λ_i$ is the dimensionless frequency parameter, and $h_i$ is the thickness. The index $i$ can take on values 1 and 2 representing the outer and inner sub-plates, respectively. The values of shear correction factor,$κ^2$, and mass density, $ρ$, are given in Table 2.

Without loss of generality, the thicknesses of the two sub-plates were considered to be equal in this study. The ratio of $aR=0.1$ was considered in all calculations. The dimensionless frequency parameter of the plate, $λ_s$, is expressed as:

$$λ_s = \omega R^2 \sqrt{\frac{ρ h}{D}} , \quad (8)$$

where,

$$D = \frac{E h^3}{12(1-ν^2)} . \quad (9)$$

Finally, the governing equations of motion for vibrating circular stepped plate, in polar coordinate, can be expressed in compact form as:

$$(\nabla^2 + \delta_{ji}^2) \Theta_{ji} = 0, \quad j = 1, 2, 3; \quad \text{for each } j : i = 1, 2 . \quad (10)$$

In which $\nabla^2$ is the Laplacian operator in the polar coordinate system and was defined as:

$$\nabla^2 (\bullet) = \frac{\partial^2 (\bullet)}{\partial χ^2} + \frac{1}{χ} \frac{\partial (\bullet)}{\partial χ} + \frac{1}{χ^2} \frac{\partial^2 (\bullet)}{\partial θ^2} . \quad (11)$$

The general solution for the Eq. (10) was expressed as:

$$\Theta_{1i} = [A_{1i} R_1 (Δ_{1i} χ) + B_{1i} S_1 (Δ_{1i} χ)] \cos nθ; \quad \Theta_{2i} = [A_{2i} R_2 (Δ_{2i} χ) + B_{2i} S_2 (Δ_{2i} χ)] \cos nθ; \quad \Theta_{3i} = [A_{3i} R_3 (Δ_{3i} χ) + B_{3i} S_3 (Δ_{3i} χ)] \sin nθ; \quad j=1, 2, 3 \text{ and } i=1, 2 ; \quad (12)$$

where, $A_{ji}$ and $B_{ij}$ are the unknown constants, which can be specified using the boundary conditions at the edges and the compatibility conditions at the crack location, $n$ is the number of nodal diameters, and $Δ_{1j}$, $R_{1n}$, and $S_{1n}$ can be expressed as:

$$Δ_{1j} = \{ \begin{cases} δ_{ji} & \text{if } δ_{ji}^2 ≥ 0 \\ \text{Im} (δ_{ji}) & \text{if } δ_{ji}^2 < 0 \end{cases} ; \quad \text{if } i=1, 2; \quad j=1, 2, 3; \quad (13a)$$

$$R_{1n} (Δ_{ji} χ) = \{ \begin{cases} J_n (Δ_{ji} χ) & \text{if } δ_{ji}^2 ≥ 0 \\ I_n (Δ_{ji} χ) & \text{if } δ_{ji}^2 < 0 \end{cases} ; \quad \text{if } i=1, 2; \quad j=1, 2, 3; \quad (13b)$$

$$S_{1n} (Δ_{ji} χ) = \{ \begin{cases} Y_n (Δ_{ji} χ) & \text{if } δ_{ji}^2 ≥ 0 \\ K_n (Δ_{ji} χ) & \text{if } δ_{ji}^2 < 0 \end{cases} ; \quad \text{if } i=1, 2; \quad j=1, 2, 3; \quad (13c)$$

In which, $J_n (•)$ and $I_n (•)$ are the first kind and the modified first kind Bessel functions of order $n$, and $Y_n (•)$ and $K_n (•)$ are the second kind and the modified second kind Bessel functions of order $n$. To avoid singularity, the arbitrary constants $B_{12}$ for sub-plate 2, must be assumed as zero for the displacement fields $\psi_{q2}$, $ψ_{q2}$ at the centre of the plate ($χ = \pi = 0$).

### 3.2. Boundary and Compatibility Conditions

The satisfaction of the following boundary and compatibility conditions were required at the edges of the plate and at the crack location:

$$M_{rr1} (1) = M_{θθ1} (1) = Q_{r1} (1) = 0 ; \quad (14)$$

$$M_{rr1} \left( \frac{b}{R} \right) = K_0 \left[ \psi_{q1} \left( \frac{b}{R} \right) - ψ_{q2} \left( \frac{b}{R} \right) \right] ; \quad (15)$$

$$M_{rr1} \left( \frac{b}{R} \right) = M_{rθ2} \left( \frac{b}{R} \right) ; \quad (16)$$

$$ψ_{q1} \left( \frac{b}{R} \right) = ψ_{q2} \left( \frac{b}{R} \right) ; \quad (17)$$

$$ψ_{q1} \left( \frac{b}{R} \right) = ψ_{q2} \left( \frac{b}{R} \right) ; \quad (18)$$

$$Q_{r1} \left( \frac{b}{R} \right) = Q_{r2} \left( \frac{b}{R} \right) ; \quad (19)$$

$$Q_{r1} \left( \frac{b}{R} \right) = Q_{r2} \left( \frac{b}{R} \right) = 0 . \quad (20)$$

In which, the radial bending moments $M_{rr1}$, the twisting moments $M_{θθ1}$, and the transverse shear forces $Q_{r1}$ are presented in Eqs. (22-a) to (22-c):

$$M_{rr1} = \frac{D}{R} \left[ \frac{∂^2 ψ_{r1}}{∂ χ^2} + \frac{v}{χ} \left( ψ_{r1} + \frac{∂ ψ_{θ1}}{∂ χ} \right) \right] ; \quad (22a)$$

$$M_{θθ1} = \frac{D}{R} \left[ \frac{1-v}{2} \left( 1 + \frac{∂ ψ_{r1}}{∂ χ} - ψ_{θ1} + \frac{∂ ψ_{θ1}}{∂ χ} \right) \right] ; \quad (22b)$$

$$Q_{r1} = κ^2 G h \left( \frac{∂ ψ_{θ1}}{∂ χ} + ψ_{r1} \right) ; \quad (22c)$$

where, $G = E/[2(1+ν)]$ is the shear modulus.

By rearranging the conditions in Eqs. (14) to (21) into matrix form, a system of 12×12 homogenous equations was obtained as follows:

$$[K]_{12×12} [X]_{12×1} = [0]_{12×1} . \quad (23)$$

The matrix $[K]$ is the stiffness matrix of the system and its components are provided in appendix A. The vector $[X]$ is the vector of unknown coefficients and is presented in Eq. (24).

$$[X] = [A_{11} A_{21} A_{31} B_{11} B_{21} B_{31} A_{12} A_{22} A_{32} B_{12} B_{22} B_{32}]^T . \quad (24)$$

The vector $[X]$ was required to determine the motion of the plate in Eq. (12). By solving the above system of equations, the natural frequencies and mode shapes of the cracked plate were obtained.
Table 3. The dimensionless natural frequencies for the un-cracked plate.

<table>
<thead>
<tr>
<th>mode order</th>
<th>((n,m))</th>
<th>(\lambda_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,0)</td>
<td>3.47664</td>
</tr>
<tr>
<td>2</td>
<td>(0,0)</td>
<td>4.23750</td>
</tr>
<tr>
<td>3</td>
<td>(2,0)</td>
<td>5.61526</td>
</tr>
<tr>
<td>4</td>
<td>(3,0)</td>
<td>12.4356</td>
</tr>
<tr>
<td>5</td>
<td>(4,0)</td>
<td>21.7965</td>
</tr>
<tr>
<td>6</td>
<td>(0,1)</td>
<td>25.2361</td>
</tr>
<tr>
<td>7</td>
<td>(1,1)</td>
<td>27.6333</td>
</tr>
<tr>
<td>8</td>
<td>(5,0)</td>
<td>33.4238</td>
</tr>
</tbody>
</table>

4. RESULTS AND DISCUSSION

The resulting natural frequencies are expressed in terms of the dimensionless parameter, \(\lambda_s\). The dimensionless crack location, \(b/R\), changed from near \(a/R\) to near 1, and the natural frequency reduction in each mode was calculated. The effects of crack location on the natural frequency were determined. The first 8 dimensionless natural frequencies for the un-cracked annular Mindlin plate are tabulated in Table 3, in which the mode shapes are denoted by \((n, m)\), \(n\) indicates the number of nodal diameters, and \(m\) indicates the number of nodal circles. The smallest dimensionless natural frequency belongs to mode \((1, 0)\), as shown in Fig. (2-a).

The natural frequencies of the plate are tabulated in Table 4 for different values of crack depth and location. The results obtained from the method provided in this study were compared with those of axisymmetric FEA in Abaqus to show the capability of the present study.

The observed discrepancies are below 2\%, indicating the merit of the present method in predicting the cracked plate vibration behaviour.

Focusing on the crack depth, it is evident that for all the vibration mode shapes, as the crack deepens, the reduction in the natural frequencies increases. In this study, the deepest crack depth of 0.7 of the plate thickness was considered. Above this limit, the plate may be considered as failed.

The effects of crack depth and crack radial location on the first eight natural frequencies and the corresponding mode shapes are shown in Fig. 2 to Fig. 9. It is apparent that the worst radial location of the circumferential crack differs for different mode shapes. For example, for the first three mode shapes \((1,0)\), \((0,0)\), and \((2,0)\), the worst case occurred when the crack was located in the inner edge of the plate, and for modes \((0,1)\) and \((1,1)\), the worst location of the crack was around the middle section.

More interesting, for mode \((5,0)\), the worst crack location was very close to the outer edge. In general, it was observable that the worst location of the crack for each mode shape was the location near the maximum curvature of that mode shape. The outlines, may be used in the design of annular plates under dynamic excitations, especially those with high frequencies.

5. CONCLUSIONS

In the present study, Mindlin plate theory was employed to study the free vibration behaviour of an annular moderately thick plate with a circumferential crack which was fixed on the inner edge and free on the outer edge. The crack was considered to remain open. An analytical method was developed to obtain the natural frequencies and mode shapes for the plate by solving a characteristic equation. The results of the current method were found to be in a good agreement with those of axisymmetric FEA. The results showed that:

1. Emerging the crack decreases the natural frequencies. However, for a certain crack depth, the natural frequency reduction is a strong function of the radial location of the crack.
2. The critical location of the crack for each mode, at which the highest reduction in natural frequency of that mode occurs, depends on the deformation style in the relevant mode.
3. Although the cracks locating near the outer edge of the plate are less effective in low frequencies, they may be quite catastrophic in higher excitation frequencies.
4. Considering the crack depth, it is evident that for all the vibration mode shapes, the reduction in the natural frequencies increases as the crack deepens.
5. For a specific mode shape, the crack induces the most reduction when it is located at the maximum curvature point of that mode shape, while it does not essentially affect the mode shape when it is located at the minimum curvature point.

The method can be easily extended to consider the effects of multiple circumferential cracks, or to account for different boundary conditions for an annular plate. The outlines, may be used in design or in crack detection of annular plates under dynamic excitations, especially those with high frequencies.
<table>
<thead>
<tr>
<th>mode order</th>
<th>( \nu )</th>
<th>( \nu = 0.3 )</th>
<th>( \nu = 0.5 )</th>
<th>( \nu = 0.7 )</th>
<th>( \nu = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present</td>
<td>3.42</td>
<td>3.46</td>
<td>3.47</td>
<td>3.47</td>
<td>3.47</td>
</tr>
<tr>
<td>% Diff</td>
<td>3.37</td>
<td>3.41</td>
<td>3.45</td>
<td>3.47</td>
<td>3.47</td>
</tr>
<tr>
<td>FEM</td>
<td>3.43</td>
<td>3.45</td>
<td>3.46</td>
<td>3.47</td>
<td>3.47</td>
</tr>
<tr>
<td>% Diff</td>
<td>3.43</td>
<td>3.45</td>
<td>3.46</td>
<td>3.47</td>
<td>3.47</td>
</tr>
</tbody>
</table>

**Table 4.** The dimensionless natural frequencies as a function of crack depth and location.
Figure 3. (a) 3D view of mode (0,0), (b) 2D view, (c) reduction in the natural frequencies as the depth and location of the crack change.

Figure 4. (a) 3D view of mode (2,0), (b) 2D view, (c) reduction in the natural frequencies as the depth and location of the crack change.

Figure 5. (a) 3D view of mode (3,0), (b) 2D view, (c) reduction in the natural frequencies as the depth and location of the crack change.

APPENDIX A

The components of the $K$ matrix of the Eq. (23) are presented here.

\[
K_{1,1} = (1 - \nu) \left[ R_n (\Delta_{11}^2 \Delta_{11} + 2 \nu R_n' (\Delta_{11}) - \nu n^2 R_n (\Delta_{11}) \right];
\]

\[
K_{1,2} = (1 - \nu) \left[ R_n (\Delta_{21}^2 \Delta_{21} + 2 \nu R_n' (\Delta_{21}) - \nu n^2 R_n (\Delta_{21}) \right];
\]

\[
K_{1,3} = n(1 - \nu) \left[ R_n (\Delta_{31}^2 \Delta_{31} - R_n (\Delta_{31})) \right];
\]

\[
K_{1,4} = (1 - \nu) \left[ S_n (\Delta_{11}^2 \Delta_{11} + 2 \nu S_n' (\Delta_{11}) - \nu n^2 S_n (\Delta_{11}) \right];
\]

\[
K_{1,5} = (1 - \nu) \left[ S_n (\Delta_{21}^2 \Delta_{21} + 2 \nu S_n' (\Delta_{21}) - \nu n^2 S_n (\Delta_{21}) \right];
\]

\[
K_{1,6} = n(1 - \nu) \left[ R_n (\Delta_{31}^2 \Delta_{31} - S_n (\Delta_{31})) \right];
\]

\[
K_{1,7} = K_{1,8} = K_{1,9} = K_{1,10} = K_{1,11} = K_{1,12} = 0;
\]

\[
K_{2,1} = 2n(1 - \nu) \left[ R_n (\Delta_{11}^2 \Delta_{11} + R_n (\Delta_{11})) \right];
\]

\[
K_{2,2} = 2n(1 - \nu) \left[ R_n (\Delta_{21}^2 \Delta_{21} + R_n (\Delta_{21})) \right];
\]

\[
K_{2,3} = -n^2 R_n (\Delta_{31}^2 \Delta_{31} + R_n (\Delta_{31}) - \Delta_{31}^2 R_n (\Delta_{31}));
\]
\[ K_{2,4} = 2n (\sigma_{11} - 1) \left[ -\Delta_{11} S_n' (\Delta_{11}) + S_n (\Delta_{11}) \right]; \]
\[ K_{2,5} = 2n (\sigma_{21} - 1) \left[ -\Delta_{21} S_n' (\Delta_{21}) + S_n (\Delta_{21}) \right]; \]
\[ K_{2,6} = -n^2 S_n (\Delta_{31}) + \Delta_{31} S_n' (\Delta_{31}) - \Delta_{31}^2 S_n'' (\Delta_{31}) ; \]
\[ K_{2,7} = K_{2,8} = K_{2,9} = K_{2,10} = K_{2,11} = K_{2,12} = 0; \]
\[ K_{3,1} = \left[ \sigma_{11} \Delta_{11} R_n' (\Delta_{11}) \right]; \]
\[ K_{3,2} = \left[ \sigma_{21} \Delta_{21} R_n' (\Delta_{21}) \right]; \]
\[ K_{3,3} = [nR_n (\Delta_{31})]; \]
\[ K_{3,4} = \left[ \sigma_{11} \Delta_{11} S_n' (\Delta_{11}) \right]; \]
\[ K_{3,5} = \left[ \sigma_{21} \Delta_{21} S_n' (\Delta_{21}) \right]; \]
\[ K_{3,6} = [nS_n (\Delta_{31})]; \]
\[ K_{3,7} = K_{3,8} = K_{3,9} = K_{3,10} = K_{3,11} = K_{3,12} = 0; \]
\[ K_{4,1} = \frac{D_1 (\sigma_{11} - 1)}{R} \left[ \Delta_{11}^2 R_n'' (\Delta_{11} \chi_1) + \frac{\nu}{\chi_1} \Delta_{11} R_n' (\Delta_{11} \chi_1) - \frac{\nu n^2}{\chi_1^2} R_n (\Delta_{11} \chi_1) \right]; \]
\[ K_{4,2} = \frac{D_1 (\sigma_{21} - 1)}{R} \left[ \Delta_{21}^2 R_n'' (\Delta_{21} \chi_1) + \frac{\nu}{\chi_1} \Delta_{21} R_n' (\Delta_{21} \chi_1) - \frac{\nu n^2}{\chi_1^2} R_n (\Delta_{21} \chi_1) \right]; \]
\[ K_{4,3} = \frac{D_1 (\sigma_{21} - 1)}{R} \left[ \Delta_{21}^2 R_n'' (\Delta_{21} \chi_1) + \frac{\nu}{\chi_1} \Delta_{21} R_n' (\Delta_{21} \chi_1) - \frac{\nu n^2}{\chi_1^2} R_n (\Delta_{21} \chi_1) \right]; \]
\[ K_{4,4} = \frac{D_1 (\sigma_{11} - 1)}{R} \left[ \Delta_{11}^2 S_n'' (\Delta_{11} \chi_1) + \frac{\nu}{\chi_1} \Delta_{11} S_n' (\Delta_{11} \chi_1) - \frac{\nu n^2}{\chi_1^2} S_n (\Delta_{11} \chi_1) \right]; \]
\[ K_{4,5} = \frac{D_1 (\sigma_{21} - 1)}{R} \left[ \Delta_{21}^2 S_n'' (\Delta_{21} \chi_1) + \frac{\nu}{\chi_1} \Delta_{21} S_n' (\Delta_{21} \chi_1) - \frac{\nu n^2}{\chi_1^2} S_n (\Delta_{21} \chi_1) \right]; \]
\[ K_{4,6} = \frac{D_1 n}{R \chi_1^2} \left[ (1 - \nu) \chi_1 \Delta_{31} S_n' (\Delta_{31} \chi_1) - (1 - \nu) S_n (\Delta_{31} \chi_1) \right]; \]
\[ K_{4,7} = + K_1 \left[ \sigma_{12} - 1 \right] \Delta_{12} R_n' (\Delta_{12} \chi_1); \]
\[ K_{4,8} = +K_t \left[ (\sigma_{22} - 1) \Delta_{22} R_n' \left( \Delta_{22}\chi_1 \right) \right]; \]
\[ K_{4,9} = +K_t \left[ \frac{n}{\chi_1} R_n \left( \Delta_{32}\chi_1 \right) \right]; \]
\[ K_{4,10} = +K_t \left[ (\sigma_{12} - 1) \Delta_{12} S_n' \left( \Delta_{12}\chi_1 \right) \right]; \]
\[ K_{4,11} = +K_t \left[ (\sigma_{22} - 1) \Delta_{22} S_n' \left( \Delta_{22}\chi_1 \right) \right]; \]
\[ K_{4,12} = +K_t \left[ \frac{n}{\chi_1} S_n \left( \Delta_{32}\chi_1 \right) \right]; \]
\[ K_{5,1} = \kappa^2 G h_1 \left[ \sigma_{11} \Delta_{11} R_n' \left( \Delta_{11}\chi_1 \right) \right]; \]
\[ K_{5,2} = \kappa^2 G h_1 \left[ \sigma_{21} \Delta_{21} R_n' \left( \Delta_{21}\chi_1 \right) \right]; \]
\[ K_{5,3} = \kappa^2 G h_1 \left[ \frac{n}{\chi_1} R_n \left( \Delta_{31}\chi_1 \right) \right]; \]
\[ K_{5,4} = \kappa^2 G h_1 \left[ \sigma_{11} \Delta_{11} S_n' \left( \Delta_{11}\chi_1 \right) \right]; \]
\[ K_{5,5} = \kappa^2 G h_1 \left[ \sigma_{11} \Delta_{21} S_n' \left( \Delta_{21}\chi_1 \right) \right]; \]
\[ K_{5,6} = \kappa^2 G h_1 \left[ \frac{n}{\chi_1} S_n \left( \Delta_{31}\chi_1 \right) \right]; \]
\[ K_{5,7} = \kappa^2 G h_2 \left[ \sigma_{12} \Delta_{12} R_n' \left( \Delta_{12}\chi_1 \right) \right]; \]
\[ K_{5,8} = \kappa^2 G h_2 \left[ \sigma_{22} \Delta_{22} R_n' \left( \Delta_{22}\chi_1 \right) \right]; \]
\[ K_{5,9} = \kappa^2 G h_2 \left[ \frac{n}{\chi_1} R_n \left( \Delta_{32}\chi_1 \right) \right]; \]
\[ K_{5,10} = \kappa^2 G h_2 \left[ \sigma_{12} \Delta_{12} S_n' \left( \Delta_{12}\chi_1 \right) \right]; \]

\[ K_{5,11} = \kappa^2 G h_2 \left[ \sigma_{12} \Delta_{22} S_n' \left( \Delta_{22}\chi_1 \right) \right]; \]
\[ K_{5,12} = \kappa^2 G h_2 \left[ \frac{n}{\chi_1} S_n \left( \Delta_{32}\chi_1 \right) \right]; \]
\[ K_{6,1} = -K_t \left[ \Delta_{11} \left( \sigma_{11} - 1 \right) R_n' \left( \Delta_{11}\chi_1 \right) \right]; \]
\[ K_{6,2} = -K_t \left[ \left( \sigma_{21} - 1 \right) \Delta_{21} R_n' \left( \Delta_{21}\chi_1 \right) \right]; \]
\[ K_{6,3} = -K_t \left[ \frac{n}{\chi_1} R_n \left( \Delta_{31}\chi_1 \right) \right]; \]
\[ K_{6,4} = -K_t \left[ \left( \sigma_{11} - 1 \right) \Delta_{11} S_n' \left( \Delta_{11}\chi_1 \right) \right]; \]
\[ K_{6,5} = -K_t \left[ \left( \sigma_{21} - 1 \right) \Delta_{21} S_n' \left( \Delta_{21}\chi_1 \right) \right]; \]
\[ K_{6,6} = -K_t \left[ \frac{n}{\chi_1} S_n \left( \Delta_{31}\chi_1 \right) \right]; \]
\[ K_{6,7} = \frac{D_2 (\sigma_{12} - 1)}{R} \left[ \Delta_{12} \Delta_{22} R_n'' \left( \Delta_{22}\chi_1 \right) + \frac{v}{\chi_1^2} \Delta_{12} R_n' \left( \Delta_{12}\chi_1 \right) - \frac{v n^2}{\chi_1^4} R_n \left( \Delta_{12}\chi_1 \right) \right] + \]
\[ K_t \left[ \sigma_{12} - 1 \right] \Delta_{12} R_n' \left( \Delta_{12}\chi_1 \right); \]
\[ K_{6,8} = \frac{D_2 (\sigma_{22} - 1)}{R} \left[ \Delta_{22} \Delta_{22} R_n'' \left( \Delta_{22}\chi_1 \right) + \frac{v}{\chi_1^2} \Delta_{22} R_n' \left( \Delta_{22}\chi_1 \right) - \frac{v n^2}{\chi_1^4} R_n \left( \Delta_{22}\chi_1 \right) \right] + \]
\[ K_t \left[ \sigma_{22} - 1 \right] \Delta_{22} R_n' \left( \Delta_{22}\chi_1 \right); \]
\[ K_{6,9} = \frac{D_2 n (1 - v)}{R^2 \chi_x} \left( \chi_1 \Delta_{32} R_n' \left( \Delta_{32} \chi_1 \right) - \frac{n^2 \chi_1^2}{\chi_x^2} S_n \left( \Delta_{32} \chi_1 \right) \right) \]

\[ R_n \left( \Delta_{32} \chi_1 \right) + K_t \left[ \frac{n}{\chi_x} R_n \left( \Delta_{32} \chi_1 \right) \right] ; \]

\[ K_{6,10} = \frac{D_2 (\sigma_{12} - 1)}{R} \left[ \frac{\chi_1}{\chi_x^2} \Delta_{32} S_n'' \left( \Delta_{32} \chi_1 \right) + \frac{\chi_1}{\chi_x^2} S_n \left( \Delta_{32} \chi_1 \right) \right] + \frac{v}{\chi_x} \Delta_{32} S_n' \left( \Delta_{32} \chi_1 \right) - \frac{v n^2}{\chi_x^2} S_n \left( \Delta_{32} \chi_1 \right) + K_t \left[ (\sigma_{12} - 1) \Delta_{32} S_n' \left( \Delta_{32} \chi_1 \right) \right] ; \]

\[ K_{6,11} = \frac{D_2 (\sigma_{22} - 1)}{R} \left[ \frac{\chi_1}{\chi_x^2} \Delta_{22} S_n'' \left( \Delta_{32} \chi_1 \right) + \frac{\chi_1}{\chi_x^2} S_n \left( \Delta_{32} \chi_1 \right) \right] + \frac{v}{\chi_x} \Delta_{22} S_n' \left( \Delta_{22} \chi_1 \right) - \frac{v n^2}{\chi_x^2} S_n \left( \Delta_{22} \chi_1 \right) + K_t \left[ (\sigma_{22} - 1) \Delta_{22} S_n' \left( \Delta_{22} \chi_1 \right) \right] ; \]

\[ K_{6,12} = \frac{D_2 n (1 - v)}{R^2 \chi_x} \left[ \chi_1 \Delta_{32} S_n' \left( \Delta_{22} \chi_1 \right) - \frac{n^2 \chi_1^2}{\chi_x^2} R_n \left( \Delta_{32} \chi_1 \right) \right] + S_n \left( \Delta_{32} \chi_1 \right) + K_t \left[ \frac{n}{\chi_x} S_n \left( \Delta_{32} \chi_1 \right) \right] ; \]

\[ K_{7,1} = \frac{D_1}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{11} - 1) \left[ -\chi_1 \Delta_{31} R_n' \left( \Delta_{31} \chi_1 \right) + R_n \left( \Delta_{31} \chi_1 \right) \right] ; \]

\[ K_{7,2} = \frac{D_1}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{21} - 1) \left[ -\chi_1 \Delta_{21} R_n' \left( \Delta_{21} \chi_1 \right) + R_n \left( \Delta_{21} \chi_1 \right) \right] ; \]

\[ K_{7,3} = \frac{D_1}{R} \left( \frac{1 - v}{2} \right) \frac{-n^2}{\chi_x^2} R_n \left( \Delta_{31} \chi_1 \right) + \frac{1}{\chi_x} \Delta_{31} R_n' \left( \Delta_{31} \chi_1 \right) - \frac{1}{\chi_x} \Delta_{31} S_n' \left( \Delta_{31} \chi_1 \right) \right] ; \]

\[ K_{7,4} = \frac{D_1}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{11} - 1) \left[ -\chi_1 \Delta_{31} S_n' \left( \Delta_{31} \chi_1 \right) + S_n \left( \Delta_{31} \chi_1 \right) \right] ; \]

\[ K_{7,5} = \frac{D_1}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{21} - 1) \left[ -\chi_1 \Delta_{21} S_n' \left( \Delta_{21} \chi_1 \right) + S_n \left( \Delta_{21} \chi_1 \right) \right] ; \]

\[ K_{7,6} = \frac{D_1}{R} \left( \frac{1 - v}{2} \right) \frac{-n^2}{\chi_x^2} S_n \left( \Delta_{31} \chi_1 \right) + \frac{1}{\chi_x} \Delta_{31} S_n' \left( \Delta_{31} \chi_1 \right) - \frac{1}{\chi_x} \Delta_{31} S_n'' \left( \Delta_{31} \chi_1 \right) \right] ; \]

\[ K_{7,7} = \frac{D_2}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{12} - 1) \left[ -\chi_1 \Delta_{32} R_n' \left( \Delta_{32} \chi_1 \right) + R_n \left( \Delta_{32} \chi_1 \right) \right] ; \]

\[ K_{7,8} = \frac{D_2}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{22} - 1) \left[ -\chi_1 \Delta_{22} R_n' \left( \Delta_{22} \chi_1 \right) + R_n \left( \Delta_{22} \chi_1 \right) \right] ; \]

\[ K_{7,9} = \frac{D_2}{R} \left( \frac{1 - v}{2} \right) \frac{-n^2}{\chi_x^2} R_n \left( \Delta_{32} \chi_1 \right) + \frac{1}{\chi_x} \Delta_{32} R_n' \left( \Delta_{32} \chi_1 \right) - \frac{1}{\chi_x} \Delta_{32} S_n'' \left( \Delta_{32} \chi_1 \right) \right] ; \]

\[ K_{7,10} = - \frac{D_2}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{12} - 1) \left[ -\chi_1 \Delta_{12} S_n' \left( \Delta_{12} \chi_1 \right) + R_n \left( \Delta_{12} \chi_1 \right) \right] ; \]

\[ K_{7,11} = - \frac{D_2}{R} \left( \frac{1 - v}{2} \right) \frac{2 n}{\chi_x^2} (\sigma_{22} - 1) \left[ -\chi_1 \Delta_{22} S_n' \left( \Delta_{22} \chi_1 \right) + S_n \left( \Delta_{22} \chi_1 \right) \right] ; \]

\[ K_{7,12} = - \frac{D_2}{R} \left( \frac{1 - v}{2} \right) \frac{-n^2}{\chi_x^2} S_n \left( \Delta_{32} \chi_1 \right) + \frac{1}{\chi_x} \Delta_{32} S_n' \left( \Delta_{32} \chi_1 \right) - \frac{1}{\chi_x} \Delta_{32} S_n'' \left( \Delta_{32} \chi_1 \right) \right] ; \]
\[ K_{12,12} = \Delta_{32} S_n \left( \Delta_{32} \chi \right). \]

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Study of Static and Dynamic Stability of an Exponentially Tapered Revolving Beam Exposed to a Variable Temperature Grade under Axial Loading

Rakesh Ranjan Chand, Pravat Kumar Behera, Madhusmita Pradhan and Pusparaj Dash

Department of mechanical Engineering, VSSUT, Burla, Odisha, India, 768018.

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This research work is concerned with the static and dynamic stability study of an exponentially tapered revolving beam having a circular cross-section exposed to an axial live excitation and a variable temperature grade. The stability is analysed for clamped-clamped, clamped-pinned, and pinned-pinned end arrangements. Hamilton’s principle is used to develop the equation of motion and accompanying end conditions. Then, the non-dimensional form of the equation of motion and the end conditions are found. Galerkin’s process is used to find a number of Hill’s equations from the non-dimensional equations. The parametric instability regions are acquired by means of the Saito-Otomi conditions. The consequences of the variation parameter, revolution speed, temperature grade, and hub radius on the instability regions are examined for both static and dynamic load case and represented by a number of plots. The legitimacy of the results is tested by plotting different graphs between displacement and time using the Runge-Kutta fourth-order method. The results divulge that the stability is increased by increasing the revolution speed; however, an increase in the variation parameter leads to destabilization in the system and for same parameters, the stability is less in the case of a variable temperature grade than that of a constant temperature grade condition.

1. INTRODUCTION

Machine components like steam turbine blades, compressor vanes, helicopter blades, propeller of aircrafts, manipulator of different robots, horn antennas, etc. rotate about a certain axis. Some of these components function at an elevated temperature condition. Therefore, the investigation of static and dynamic stability of revolving beam subjected to temperature grade is very crucial for the above-mentioned real-world applications.

Generous numbers of research articles are concerned with the investigation of revolving beams. Boyce, Diprima, and Handleman used the Rayleigh-Ritz and Southwell processes to define the higher and lower constraints of natural frequencies and concluded that the relationship between the frequency and hub radius is almost linear for different revolution speeds. Niordson and Horway investigated the effects of hinged ends on the natural frequencies of revolving blades. Schillhansl used a sequential guessestimate to find out the strengthening influence of the centrifugal forces on the frequencies of certain revolving blades of a cantilever type. The first three modes of a revolving beam were found by Yntema using the bending mode of non-rotating beam. Influence of revolving speed, disc diameter, and the stagger angle of a blade was studied by Rao by using Galerkin’s method. The nonlinear response of a cantilever blade was determined by Rao and Carnegie using the Rayleigh-Ritz energy formulation to solve the differential equation formed by Carnegie. The formulation of the fundamental frequency of certain narrowing cantilever beams was done by Rao with the help of Galerkin’s process. Transverse vibration of a cantilever beam with a varying cross-section was studied by Ward, Nicholson, Wrinch, Wang, and Meyer. Taylor proposed the power series solution to the blade natural frequencies for a uniform and a completely tapered beam. Rao and Carnegie used computer programming to get first three frequencies of a certain tapered beam of a cantilever type by using Galerkin’s process. Rao, Belgaumkar, and Carnegie took the help of the collocation method to find out the fundamental mode of tapered cantilever under torsional vibration. The fundamental frequencies and mode shapes of certain revolving beam having end weight for various revolving speeds, disc diameter, and setting direction was investigated by Bhat using the Rayleigh-Ritz process. Non-uniform beams with a restrained base were studied by Liu and Yeh. Kar and Sujata investigated the dynamic stability of a revolving non-uniform beam with different end conditions for ordinary, as well as pre-twisted cases. Dwivedy studied the parametric instabilities of a sandwich beam for different end conditions with the help of Hsu’s method and concluded that the shear factor, core loss parameter, and the proportion of thickness influence the instability regions. Parida and Dash investigated the instability regions for a rotating tapered beam having circular cross section under an axial pulsating load and a constant temperature grade.

The literature assessment divulges that selected study has been conducted on the parametric instability of symmetric revolving beams, non-uniform beams with constant temperature grade, and sandwich beams with several end situations. Nevertheless, no research has been conducted on the study of the stability of an exponentially tapered revolving beam having a circular cross section imperilled to an axial live load at one end for different edge conditions. A steady flow of heat is considered through the beam, so the temperature will vary from point to point along the length of the beam. Therefore, a variable temperature grade is considered along the length of the beam, which is more appropriate than the constant temperature grade as was considered by the previous authors. Research is con-
ducted on this type of beam because it will be economical, as less quantity of material will be required due to the exponentially varying profile and will be more efficient because there will be less drag force due to the curved profile.

2. SYSTEM MODELLING

2.1. Kinematics and Equation of Motion

As shown diagrammatically in Fig. 1, an exponentially converging beam having a circular cross-section is rotating about Z-axis and capable of vibrate in X-Z plane. The beam had a length l, base diameter D, was attached to a hub of radius $B_0$, which was subjected to an axial live load of $W(t) = W_0 + W_1 \cos(\omega t)$ at $x' = B_0 + l$ along the centroid of the cross-section.

The succeeding assumptions were made for the analysis of the system:

1. The beam material was homogeneous and isotropic;
2. The transverse displacement $\Delta(x, t)$ was small and equal for all points of a cross-section;
3. The beam followed the Euler-Bernoulli beam theory;
4. Longitudinal displacement of the beam was abandoned;
5. Stable-one dimensional variable temperature grade was assumed to be present along the centroidal axis of the beam and the variation of temperature grade along vertical direction was neglected;
6. Extension and rotary inertia influences were insignificant.

The terminologies for total kinetic energy, total potential energy and work done were as follows:

$$T = \frac{1}{2} \int_0^l \rho A(x) \left( \frac{\partial \Delta}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^l \rho A(x) N^2 \Delta^2 \partial x; \quad (1)$$

$$U = \frac{1}{2} \int_0^l E(x) I(x) \left( \frac{\partial^2 \Delta}{\partial t^2} \right)^2 dx + \frac{1}{2} \int_0^l \rho A(x) N^2 (B_0 + x) \left( \int_0^x \left( \frac{\partial \Delta}{\partial x} \right)^2 dx \right) \partial x; \quad (2)$$

$$W = \frac{1}{2} \int_0^l W(t) \left( \frac{\partial \Delta}{\partial x} \right)^2 dx. \quad (3)$$

Hamilton’s principle was used to derive the boundary conditions and equation of motion as follows:

$$\delta \int_{t_1}^{t_2} (T - U + W) = 0. \quad (4)$$

Using the Eqs. (1), (2), and (3) in the Eq. (4), the equation of motion was obtained as:

$$[E(x) I(x) \Delta_{xx}]_{xx} + \rho A(x) \Delta_{tt} + \rho N^2 I(x) \Delta_{xx} - [V(x_1) \Delta_{xx}]_{x} + W(t) \Delta_{xx} = 0. \quad (5)$$

In that case,

$$V(x_1) = \frac{1}{2} \rho A(x) N^2 [B_0 + l]^2 - (B_0 + x)^2. \quad (6)$$

The edge conditions at $x' = B_0$ and $x' = B_0 + l$ were:

$$[E(x) I(x) \Delta_{xx}]_{x} = 0; \quad \text{and} \quad \Delta_{x} = 0 \text{ or } \Delta = 0; \quad (7)$$

where $\Delta_{xx} = \frac{\partial \Delta}{\partial x}, \Delta_{tx} = \frac{\partial^2 \Delta}{\partial x \partial t}, \Delta_{tt} = \frac{\partial^2 \Delta}{\partial t^2}.$

Using the various dimensionless parameters, the dimensionless equation of motion was expressed as:

$$[T(\eta) S(\eta) \zeta''(\eta) + m(\eta) \zeta + \int[r_g N_0^2 + (w(\tau) \zeta''(\eta)] = 0; \quad (8)$$

where:

$$r_g = \frac{r_0(\eta)}{A(\eta)} N_0^2 = \frac{\rho A(x) N^2 l^4}{E l^4},$$

$$N_0^2(\eta) = \frac{V(x_0 l)^2}{E l^4}, \quad I(\eta) = I(\eta),$$

$$E(\eta) = E_0 S(\eta) \text{ and } A(\eta) = A_0 m(\eta).$$

The boundary conditions were:

$$\{ [T(\eta) S(\eta) \zeta''(\eta)] + \int[r_g N_0^2 + (w(\tau) \zeta''(\eta)] \}_{\eta = 1} = 0; \quad \text{or} \quad [T(\eta) S(\eta) \zeta''(\eta)] \}_{\eta = 0} = 0; \quad (9)$$

In order to obtain the Eqs. (8) & (9), the non-dimensional parameters used were:

$$\eta = \frac{x}{l}, \quad \zeta = \frac{x}{l}, \quad \tau = \frac{t}{l};$$

where $c^2 = \frac{E(x) I(x)}{\rho A(x) N^2 l^4}, \quad \frac{\partial \Delta}{\partial x} = \frac{\partial \zeta}{\partial x}, \quad \left( \frac{\partial^2 \Delta}{\partial x \partial t} \right)^2 = \left( \frac{\partial \zeta}{\partial x} \right)^2,$

$$\frac{\partial^2 \Delta}{\partial t^2} = \frac{1}{c^2} \left( \frac{\partial \zeta}{\partial x} \right)^2, \quad \frac{\partial \Delta}{\partial x} = \frac{\partial \zeta}{\partial x}, \quad \left( \frac{\partial \Delta}{\partial x} \right)^2 = \left( \frac{\partial \zeta}{\partial x} \right)^2,$$

$$w(\tau) = \frac{w(t)}{E_0 l^4}, \quad w(t) = w_0 + w_1 \cos(\omega t) \text{ and } b_0 = \frac{B_0}{l}.$$

2.2. Series Solution to the Equation of Motion

The inexact solution could be presumed as:

$$\zeta(\eta, \tau) = \sum_{r=1}^{N} \zeta_r(\eta) s_r(\tau). \quad (10)$$

In this case, we had to choose the function of time $s_r(\tau)$ and the coordinate function $\zeta_r(\eta)$ in such a way that most of the edge conditions in Eq. (9) and the equation of motion would be satisfied. It was further anticipated that $\zeta_r(\eta)$ could be characterized by using a number of functions (Eq. (10)), which must fulfill the conditions acquired from Eq. (8) by cancelling the expressions comprising of $w_0$ and $w(\tau).$ The coordinate function for the different edge conditions was estimated by the expressions specified in the Table 1. 

The subsequent matrix equation of motion in Eq. (11) was obtained by applying the Galerkin’s method and replacing a series of solutions in the non-dimensional equation.

$$[M] \{ \ddot{s} \} + [K] \{ s \} - \begin{pmatrix} w_0[H] \frac{\cos(\tau)}{H} \end{pmatrix} \{ s \} = 0. \quad (11)$$

In the Eq. (11), $\ddot{s} = \frac{\partial^2 s}{\partial \tau^2}$, $s = \{ s_1, s_2, ..., s_n \}^T$ and the different matrix components could be written as:

$$M_{ij} = \int_0^1 m(\eta) \zeta_i(\eta) \zeta_j(\eta) d\eta;$$

$$K_{ij} = \int_0^1 \left[ T(\eta) S(\eta) \zeta_i(\eta) \zeta_j(\eta) \right] d\eta;$$

$$H_{ij} = \int_0^1 \left\{ \zeta_i(\eta) \zeta_j(\eta) \right\} d\eta \quad \text{for } i, j = 1, 2, 3, ..., N.$$
2.3. Formulation for Static Buckling Load

By substituting \( \{ \tilde{s} \} = \{ 0 \} \) and \( w_1 = 0 \) in Eq. (11), we had an eigenvalue problem of:

\[
[K]^{-1}[H]\{ s \} = \frac{1}{w_0}\{ s \}. \tag{12}
\]

The static buckling loads \( w_0 \) for the first few modes were found as the real parts of the reciprocal of the eigenvalues of \([K]^{-1}[H]\). Then a number of graphs were plotted between \( w_0 \) and the rotational speed, taper parameter, and temperature grade.

2.4. Formulation for Dynamic Instability Regions

By considering \([L]\) to be the modal matrix of \([M]^{-1}[K]\), and by introducing the linear coordinate conversion \( \{ s \} = [L]\{ u \} \), we had:

\[
\{ \ddot{u} \} + [f_2^0]\{ u \} + w_1 \cos(\bar{f}_\tau) [B]\{ u \} = \{ 0 \}. \tag{13}
\]

In this case, \( \{ u \} \) was a number of new generalized coordinates, \( [f_2^0] \) was a special matrix analogous to \([M]^{-1}[K]\) and \([B] = -[L]^{-1}[M]^{-1}[H][L]\). The Eq. (13) could be expressed as:

\[
\ddot{u}_n + f_{2n}^0 + w_1 \cos(\bar{f}_\tau) \sum_{m=1}^{N} b_{mn}u_m = 0; \tag{14}
\]

where \( n = 1, 2, 3, \ldots, N \).

The Eq. (14) characterized a system of \( N \) that combined Hill’s equations with complex factors. The complex terms are:

\[
f_n = f_{n,R} + jf_{n,I};
\]

\[
b_{n,m} = b_{n,m,R} + jb_{n,m,I}.
\]

Using the Saito-Otomi conditions, the constraints of the instability regions for simple and combination resonances were obtained for an undamped case.\(^{27,28}\)

1. Simple resonance

\[
\frac{\bar{T}}{2} - \bar{T}_{\mu,R} < 1 \left| \frac{w_1 b_{\mu,R}}{\bar{T}_{\mu,R}} \right|; \tag{15}
\]

2. Combination resonance of sum type (\( \nu \neq \mu \))

\[
\left| \frac{\bar{T}}{2} - \frac{1}{2} \left( \bar{T}_{\mu,R} + \bar{T}_{\nu,R} \right) \right| < \frac{w_1}{4} \sqrt{\frac{b_{\mu,R}b_{\nu,R}}{\bar{T}_{\mu,R}\bar{T}_{\nu,R}}}; \tag{16}
\]

3. Combination resonance of subtraction type

\[
\left| \frac{\bar{T}}{2} - \frac{1}{2} \left( \bar{T}_{\nu,R} - \bar{T}_{\mu,R} \right) \right| < \frac{w_1}{4} \sqrt{\frac{-b_{\mu,R}b_{\nu,R}}{\bar{T}_{\mu,R}\bar{T}_{\nu,R}}}. \tag{17}
\]

The system stability was investigated for the various boundary conditions with the help of Eqs. (15), (16), and (17). Then the regions of instability were analysed.

3. RESULTS AND OBSERVATION

The diameter of the exponentially tapered beam having circular cross-section is supposed to change, agreeing to the expression \( d(\eta) = D(e^{-\beta\eta}) \). In this case, \( D \) is base diameter and \( \beta^* \) is variation parameter. Subsequently, the mass distribution is \( m(\eta) = e^{-2\beta^*\eta} \) and \( A(\eta) = A + m(\eta) \). The temperature variation is supposed to follow the relation \( \delta(\eta) = \delta(1 - \eta) \). Considering \( \delta \) as the reference temperature, which is the temperature at the base, variation of the modulus of elasticity will follow the relation, \( E(\eta) = E[1 - \psi(\eta)(1 - \eta)] \), \( E(\eta) = E \times S(\eta), 0 \leq \alpha \delta \leq 1 \), where \( \psi = \alpha \delta \) is the thermal grade parameter and \( \psi(\eta) = \psi/e^{-2\beta^*\eta} \). Numerical results are acquired for different values of the parameters like revolution speed parameter, temperature grade, variation parameter and hub radius.
If a certain change in the values of the parameter leads to narrowing the instability regions shifts the regions towards the higher values of excitation frequencies or reduces the number of instability regions, then it can be concluded that the stability of system has improved. Otherwise, the stability is reduced.

3.1. Dynamic Stability Plot

In order to avoid the clumsiness of the figures, only two values of parameters are shown, but the trend remains same for other values of the parameters.

The increase in the variation parameter moves the instability areas of a beam with clamped-pinned (C-P), clamped-clamped (C-C), and pinned-pinned (P-P) conditions toward a lower frequency region, as shown in Figs. 2, 3, and 4.

For all the three end arrangements, a surge in the revolving speed parameter reduces the width of the instability areas significantly and moves those to a higher excitation frequency ($\bar{f}$) region, as shown in Figs. 5, 6, and 7.

For all three end conditions, it is found that the instability of the system is more in the case of a variable temperature grade in comparison to a constant temperature grade, as shown in Figs. 8, 9, and 10.

Further in the dynamic analysis of the system, it is found that the stability of the system is more in a C-C case than that of a C-P and P-P case for all the parameters considered.

Change in the hub radius has no effect on the instability regions for all three edge conditions because the change in the hub radius does not affect the centre of gravity of the hub, which always lies on the central axis of revolution and hence, the stabilizing centrifugal force is mainly due to the rotating beam.

3.2. Static Stability Plot

It is observed that a surge in the revolving speed parameter increases the static load factor, as shown in Fig. 11.

The rise in the variation parameter as well as the thermal grade parameter decreases the static load factor for all three situations, as shown in Figs. 12 and 13.

A number of graphs were plotted for different edge arrangements and different modes of frequency by solving Eq. (13)
4. CONCLUSIONS

The static and dynamic stability of an exponentially tapered revolving beam having a circular cross-section is exposed to an axial live excitation and constant as well as a variable temperature grade under several edge arrangements has been investigated computationally by developing a MATLAB code. The instability areas are analysed using the Runge-Kutta fourth-order method, which certified the results to be true. The investigation leads to the following conclusions.

Figure 6. Stability plot for C-P beam with $D = 5, L = 40, B_0 = 1, \beta^* = 0.5$.

Figure 7. Stability plot for P-P beam with $D = 5, L = 40, B_0 = 1, \beta^* = 0.5$.

Figure 8. Stability plot for C-C beam with $D = 5, L = 40, B_0 = 1, \beta^* = 0.5$ and $N_0 = 1$.

Figure 9. Stability plot for C-P beam with $D = 5, L = 40, B_0 = 1, \beta^* = 0.5$ and $N_0 = 1$.

Figure 10. Stability plot for P-P beam with $D = 5, L = 40, B_0 = 1, \beta^* = 0.5$ and $N_0 = 1$.

Figure 14 shows the variation of amplitude with respect to time. When the frequency is chosen from the instability zone (from first mode ($2f_1$) of Fig. 3), the amplitude continues increasing.

When the frequency is chosen from outside the instability area (from first mode ($2f_1$) of Fig. 3), the amplitude continues decreasing, as shown in Figure 15.
For all end arrangements, an increase in the revolving speed parameter makes the beam more stable due to an increase in the centrifugal force which decreases the effect of the external load but the stability of clamped-clamped beam increases more rapidly than clamped-pinned and pinned-clamped beams. The increase in the variation parameter leads to a decrease in the stability of the lower excitation frequency regions less significantly than the higher excitation regions, but the reverse occurs in the case of diverging beams due to an increase of the variation parameter; the rate of decrease of flexural rigidity is less for lower excitation frequency regions and the rate of decrease is more for higher frequency regions. For a certain temperature grade parameter, stability decreases if it is taken as a constant, and again it is further decreases if taken as a variable for a converging taper; this is due to a decrease of stiffness for the variable temperature grade in comparison to a constant temperature grade. It is also concluded that the hub radius has no effect on the dynamic instability. The static load factor increases with an increase in the rotational speed. The increase in the temperature grade along the positive $X$-axis decreases the stiffness of the system and an increase in the variation parameter along a positive $X$-axis decreases the flexural rigidity of the system. Due to these two reasons, the static load factors decrease with an increase in the temperature grade as well as in the variation parameter. This research can be useful for the vibration isolation of a rotating non-uniform beams with a high surrounding temperature and moderate rotational speeds as well as the design of rotor blades with a high strength to weight ratio, by choosing the suitable parameters obtained from the computational analysis.

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Figure 15. Displacement-time plot for C-P edge condition with \( w_1 = 1.8 \) and \( f = 13 \) using RK 4th method.


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Vibration Modelling and Verification for Rotor-Support-Casing Coupling System with Misalignment Fault

Nanfei Wang, Dongxiang Jiang and Yizhou Yang

State Key Laboratory of Control and Simulation of Power System and Generation Equipment, Department of Energy and Power Engineering, Tsinghua University, Beijing 100084, China.

Kamran Behdinan

Department of Mechanical & Industrial Engineering, University of Toronto, Toronto M5S3E3, Canada.

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Misalignment is one of the common malfunctions that occur in rotating machines. Effects of misalignment on the casing vibration response of a rotor-support-casing (RSC) coupling system is investigated in detail. The model of an RSC coupling system is established using the lumped mass method. The coupling effects between the rotor, support and casing are fully considered. A misalignment model is proposed, and equivalent misalignment force is applied on corresponding lumped mass points. Nonlinear factors of bearing, such as the clearance of bearing, oil film force, nonlinear Hertzian contact force, and the varying compliance vibration are developed. The influences of oil film thickness and bearing size are considered in the nonlinear oil film force model. By using a numerical method, the governing equations of the system are solved to obtain the steady-state vibration. The simulation results from a coupling model are compared with the experimental results and the effectiveness of the new model is verified. It has been found that spectrums and orbit plots are effectively used to reveal the unique nature of misalignment faults, leading to reliable misalignment diagnostic information.

1. INTRODUCTION

Rotating machines have been widely used in many industrial fields, such as aero-engines, gas turbines and nuclear reactors. Rotating machines often have different malfunctions. After unbalance, misalignment is accepted as the second most common source of vibration. Misalignment may lead to the excessive maintenance of engines and possible engine failure. Shaft misalignment is a situation in which the shafts of the driving and driven machines are not in the same centreline due to improper machine assembly. It is very difficult to obtain a perfect alignment between two shafts in rotating machines. Even if an accurate alignment is secured, misalignment faults also occur over a period of continuous operations because of either unequal foundation movement or uneven thermal heating in the rotor system. Therefore, both an in-depth study and an accurate knowledge of the vibration characteristics are very helpful for understanding and diagnosing rotor misalignment in order to avoid any failure or damage that may arise.

The misalignment of machinery shafts results in both reaction forces and moments at coupling locations. The misalignment of machinery shafts influences the vibration behaviour of the rotors. The accurate prediction of the vibration response of misaligned rotors strongly depends on the realistic modeling of the misalignment effect. Gibbons first proposed the misalignment reaction forces generated in different types of couplings. Later, many scholars used force equations based on the model proposed by Gibbons. Sekhar and Prabhu revealed the influences of coupling misalignment on rotor vibration by using a numerical method, which demonstrated that an increase in misalignment results in changes to the second and third harmonics of the dynamic responses. Lee developed a model for a flexible coupling-rotor-bearing system that considers the reaction loads from the deformations of rolling elements of bearings and the coupling elements as the misalignment effects, and vibration at twice the rotor speed due to the presence of misalignment.

The other method often employed to simulate a misalignment effect in rotor systems comes from the kinematics of the couplings. Xu and Marangoni derived misalignment force based on the kinematics of the Hooke’s joint and formulated the generalized equations for rotor systems with misalignment. The generalized equations indicate that the forcing frequencies due to shaft misalignment are even multiple frequencies of the motor rotational speed. Al-Hussain confirmed the relationship between the frequency and transient response of the dimensionless by using a numerical misalignment method to solve nonlinear equations. Lees investigated the effect of misalignment on rigidly coupled rotors.

Some researchers pay more attention to the dynamic responses of rotor systems with misalignment faults by using numerical and experimental methods. Bouaziz et al. presented the effect of angular misalignment on the dynamic behavior
of a rotor supported by two hydrodynamic journal bearings by means of analytical and numerical methods. Li et al. investigated the dynamic mechanism of a rotor system connected by the gear coupling and physical properties of faults. Han et al. conducted a study on the misalignment mechanism of a rotor system connected by gear coupling from the prospective of kinematics. Wang investigated the shaft misalignment dynamics and characteristics of vibration malfunction in detail. Bouaziz et al. developed a theoretical model to quantify the effect introduced by the presence of shaft misalignment and investigated the dynamic behaviour of a spatial misaligned rotor mounted on two identical active magnetic bearings (AMBs). Tejas evaluated the influence of misalignment and its type on the forcing characteristics of flexible couplings by means of the experimental investigation of the vibration response of misaligned coupled rotors. Saavedra and Ramirez carried out experimental research on a misalignment rotor system, and revealed a frequency spectrum with a series of harmonics of rotational speeds under misalignment conditions. Tejas developed a rotor model with misalignment faults by using Timoshenko beam elements, and the effects of misalignment are simulated by using nodal force vector. Based on the experimental examination of force-displacement relations for different types of couplings, Saavedra and Ramirez proposed a coupling stiffness matrix and used it in the FE model of a misaligned rotor system. However, even in the absence of a misalignment, the derived stiffness coefficients vary with shaft rotation. Li et al. presented the quantification of uncertainty effects when the shaft misalignment and the mass unbalance existed.

Due to the complex structure of aero-engines and the limitations of computer processing capability, it’s very difficult to establish a detailed whole aero-engine model. For an aero-engine, only acceleration signals on casings can be picked up and it’s very challenging to obtain the displacement signals. The existing literature mainly investigates the effects of misalignment on rotor-bearing systems from the prospective of displacement responses and the casing is not included in the dynamic model. Research on the acceleration response of casings under misalignment conditions are limited. In this case, a new RSC coupling dynamic model using a lumped mass method is proposed. The misalignment model is proposed and the nonlinear bearing model, including oil film force, is considered. The vibration responses of the system are obtained by means of numerical integration method, and vibration characteristics are studied. Finally, the correctness of the proposed model is verified by experimental results.

2. FORMULATION OF MISALIGNMENT EFFECT IN COUPLING SYSTEM

2.1. Introduction to the New Model

Based on the actual structure and size parameters of test rig in the laboratory, a RSC coupling system was introduced, as shown in Figure 1. The RSC coupling system consisted of one disk with mass unbalance and one rotor supported by bearing, which had a large stiffness value to simulate the support structure of a fan in an aero-engine. A flexible coupling located in the right end of the RSC coupling system was employed to connect both the motor and the rotor system. The misalignment fault was simulated in the coupling.

The model of the RSC system was achieved by utilizing five equivalently lumped mass elements. \( M_p \) and \( M_r \) denoted the masses of disks with blades and casing, respectively. \( M_w \) and \( M_b \) were the corresponding lumped masses of bearing outer rings and bearing pedestals, respectively. \( K \) and \( C \) represented the connection stiffness and damping between shaft and disk, respectively. \( C_{rb} \) represented connection damping between the shaft and the bearing. \( K_{fH} \) and \( K_{fV} \) represented the horizontal and vertical elastic support stiffness between the bearing outer-ring and the bearing pedestal, respectively. \( C_{fH} \) and \( C_{fV} \) were the horizontal and vertical damping coefficients between the bearing outer-ring and the bearing pedestal, respectively. \( K_{cH} \) and \( K_{cV} \) represented the horizontal and vertical elastic support stiffness between the casing and the bearing pedestal, respectively. \( C_{cH} \) and \( C_{cV} \) were the horizontal and vertical damping coefficients between the casing and base, respectively. \( O_1 \) and \( O_2 \) were the geometric centre of rotor and centroid of rotor, respectively.

3. THE DYNAMIC MODEL OF MISALIGNMENT

In practice, the coupling of parallel and angular misalignment is one of the most commonly observed faults in rotor systems. Therefore, in order to reveal the fault mechanism, the combination integrated misalignment was considered in detail. The relative motion diagram of coupling housing and two parts under integrated misalignment condition were illustrated, in which \( O \) and \( P \) were the geometric centre of the coupling housing at rest and in circular motion, respectively. As shown in Figure 2, \( O_{01} \) and \( O_{02} \) were the rotation centres of left and right half-couplings, respectively. When integrated misalignment occurred in the coupling used to connect the driving and driven rotor axis, the coupling housing was forced to do circular motion around its centre at the rotating speed equal to the rotating angular speed \( \omega \) of rotor, namely \( d\theta/dt = \omega \). The displacement components of \( P \) in \( x \) and \( y \) directions were obtained as follows:

\[
\begin{align*}
x &= \Delta E \sin \theta \cos \alpha = \\
y &= \Delta E \cos \theta \cos \alpha - \frac{1}{2} \Delta E = \\
\end{align*}
\]

where \( \Delta E \) was the equivalent amount of misalignment, which was determined by the coupling spacing \( \Delta L \), parallel mis-
alignment amount $\Delta y$ and misalignment angle $\Delta \alpha$, $\Delta E = \Delta y + \Delta L \cdot \tan(\Delta \alpha/2)$.

The linear velocity of $P$ was expressed in the following formulas:

$$v_P = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \omega_1 \cdot (\Delta y + \Delta L \cdot \tan(\Delta \alpha/2)).$$ (2)

The angular velocity of $P$ was further written as:

$$\omega_P = \frac{v_p}{\Delta E/2} = 2\omega_1.$$ (3)

The acceleration of $P$ was defined as:

$$a_P = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = -2\omega_1^2 \cdot (\Delta y + \Delta L \cdot \tan(\Delta \alpha/2)).$$ (4)

Let $F_L$ and $F_R$ represented the exciting forces exerted by the left and right half-couplings on the rotor, respectively. The resultant force was $F$, as shown in Figure 2. Based on Newton’s second law of motion and the acceleration of the couplings, the exciting force $F$ was decomposed into two components in $x$ and $y$ directions as follows:

$$\begin{cases} 
F_{xC} = m_0 \cdot (\Delta y + \Delta L \cdot \tan(\Delta \alpha/2)) \cdot \omega_1^2 \cdot \sin(2\omega_1 t); \\
F_{yC} = m_0 \cdot (\Delta y + \Delta L \cdot \tan(\Delta \alpha/2)) \cdot \omega_1^2 \cdot \cos(2\omega_1 t);
\end{cases}$$ (5)

where $m_0$ denoted the mass of the coupling housing.

The misalignment forces changed with the coupling space and parallel misalignment were presented in Figure 3 under different misalignment angles. Figure 3a showed that, when the coupling space was smaller, the misalignment force changed slightly with the increase of the misalignment angle. However, the misalignment force rapidly increased under larger coupling space with the increased misalignment angle. Figure 3b showed that the misalignment force increased with the increasing parallel misalignment, and presented the linear relationship. These changes were explained by Eq. (5).

4. BEARING NONLINEAR MODEL

A schematic map of the rolling-element bearing with its fixed frame of reference and the rotation direction of the rotor is shown in Figure 4. It was assumed that the balls are equispaced between the surfaces of the inner and outer races, and pure rolling occurred between the balls and the surfaces. It was supposed that $v_{out}$ was the tangent velocity of the point of ball contacting the outer race, $v_{in}$ denoted the tangent velocity of the point of ball contacting the inner race, $\omega_{out}$ was the rotational angular velocity of the bearing outer race, $\omega_{in}$ was the rotational angular velocity of the bearing inner race, $R$ was the radius of the outer race, and $r$ was the radius of the inner race.

respectively.\textsuperscript{19} Therefore,

\[ v_{\text{out}} = \omega_{\text{out}} \times R, \quad v_{\text{in}} = \omega_{\text{in}} \times r. \]  

(6)

The tangent velocity of the cage (the centre of the ball) was expressed by \( v_{\text{cage}} = (v_{\text{out}} + v_{\text{in}})/2 \). The outer race of the bearing was assumed to be fixed to the flexible support, namely, \( \omega_{\text{out}} = 0 \), \( \omega_{\text{cage}} = v_{\text{in}}/2 = (\omega_{\text{in}} \times r)/2 \). The angular velocity of the cage was written by:

\[ \omega_{\text{cage}} = \frac{v_{\text{cage}}}{(R + r)/2} = \frac{(\omega_{\text{in}} \times r)/2}{(R + r)/2} = \frac{\omega_{\text{in}} \times r}{(R + r)}. \]  

(7)

The inner race was assumed to be fixed to the shaft and rotate with the shaft, namely \( \omega_{\text{in}} = \omega_{\text{rotor}} \). \( N_b \) was defined as the number of the balls, and bearing ball passing frequency was obtained by:

\[ \omega_{\text{bp}} = \omega_{\text{cage}} \times N_b = \omega_{\text{rotor}} \times \left( \frac{r}{R + r} \times N_b \right) = \omega_{\text{rotor}} \times B_N; \]  

(8)

where \( B_N \) was the ratio of the bearing passing frequency and the rotating frequency. Assuming that the angle location of the \( j \)th ball was \( \theta_j \), which was formulated by \( \theta_j = \omega_{\text{cage}} \times t + \frac{2\pi}{N_b}(j - 1) \), \( j = 1, 2, ..., N_b \), and \( x \) and \( y \) were the separate vibration displacements of the inner centre in \( x \)-axis and \( y \)-axis, and bearing clearance was defined as \( r_0 \). The normal contact deformation between the \( j \)th ball and the raceway was expressed as follows:

\[ \delta_j = x \cos \theta_j + y \sin \theta_j - r_0. \]  

(9)

Based on the Hertzian theory, the rolling contact allowed the contact force \( f_j \) between the ball and the race to be obtained. Since the positive force was generated only when \( \delta_j > 0 \) under Hertzian contact condition, the force was given by using the Heaviside function \( H \) as follows:

\[ f_j = C_b[\delta_j]^2 = C_b(x \cos \theta_j + y \sin \theta_j - r_0)^2 H(x \cos \theta_j + y \sin \theta_j - r_0); \]  

(10)

where, \( C_b \) was the Hertzian contact stiffness determined by contact material and shape. The components of \( f_j \) in the \( x \) and \( y \) directions were:

\[ f_{jx} = f_j \cos \theta_j; \]
\[ f_{jy} = f_j \sin \theta_j. \]  

(11)

Therefore, the bearing reaction was obtained by summing the individual restoring force generated by bearings:

\[ F_{x1} = \sum_{j=1}^{N_b} f_{jx} = \sum_{j=1}^{N_b} f_j \cos \theta_j; \]
\[ F_{y1} = \sum_{j=1}^{N_b} f_{jy} = \sum_{j=1}^{N_b} f_j \sin \theta_j. \]  

(12)

In this paper, the nonlinear oil film force model was adopted under the assumption of short bearing with Capone.\textsuperscript{20} The results indicate that the model has better accuracy and convergence. In short bearing assumptions, dimensionless Reynolds
\[
\left( \frac{r}{L} \right)^2 \frac{\delta}{\delta z} \left( \frac{h^3 \delta p}{\delta z} \right) = x \sin \theta - y \cos \theta - 2(\dot{x} \cos \theta + \dot{y} \sin \theta).
\] (13)

The dimensionless pressure \( p \) was obtained by integration Eq. (13):
\[
p = \frac{1}{2} \left( \frac{L}{D} \right)^2 \frac{(x - 2\dot{y}) \sin \theta - (y + 2\dot{x}) \cos \theta}{(1 - x \cos \theta - y \sin \theta)^3}(4z^2 - 1).
\] (14)

The final expression for the nonlinear oil film force was:
\[
\begin{align*}
\begin{cases}
    f_x = -A \times \left( 3xV(x, y, \alpha) - 3yV(x, y, \alpha) + G(x, y, \alpha) \sin \alpha - 2S(x, y, \alpha) \cos \alpha \right), \\
    f_y = -A \times \left( 3yV(x, y, \alpha) + 3xV(x, y, \alpha) + G(x, y, \alpha) \cos \alpha - 2S(x, y, \alpha) \sin \alpha \right)\
\end{cases}
\end{align*}
\] (15)

where \( f_x \) and \( f_y \) were the dimensionless nonlinear oil film force. Their expressions were \( f_x = F_{x2}/\sigma \) and \( f_y = F_{y2}/\sigma \). \( F_{x2} \) and \( F_{y2} \) were the oil film force of the bearing in direction \( x \) and \( y \) respectively, where \( \sigma = \mu \omega L (r/c)^2 (2r)^2 \). \( V(x, y, \alpha), S(x, y, \alpha), A, G(x, y, \alpha) \) and \( \alpha \) were expressed as follows:
\[
V(x, y, \alpha) = \frac{2 + (y \cos \alpha - x \sin \alpha)G(x, y, \alpha)}{1 - x^2 - y^2};
\] (16)
\[
S(x, y, \alpha) = \frac{x \cos \alpha + y \sin \alpha}{1 - (x \cos \alpha + y \sin \alpha)^2};
\] (17)
\[
A = \frac{[(x - 2\dot{y})^2 + (y + 2\dot{x})^2]^\frac{3}{2}}{1 - x^2 - y^2};
\] (18)
\[
G(x, y, \alpha) = \frac{2}{(1 - x^2 - y^2)^\frac{3}{2}} \times \left[ \frac{\pi}{2} + \arctan \left( \frac{y \cos \alpha - x \sin \alpha}{(1 - x^2 - y^2)^\frac{1}{2}} \right) \right];
\] (19)
\[
\alpha = \arctan \left( \frac{y + 2\dot{x}}{x - 2\dot{y}} - \frac{\pi}{2} \frac{y + 2\dot{x}}{x - 2\dot{y}} \right) - \frac{\pi}{2} \cdot \text{sign}(y + 2\dot{x});
\] (20)

where \( c \) represented the oil film thickness, \( L \) was the bearing length, and \( \mu \) represented the lubricant’s density.

5. EQUATIONS OF MOTION OF MISALIGNED COUPLING SYSTEM

Based on the above analysis and the mechanical model of the rotor-support-casing coupling system exhibited in Figure 1, utilizing the D’Alembert principle, the system’s governing equations of motion under misalignment and initial unbalance in the Cartesian coordinates were written as:
\[
\begin{align*}
M_{rp}\ddot{x}_{rp} + K(x_{rp} - x_r) + C\dot{x}_{rp} &= 0; \\
M_{rp}\ddot{y}_{rp} + K(y_{rp} - y_r) + C\dot{y}_{rp} &= 0; \\
M_{b}\ddot{x}_b + K_{fH}(x_b - x_c) + C_{fH}(\dot{x}_b - \dot{x}_c) + K_{fV}(y_b - y_c) + C_{fV}(\dot{y}_b - \dot{y}_c) &= 0; \\
M_{b}\ddot{y}_b + K_{fV}(y_b - y_c) + C_{fV}(\dot{y}_b - \dot{y}_c) &= 0.
\end{align*}
\] (21)

where \( F_{x1} \) and \( F_{y1} \) were the nonlinear bearing restoring reaction forces of contact layer in the \( x \)-direction and \( y \)-direction, \( F_{x2} \) and \( F_{y2} \) were the nonlinear oil-film forces in the \( x \)-direction and \( y \)-direction, \( F_{xC} \) and \( F_{yC} \) were the misalignment forces in the \( x \)-direction and \( y \)-direction.

The above motion equations of the rotor system were expressed in following matrix form:
\[
M \ddot{X} + C \dot{X} + K X = F(t);
\] (22)

where \( X = \{x_{rp}, y_{rp}, ..., x_r, y_r, x_c, y_c\} \) represented the displacement vector corresponding to each mass. \( M, C \) and \( K \) were the mass, damping and stiffness matrices, respectively. \( F(t) \) comprised unbalance excitation force, bearing force, forces due to fault in rotor system and other external forces.

6. THE RESULTS AND DISCUSSION

6.1. Numerical Simulation of Misalignment

An RSC coupling system, as illustrated in Figure 1, has been taken into account for analysis. The parameters employed to describe the model are given in Table 1. The main parameters of bearing are listed in Table 2. The Runge-Kutta numerical method is used to calculate the casing dynamic response of the coupling system with misalignment fault. Based on calculation and operation conditions, the rotating speed should be higher than the first order critical speed and is set to 35 Hz for this analysis.21–23 The steady-state responses of the casing for an RSC coupling system with different mass unbalance and misalignment, including acceleration time waveforms, orbits and corresponding spectrums, are shown in Figures 5–7, respectively.
It can be clearly seen from Figure 5a and Figure 6a that the vibration amplitude increases due to the enlargement of unbalanced mass, and some vibration waveforms with small amplitudes and high frequencies. Due to the effects of gravity and different equivalent support stiffness, the orbits present ellipse shape, as illustrated in Figure 5b and Figure 6b. Figure 5c and Figure 6c show that the amplitude of rotating frequency components are most prominent under the excitation of unbalanced mass and that some small peaks of \( n \times \) rotating frequency components are also excited in the spectrums. Figure 7a and 7b explicitly show that the amplitudes increase and that the orbits present complicated shapes under misalignment faults. Figure 7c shows that, due to the coupling effect of rotor-support-casing system, the spectrum contains more frequency components, including some \( n \times \) running speed components in the range of \([0, 350]\) Hz, which is in accordance with the results of literature.

### 6.2. Experiment Facilities

In order to verify the effectiveness of the proposed rotor-support-casing model, a rotor-disk-bearing-casing test rig is designed and manufactured. The test rig is composed of key parts, such as disk, shaft, bearing and motor. When compared with an actual aero-engine, the test rig is simplified and has the following several features: (1) the material properties of casing are consistent with those of actual compressor casings; (2) the multistage compressor is simplified as a single-stage disk structure; (3) in order to adjust the system dynamic characteristics, the support structure is simplified as bearing with oil film force; (4) 12 straight blades are used and lateral installed equidistantly on the disk, which is mainly to reduce the influence of air resistance and highlight the dynamic characteristics of the mechanical structure. The acceleration transducers are mounted on the casing to pick up the acceleration vibration signals in horizontal (\(x\)-axis) and vertical (\(y\)-axis) directions, and the rotor is connected to the motor by a coupling, as shown in Figure 8a. According to the design of the motor, the highest rotational speed is about 3000 rpm, and operation speed of rotor is set to 2100 rev/min (35 Hz). The fault is simulated by adjusting the thickness and number of metal gasket to achieve the misalignment between experimental rotor and motor rotor, as illustrated in Figure 8b.

### 6.3. Results

The time waveforms and corresponding spectrum of casing vibration acceleration under coupling faults of unbalance and misalignment is obtained, as shown in Figure 9. By comparing Figure 7 and Figure 9, it can be seen that both results from simulations and experiments both reflect super harmonics, such as the second harmonic frequency component resulted from misalignment fault. Due to the influence of high frequency noise, there are some differences in the time waveforms. Rotating frequency and other \( n \times \) running speed components can also be observed under the excitation of unbalance and misalignment forces. The qualitative consistency of simulation and experimental results to some extent verifies the correctness of the new rotor-support-casing coupling model in the paper.

### 7. CONCLUSIONS

In this work, a new rotor-support-casing (RSC) coupling model with misalignment faults is proposed and the integrated misalignment model is developed. The influences of integrated misalignment on the casing vibration characteristics are investigated in detail. In addition to the vibration waveforms, orbit plots and spectrums are effectively employed to detect unique vibration features relayed to the misalignment fault. The main conclusions are as follows:

1. According to the theoretical model, it shows that the casing response under the coupling effects of misalignment force and unbalance mass rich in harmonics components, on which \(1 \times\) and \(2 \times\) rotating frequency component are based. This misalignment results in a major super-harmonic at \(2 \times\) frequency component.

2. Multi-looped and stretched orbits are typical misalignment fault indicators. This is an important feature which can be used along with the spectrum to identify misalignment fault in the system.
(3) Combined with the test conclusion, the theoretical simulation has the same frequency and waveforms with practical experiment. It verified that the theoretical simulations are basically correct.

CONFLICT OF INTEREST

There is no conflict of interest regarding the publishing of this paper.

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Figure 7. Acceleration time waveform, orbit and corresponding spectrum for rotor-support-casing system with 3 mm parallel misalignment, 0.5° angular misalignment and 0.8 mm unbalance.

Figure 8. (a) Rotor-disk-bearing-casing experimental rig; (b) The bottom of base.

Figure 9. Time waveforms and corresponding spectrum of casing vibration acceleration.
REFERENCES


Effects of Elastic Restraints on the Fundamental Frequency of Nonlocal Nanobeams with Tip Mass

Malesela K. Moutlana
Department of Mechanical Engineering, Durban University of Technology, Durban 4001, South Africa.

Sarp Adali
Discipline of Mechanical Engineering, University of KwaZulu-Natal, Durban 4041, South Africa.

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The fundamental frequencies of an elastically restrained nanobeam with a tip mass are studied based on the nonlocal Euler-Bernoulli beam theory. The nanobeam has a torsional spring at one end and a translational spring at the other end where a tip mass is attached. The aim is to model a tapping mode atomic force microscope (TM-AFM), which can be utilized in imaging and the manufacture of Nano-scale structures. A TM-AFM uses high frequency oscillations to remove material, shape structures or scan the topology of a Nano-scale structure. The nonlocal theory is effective at modelling Nano-scale structures, as it takes small scale effects into account. Torsional elastic restraints can model clamped and pinned boundary conditions, as their stiffness values change between zero and infinity. The effects of the stiffness of the elastic restraints, tip mass and the small-scale parameter on the fundamental frequency are investigated numerically.

1. INTRODUCTION

The vibrations of carbon nanotubes have been studied extensively due to their use as sensors in a number of applications such as an atomic force microscope (AFM), biosensors and Nano-resonators.1–3 The atomic force microscope, discovered in the late eighties by Binnig et al.,4 has become more accurate with the discovery of carbon nanotubes which are used as sensor tips in the form of a cantilever.5–8 An AFM cantilever provides dynamic interaction with the surface leading to transverse vibrations. Contact forces represent the dominant component of the forces when the interaction takes place. In the fabrication of nano-scale structures, the high frequency oscillations of a tip mass attached to a beam are employed with the purpose of deforming or shaping a material into a desired shape.9,10 This process is widely known as either dynamic atomic force microscopy (DAFM)7 or TM-AFM.11 Cantilever nanotubes are also used as biosensors as their vibration patterns are analysed to obtain sensing data.12–16 Nanotubes are also used as Nano-resonator sensors that are based on detecting shifts in resonant frequencies caused by a mass attached to the nanotube tip.17–23 Nano-resonators improve the accuracy of the measurements by providing a high level of sensitivity as compared to conventional sensors. A review of the applications of carbon nanotubes as nano-resonator sensors is given in Wang and Arash.24

The vibration characteristics of Nano-scale beams can be analysed by modelling them within the framework of nonlocal continuum theory.25,26 Nonlocal theory has been applied to the vibration modelling of carbon nanotubes in several studies27–34 and, in particular, vibrations of carbon nanotubes with a tip mass have been studied extensively due to their use in sensor applications.18,21,23,35–37 In these studies the flexibility of the boundaries were not taken into account and the classical boundary conditions were considered. The vibrations of nanotubes with elastically restrained boundaries have been analysed in a number of studies.38–41

In the present study, an elastically restrained nanobeam with a tip mass was modelled as a Euler-Bernoulli beam based on the nonlocal theory25,26 to take into account the small scale effects.42,43 The restraint at one support was specified as a torsional spring which replaced the clamped end boundary condition used in a number of studies. A translational spring was attached to the other end where a tip mass was attached. The governing differential equation of motion was solved analytically to determine the fundamental vibration frequency. The effects of the elastic restraints, tip mass and the small-scale parameter on the fundamental frequency were investigated numerically. Similar problems have been solved in the case of beams based on the local (classical) theory of elasticity. Elastically supported beams with a tip mass had been studied in literature44–48 and a beam with torsional and translational springs with a tip mass had been studied in Zhou49 based on the local theory of Euler-Bernoulli beams.

Most researchers model TM-AFM as a clamped-free beam in which case the clamped end can be considered ‘ideal’.6,8,50,51 A model replacing the clamped end with a torsional spring represents a more ‘realistic’ boundary condition, allowing for the flexibility of the attachment. This class of AFM is commonly referred to as torsion cantilevers and has been proven to exhibit some advantageous properties in subjects of research involving hydrated or soft biological samples and improved resolution and sensitivity of force in AFM probes subject to liquid environments.52,53 Torsional cantilevers are important in the field of Nano-manufacturing because they allow for compliance matching or mismatching between the machining tool and the sample to be shaped. Matching the compliance of the machining tool and the sample prevents the destruction of the sample during manufacturing. Probing and compliance matching allows for the ‘effortless’ shaping of a sample. In TM-AFM, the free end carries a fabrication/cutting tool which is modelled as concentrated mass
such that the centre of the mass of the tool coincides with the sharp end of the cutting tool and the tip of the beam. A translational linear spring was attached to the tip of the mass and modelled the contact force with the sample of interest.

2. ELASTICALLY RESTRAINED NANOBEBEAM WITH TIP MASS

The beam under consideration was a Nano cantilever with the attachment at \( x = 0 \) modelling an elastic restraint due to a torsional spring of constant \( k_1 \) as shown in Fig. 1. A machining tool that modelled a concentrated mass with a sharp tip was attached to the free end of the beam at \( x = L \). Figure 1 shows that the tip mass’s centre of gravity coincided with the end of the beam. During the machining process, a contact force was generated between the tip mass and the object to be shaped. This contact force was modelled as a linear spring with a spring constant \( k_2 \). Figure 1 shows the configuration of the system where the beam was mounted by using a rectangular pin. The rectangular pin in Fig. 1(a) restricted vertical motion in the \( y \)-direction while restraining the lateral motion of the beam as it rotated about the rectangular pin. Therefore, the boundary conditions were that of a pinned and torsionally restrained end.

The constitutive relation of stress-strain, which was based on the nonlocal theory of elasticity, was expressed as

\[
\sigma_{xx} - \overline{\mu} \frac{d^2 \sigma_{xx}}{dx^2} = E \varepsilon_{xx};
\]

where \( E \) was the Young’s modulus, \( \overline{\mu} = e_0 l_i \) was the small scale parameter with \( e_0 \) denoting a material constant and \( l_i \) was the characteristic length. The expression for moment \( M(x) \) was given by

\[
M(x) = \overline{\mu} \frac{d^2 M(x)}{dx^2} = -EI \frac{d^2 w}{dx^2};
\]

where \( I \) was the moment of inertia. The equation of motion for a nonlocal nanobeam undergoing bending vibrations was given in Reddy\(^{42,43} \) and was expressed as

\[
EI \frac{d^4 w}{dx^4} = \overline{\mu}^2 \rho A \frac{d^2 w}{dx^2} + \rho Aw = F_0(x);
\]

where \( I \) was the moment of inertia, \( \rho \) was the density, \( A \) was the cross-sectional area and \( F_0(x) \) was the forcing function which was taken as \( F_0(x) = 0 \) for a freely vibrating beam.

3. METHOD OF SOLUTION

The solution for the governing Eq. (3) was obtained by the eigenfunction expansion of the displacement function as

\[
w(x) = \sum_{n=1}^{\infty} X_n(x).
\]

Inserting Eq. (4) into Eq. (3) and after rearrangement, we obtained the equation in the modal domain given by

\[
X_n^{\prime\prime\prime}(x) + \overline{\mu}^2 a_n^4 X_n^\prime\prime(x) - a_n^4 X_n(x) = 0.
\]

The frequency parameter \( a_n \) was defined as

\[
a_n^4 = \frac{\rho A w_0^2}{EI};
\]

where \( \omega_n \) was the natural frequency for the \( n^{th} \) mode of vibration. The general solutions of Eq. (5) were given by

\[
X_n(x) = A_n \cos p_{2n} x + B_n \sin p_{2n} x + C_n \cosh p_{1n} x + D_n \sinh p_{1n} x;
\]

where \( p_{1n} \) and \( p_{2n} \) were

\[
p_{1n,2n} = \sqrt{\frac{a_n^6 \overline{\mu}^2 \pm \sqrt{a_n^6 \overline{\mu}^2 + 4 a_n^4}}{2}}.
\]

The constants \( A_n, B_n, C_n \) and \( D_n \) were determined from the boundary conditions. The boundary conditions at \( x = 0 \) were zero displacement and the moment induced by the torsional spring and was expressed as

\[
w(0) = 0;
\]

\[
EI \frac{d^2 w(0)}{dx^2} - \overline{\mu}^2 \rho Aw - k_1 \frac{d w(0)}{dx} = 0.
\]

From Eq. (9b), it followed that, if the torsional spring constant is infinite, only the third term became nonzero. This corresponded with the clamped boundary condition. For a beam with a tip mass \( M_T \), the mass’s centre of gravity was located at \( x = L \). At the free end, taking into account the small scale
where the tip mass and the linear spring, the moment and shear boundary conditions were expressed as
\[ EI \frac{d^2w(L)}{dx^2} + \kappa_1 \frac{\partial w(L)}{\partial x} = 0; \quad (10a) \]
\[ E \frac{d^3w(L)}{dx^3} - \kappa_1 \rho Aw(L) - MTw(L) + k_2w(L) = 0. \quad (10b) \]

Eigenfunction expansions at \( x = 0 \) and \( x = L \) were given by
\[ w(0) = \sum_{n=1}^{\infty} X_n(0); \quad w(L) = \sum_{n=1}^{\infty} X_n(L). \quad (11) \]

By substituting \( w(0) \) in Eq. (11) into the boundary conditions at \( x = 0 \), we obtained
\[ X_n(0) = 0; \quad (12a) \]
\[ X_n'(0) + \mu a_n^\prime \eta X_n(0) = 0; \quad (12b) \]
where \( \kappa_1 = \frac{k_1L}{EI} \) was the dimensionless torsional spring constant. By substituting \( w(L) \) in Eq. (11) into Eqs. (10a) and (10b), moment and shear force expressions at the boundary \( x = L \) were obtained as
\[ X_n''(L) + \mu a_n^\prime \eta X_n(L) = 0; \quad (13a) \]
\[ X_n''(L) + \mu a_n^\prime \eta X_n(L) + \kappa_2 X_n(L) = 0; \quad (13b) \]
where
\[ \eta = \frac{MTL}{\rho A}; \quad \text{(dimensionless mass ratio);} \]
\[ \kappa_2 = \frac{k_2L^3}{EI}; \quad \text{(dimensionless linear spring constant);} \]
\[ \mu = \left( \frac{\kappa_1L}{EI} \right)^2 L^2; \quad \text{(dimensionless small scale parameter).} \]

With the substitution of Eq. (7) into the boundary condition for Eq. (12a), we obtained
\[ A_n - C_n = 0. \quad (14) \]
The general solution for Eq. (7) was be expressed as
\[ X_n(x) = D_n \sinh p_1n x - B_n \sin p_2n x + C_n (\cos p_2n x - \cosh p_1n x). \quad (15) \]

Substituting Eq. (15) into Eq. (12b) gave
\[ \kappa_1 p_1 D_n - (p_2^2 + p_1^2) C_n + \kappa_1 p_2 B_n = 0. \quad (16) \]
Solving for \( B_n \) and substituting into Eq. (15) allowed for the general solution to be expressed in terms of constants \( C_n \) and \( D_n \) alone. This result was substituted into the moment boundary conditions Eq. (10a) at \( x = L \) to obtain
\[ C_n \cdot \Gamma_1 + D_n \cdot \Gamma_2 = 0; \quad (17) \]
where
\[ \Gamma_{1n} = \left[ \left(p_2^2 + p_1^2\right) \sin p_2n L - \kappa_1 p_1 \left( \cos p_2n L - \cosh p_1n L \right) \right] \mu a_n^\prime + \left( p_3^2 + p_1^2 p_2^2 \right) \sin p_2n L - \kappa_1 p_1 \left( p_2^2 \cos p_2n L + p_1^2 \cosh p_1n L \right); \]
\[ \Gamma_{2n} = \frac{1}{p_2n} \left[ a_n^\prime (p_2n \sinh p_1n L - p_1n \sin p_2n L) \mu - p_1n p_2n (p_2n \sin p_2n L - p_1n \sin p_1n L) \right]. \]
Shear boundary condition Eq. (10b) at \( x = L \) was expressed as
\[ C_n \cdot \Gamma_3 + D_n \cdot \Gamma_4 = 0; \quad (18) \]
where
\[ \Gamma_{3n} = \left[ \kappa_2 a_n^4 \left( p_1n p_2n \sinh p_1n L + p_2^2 n \sin p_2n L \right) + \left( p_2n^2 + p_1n^2 p_2n \right) \sin p_2n L \right] \mu + \left[ (p_2^2 + p_1^2) a_n^\prime \right] \sinh p_2n L + p_2n \left( \kappa_4 a_n^4 \mu + p_2^2 + p_1n^2 p_2n + \kappa_1 \kappa_2 \right) \cos p_2n L - \kappa_1 p_1n p_2n \sinh p_1n L + \left( \kappa_1 p_2n a_n^\prime \mu - \kappa_1 \kappa_2 p_1n^2 \sinh p_1n L \right) \cosh p_1n L; \]
and
\[ \Gamma_{4n} = \kappa_1 \left[ p_1n p_2n \left( a_n^\prime \mu \cosh p_1n L - a_n^4 \mu \cos p_2n L \right) \mu + p_1n \left( \kappa_2 a_n^4 \mu \sin p_2n L \right) \right] + \kappa_1 \left[ p_2n \left( a_n^4 \mu - \kappa_2 \right) \sinh p_1n L - p_1n p_2n \left( p_2^2 \cos p_2n L - p_1n \sin \cosh p_1n \mu \right) \right]. \]
The results from the moment and shear boundary conditions were given in Eqs. (17) and (18) was expressed in matrix form as
\[ \begin{bmatrix} \Gamma_{1n} & \Gamma_{2n} \\ \Gamma_{3n} & \Gamma_{4n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad (19) \]
and the characteristic equation was obtained from the determinant of Eq. (19) as:
\[ \Gamma_{1n} \Gamma_{4n} - \Gamma_{2n} \Gamma_{3n} = 0. \quad (20) \]
The characteristic Eq. (20) was solved numerically to compute the roots. For the cases of infinite torsional spring constant at \( x = 0 \), \( \kappa_1 \to \infty \), zero tip mass, \( \eta = 0 \), zero linear spring constant at \( x = L \), \( \kappa_2 = 0 \), and zero small scale parameter \( \mu = 0 \), Eq. (20) was reduced to the frequency equation of a cantilever given in Magrab and Bokaian.\text{54,55}

4. NUMERICAL RESULTS

Using the characteristic Eq. (20) and substituting \( a_n = R_n/L \) allow for the dimensionless roots \( R_n \) of the characteristic equation to be computed. Equation (6) shows that
these roots are directly associated with the natural frequencies $\omega_n$ by the relation $R_n = L\omega_n^{1/2}(\rho A/EI)^{1/4}$. Numerical results are given using dimensionless $R_n$ values. Figure 2 shows the fundamental frequencies of the system with respect to the torsional spring constant $\kappa_1$ and the linear spring constant $\kappa_2$ for different values of the small scale parameter $\mu = 0, 0.2, 0.4, 0.6$. Figure 2(a) corresponds to a local beam with $\mu = 0$, which is based on the classical continuum model. The results in Figs. 2(b) to 2(d) take nonlocal effects into account and clearly demonstrate that these effects are important in the analysis of the nanobeams.

The results of Fig. 2(a) are in accordance with the results obtained by Magrab for different boundary conditions. The boundary conditions include clamped-pinned, simply supported, clamp free, pinned-free beams. When the spring constants for the torsional spring at $x = 0$ and the linear spring at $x = L$ are infinite, i.e., $\kappa_1, \kappa_2 \to \infty$, the nanobeam behaves like a clamped-pinned beam. The frequency for this case is $R_1 = 3.9266$ and the corresponding result in Magrab is $1.25\pi$. For the case when $\kappa_1 = 0$ and $\kappa_2 \to \infty$, the nanobeam becomes simply-supported with the frequency given by $R_1 = \pi$. For a clamped-free beam, $\kappa_1 \to \infty$ and $\kappa_2 \to 0$. For this case and with a tip mass of $\eta = 0.1$, the frequency is $R_1 = 1.7227$ which is the same result ($0.5484\pi$) as the one given in Ma-
Figure 4. Fundamental frequency plotted against $\kappa_2$ for various values of $\kappa_1$ and $\mu$ with tip mass ratio $\eta = 0.1$, a) $\kappa_1 = 0$, b) $\kappa_1 = 10$, c) $\kappa_1 = 10^2$, d) $\kappa_1 = 10^4$.

For zero torsional and linear spring constants, i.e., $\kappa_1, \kappa_2 \to 0$, the frequencies approach zero ($R_1 \to 0$) for all values of the small scale parameter $\mu$. A similar observation was made by Behera and Chakraverty. For zero torsional and linear spring constants, i.e., $\kappa_1, \kappa_2 \to 0$, the frequencies approach zero ($R_1 \to 0$) for all values of the small scale parameter $\mu$. A similar observation was made by Behera and Chakraverty. Frequencies of a cantilever nanobeam given by Ehteshami and Hajabasi, Murmu and Adhikari indicate that, as the small-scale parameter increases, the fundamental frequency of the cantilever nanobeam also increases. When the boundaries have a finite stiffness, the frequencies may increase or decrease depending on the problem parameters as the small-scale parameter increases. The graphs of the natural frequencies shown in Figs. 2(a) to 2(d) indicate this phenomenon.

For a simply supported nanobeam, the torsional spring constant at $x = 0$ is zero ($\kappa_1 = 0$) and the linear constant at $x = L$ is infinity ($\kappa_2 \to \infty$). For this case, Figs. 2(a) to 2(d) indicate that the frequencies decrease as the small scale parameter increases. Figures 2(a) to 2(d) indicate that, for a clamped-pinned case ($\kappa_1, \kappa_2 \to \infty$), the frequencies decrease as the small scale parameter increases. The effect of the small scale parameter on frequency is investigated in Fig. 3, which shows the trend in the frequencies as the small scale parameter increases for classical boundary conditions. It is observed that frequencies decrease for clamped-free (C-F) and simply-supported-simply-supported (S-S) boundary conditions, but increase for the clamped-pinned (C-P) case. These results are supported by the results obtained by Behera and Chakraverty. For a pinned-free (P-F) nanobeam, i.e., $\kappa_1, \kappa_2 \to 0$, the frequencies are extremely small and approach zero in the limit.

The effect of the linear spring constant $\kappa_2$ at $x = L$ is investigated in Fig. 4, which shows the curves of frequency plotted against $\kappa_2$ for $\kappa_1 = 0, 10, 100, 10000$ and for different values of the small scale parameter. The curves represent the frequency profile for a particular torsional spring constant at $x = 0$ and small scale parameter. In Fig. 4, the frequency increases steadily as the stiffness $\kappa_2$ increases while $\kappa_1$ at $x = 0$ is kept constant. It is noted that the boundary condition at $x = L$ becomes a simple support when $\kappa_2 \to \infty$.

Figure 5 shows the curves of frequency plotted against the small scale parameter for $\kappa_1 = 0, 10, 100, 10000$ and for different values of $\kappa_2$. It is observed that the frequency increases as the small scale parameter increases at low values of $\kappa_2$, but decreases at high values of $\kappa_2$ indicating the effect of the spring.
Figure 5. Fundamental frequency plotted against \( \mu \) for various values of \( \kappa_1 \) and \( \kappa_2 \) and with tip mass ratio \( \eta \) = 0.1. a) \( \kappa_1 = 0 \), b) \( \kappa_1 = 10 \), c) \( \kappa_1 = 10^2 \), d) \( \kappa_1 = 10^4 \).

at \( x = L \). Figure 5 shows that, for \( \kappa_1 = 0 \) and \( \kappa_2 = 0 \), the frequency is very small and its value is independent of the small scale parameter. As \( \kappa_1 \) increases from \( \kappa_1 = 10 \) to \( \kappa_1 = 10^4 \) (Figs. 5(b) to 5(d)) for \( \kappa_2 = 0 \), frequency increases from \( R_1 = 1.5912 \) to \( R_1 = 1.7070 \) to \( R_1 = 1.7227 \). The frequency \( R_1 = 1.7227 \) is widely published as the natural frequency of a local cantilever beam (\( \mu = 0 \)) with tip mass ratio of \( \eta = 0.1 \). From Figs. 5(b), 5(c) and 5(d), we note that frequencies increase as \( \mu \) increases for \( \kappa_2 = 0 \), and they decrease for \( \kappa_2 = 10^2, 10^4 \). For cantilever nanobeam (\( \kappa_1 = 10000 \)) in Fig. 5(d), the results in Li et al. show similar trends for the frequencies.

In Fig. 6, the frequency is plotted against the linear spring constant \( \kappa_2 \) for different values of \( \kappa_1 \) and \( \mu \). Figure 6(a) shows that, when the end of the beam is free (\( \kappa_2 = 0 \)), the small scale parameter has minimal effect on the frequency. However, Fig. 6(d) shows that its effect increases as the end \( x = L \) becomes simply supported when \( \kappa_2 \rightarrow 10^4 \). The frequency increases as the boundaries at \( x = 0 \) and \( x = L \) become stiffer, i.e., as \( \kappa_1, \kappa_2 \rightarrow 10^4 \) with the amount of increase depending on the small scale parameter.

Next, the effect of the rigidity of the end \( x = L \) is investigated by plotting the frequency against \( \mu \) in Fig. 7. It is observed that the frequency may increase or decrease as \( \mu \) increases depending on \( \kappa_1 \) and the rigidity of the end point as measured by \( \kappa_2 \). As \( \kappa_2 \) increases, decrease in the frequency with increasing \( \mu \) becomes more pronounced.

Next, the effect of the tip mass on frequency is studied in Fig. 8 for the following boundary conditions: 1) \( \kappa_1 = 10 \) and \( \kappa_2 = 0 \), restrained-free (R-F), 2) \( \kappa_1 = 10 \) and \( \kappa_2 = 10 \), restrained-restrained (R-R), 3) \( \kappa_1 \rightarrow \infty \) and \( \kappa_2 = 0 \), clamped-free (C-F) and 4) \( \kappa_1 \rightarrow \infty \) and \( \kappa_2 = 10 \), clamped-restrained (C-R). It is observed that as the tip mass ratio increases, frequency decreases as observed in Moutlana and Adali. It is observed that, for a C-F local cantilever (\( \mu = 0 \)), the fundamental frequencies are \( R_1 = 1.875 \), \( R_1 = 1.723 \), \( R_1 = 1.420 \) and \( R_1 = 1.248 \) corresponding to the tip mass ratios \( \eta = 0 \), \( \eta = 0.1 \), \( \eta = 0.5 \) and \( \eta = 1.0 \), respectively, which are the same as the results obtained by Balachandran and Magrab, which were based on classical continuum theory. It is also noted that
Figure 6. Fundamental frequency plotted against $\kappa_1$ for various values of $\kappa_2$ and $\mu$ with tip mass ratio $\eta = 0.1$, a) $\kappa_2 = 0$, b) $\kappa_2 = 10$, c) $\kappa_2 = 10^2$, d) $\kappa_2 = 10^4$.

It is observed from Fig. 8 that when the support at $x = 0$ is clamped, the frequency increases with increasing $\mu$. However, when the support at $x = 0$ is restrained with $\kappa_1 = 10$, the frequency decreases with increasing $\mu$ when $\eta \leq 0.1$. Similar observations were made in Azrar et al.\textsuperscript{61} for the case $\eta = 0$. When $\eta \geq 0.5$, frequency increases with increasing $\mu$ for all boundary conditions. As expected frequencies are reduced when the tip mass ratio increases. It is noted that for $\eta = 0$ with $\kappa_1 \to \infty$ and $\kappa_2 = 10$ (clamped-restrained case), $R_1 = 2.5954$ for $\mu = 0.565$, and for $\mu > 0.565$ no non-trivial real eigenvalue ($R_1$) does not exist as also observed in Lu.\textsuperscript{62} A similar situation is encountered for the same boundary conditions when the tip mass ratio is $\eta = 0.1$ where $R_1 = 2.4627$ for $\mu = 0.58$ and there are no real non-trivial eigenvalues beyond this point. It is noted that for a clamped-free beam, there are no real non-trivial eigenvalues beyond $\mu = 0.62$.\textsuperscript{60}

5. CONCLUSIONS

In the present paper, small scale effects on the fundamental frequency are investigated for a nanobeam with elastically restrained end conditions and carrying a tip mass. The solution for the beam is obtained analytically by expanding the deflection in terms of its eigenfunctions and solving the resulting characteristic equation numerically. The results are given in the form of three-dimensional plots in terms of the fundamental frequencies, small scale parameter, torsional and linear spring constants. Further numerical results are presented for parametric studies of the effect of support elasticity and tip mass on the fundamental frequencies of the nanobeam.

It is observed that the effect of the small-scale parameter depends on several problem parameters and may lead to an increase or decrease of the fundamental frequency depending on the support flexibility. Boundary conditions are expressed in terms of a torsional spring at $x = 0$ and a linear spring at...
$x = L$ and the classical boundary conditions correspond to setting the torsional and linear spring constants to zero ($\kappa_{1,2} \to 0$) or infinity ($\kappa_{1,2} \to \infty$). It was observed that low torsional spring stiffness and high linear spring stiffness leads to a decrease in the fundamental frequency and high torsional spring stiffness and low linear spring stiffness to an increase in the fundamental frequency as the small-scale parameter increases (Figs. 5 and 7). The rates of decrease and increase depend on the relative values of the spring constants. The effect of the tip mass on the frequencies are studied in Fig. 8 and it is observed that in some cases a real eigenvalue cannot be obtained as also observed in Lu et al. and Lu.60,62

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Figure 8. Fundamental frequency plotted against $\mu$ for various boundary conditions, a) $\eta = 0$, b) $\eta = 0.1$, c) $\eta = 0.5$, d) $\eta = 1.0$.


Chowdhury, R., Adhikari, S., and Mitchell, J. Vibrating car-


Vibration Characteristics of Plate Structures Embedded with Acoustic Black Holes and Distributed Dynamic Vibration Absorbers

Xiuxian Jia, Yu Du and Ye Yu
School of Automotive Engineering, Dalian University of Technology, Dalian, China.

Kunmin Zhao
School of Automotive Engineering, Dalian University of Technology, Dalian, China.
Institute of Industrial and Equipment Technology, Hefei University of Technology, Hefei, China.

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This study discusses a method of combining the acoustic black hole (ABH) concept and dynamic vibration absorbers (DVAs) together as a lightweight passive control approach for structural vibration and noise attenuation. Finite element (FE) simulations and experiments are used to compare vibration response levels of plate structures. The plate structures have been integrated with various vibration attenuation treatments including damping layers, DVAs, ABHs and ABH-DVA pairs. It is demonstrated experimentally that the plate structure integrated with two ABH-DVA pairs has the lowest overall vibration response level in the frequency range below 800 Hz. More interestingly, the total structural mass of the plate structure integrated with ABH-DVA pairs is 8.24% less than that of the uniform thickness plate. The experimental observations are further verified with simulation results. With the help of the FE model, plate structures integrated with more than two ABH–DVA pairs targeted at the simultaneous attenuation of multiple resonances are studied and compared with traditional uniform thickness plates. Under the design constraint of the total structural mass being equal, it is shown that plates integrated with DVA-ABH pairs always have lower vibration response levels in the low-mid frequency range where mechanical vibration commonly occurs.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>ABH</td>
<td>Acoustic Black Hole</td>
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<tr>
<td>DVA</td>
<td>Dynamic Vibration Absorber</td>
</tr>
<tr>
<td>UTP</td>
<td>Uniform Thickness Plate</td>
</tr>
<tr>
<td>UTP-Damp</td>
<td>Uniform Thickness Plate integrated with a free Damping layer</td>
</tr>
<tr>
<td>UTP-DVA</td>
<td>Uniform Thickness Plate integrated with DVA</td>
</tr>
<tr>
<td>VTP-ABH</td>
<td>Variable Thickness Plate embedded with ABH features</td>
</tr>
<tr>
<td>VTP-ABH-Damp</td>
<td>Variable Thickness Plate embedded with ABH features that are filled with free Damping layers</td>
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<tr>
<td>VTP-ABH-DVA</td>
<td>Variable Thickness Plate embedded with ABH features that are attached with DVAs</td>
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1. INTRODUCTION

The passive control of vibration and structure-borne noise is usually achieved by adding dynamic vibration absorbers (DVAs) or damping layers to the primary structure whose responses are to be attenuated.\(^\text{1-6}\) Despite the effectiveness of these two approaches in structural vibration and noise control that has been extensively studied and demonstrated,\(^\text{6-8}\) both approaches inevitably add additional masses (i.e., weight) to the primary structure. In many applications, the increased structural weight due to the additional mass may negatively influence other performances of the system. For example, it is well known that reducing the gross weight of a vehicle can dramatically reduce the fuel consumption as well as emission. It is estimated that fuel consumption can be reduced by 6%–8% with every 10% reduction in vehicle weight.\(^\text{9}\) Adding mass to achieve the desired noise and vibration performance would directly contradict with the goal of reducing fuel consumption and emission. Therefore, it is highly desirable to find ways that add as less weight as possible to the primary structure while still being able to effectively control the noise, vibration and harshness (NVH) performance of vehicles. Unfortunately, less weight normally leads to degraded NVH performance.\(^\text{10}\)

In the past decades, an interesting approach based on ABH theory has attracted significant amounts of attention in the literature.\(^\text{11-13}\) This approach can effectively reduce the vibration response of a structure without adding a significant amount of mass to the primary system. The ABH phenomenon was first discussed by Mironov, who showed that, in theory, when the thickness of a plate structure decreases smoothly to zero towards its edge, the speed of the flexural wave inside the plate slows down and approaches zero at the edge.\(^\text{14}\) This means that the wave can never reach the edge, thus no reflection occurs. This phenomenon is commonly referred to as the ABH effect in the literature. To achieve the ABH effect, the variation of the plate thickness needs to both follow a certain power-law profile and approach zero at the edge.\(^\text{14}\) This means that the wave can never reach the edge, thus no reflection occurs. This phenomenon is commonly referred to as the ABH effect in the literature. To achieve the ABH effect, the variation of the plate thickness needs to both follow a certain power-law profile and approach zero at the edge. Apparently, the second condition can never be met in a practical structure since a real structure needs to be truncated at the edges with specified thicknesses to maintain its necessary structural strength. Therefore, the ABH effect can never be fully realized in ac-
tual structures. In homogeneous structures, the deviation of thickness profiles of actual structures from the ideal power-law shapes (largely due to the thickness truncation at the edges) can lead to an energy reflection rates as high as 50–70%. To reduce the amount of energy reflection, some researchers have discussed the approach of adding damping layers to the ABH structure and showed its effectiveness. In this study, the approach of combining another common vibration control treatment—the dynamic vibration absorber (DVA)—with the ABH structure was proposed and studied.

When the plate thickness in an ABH structure changes only in one direction (e.g., in length or width) following a power-law, it is known as the one-dimensional ABH (e.g., a wedge). Due to the ABH effect described previously, vibration energy resulted from external excitations could be “trapped” near the curved edges, thus significantly attenuating response levels in areas away from the edges. However, any truncation to the edge thickness of a wedge results in energy reflections. To reduce the amount of reflections, one can typically consider covering the wedge surfaces with thin damping layers. It has been already demonstrated both theoretically and experimentally that the presence of thin absorbing damping layers on the surfaces of wedges can significantly reduce the reflection of flexural waves from their truncated edges.

When an ABH structure changes its thickness in both the length and width directions, it is called two-dimensional ABH. A common two-dimensional ABH structure can be configured in the form of a circular indentation embedded in a plate structure. Compared to the one-dimensional ABH wedge, a two-dimensional ABH structure in the form of a circular indentation built in a plate was found to be less effective in attenuating the vibration energy. This was due to the fact that the total length of the thin edge of an embedded two-dimensional ABH structure is usually short and limited by the size of the primary plate structure. One way to effectively enlarge the length of the thin ABH edge, or the absorption cross-section, is to open a hole at the center of the two-dimensional power-law indentation. This can significantly increase the vibration damping performance of a two-dimensional ABH structure.

In real applications, the two-dimensional ABH structure may be considered more practical than the one-dimensional one as it can be easily embedded into any plate-like structures, for example machine enclosures and vehicle bodies. It was found that, when multiple two-dimensional ABH structures are used, the ability of damping out vibration energy increases substantially and becomes comparable if not greater than that achieved by one-dimensional wedges of power-law profiles. In some recent studies, Conlon et al. investigated the structural vibration and noise control effectiveness when different types and numbers of ABHs were used. They demonstrated the potential of using a periodic grid of the two-dimensional ABH features for vibration and noise attenuation. Panels embedded with periodic ABH features showed better reduction in the radiated sound power with a total structural weight that was even lighter than the baseline uniform thickness panel without ABH features. More interestingly, their results also demonstrated that as the number of the ABH features increases, so did the vibration attenuation effectiveness.

Existing literature has focused on the method of adding damping layers to the ABH structures. However, it is also known that controlling structural vibration with damping is not as effective at low frequencies as at high frequencies. In this study, the vibration control effectiveness of adding DVAs to two-dimensional ABH structures was discussed using simulation models and experiments. Section 2 presents the experimental investigation comparing the vibration response levels of four different types of plate structures. They were a uniform thickness plate (UTP), a variable thickness plate with embedded ABH features (VTP-ABH), a VTP-ABH plate with damping layers covering the ABH features (VTP-ABH-Damp) and a VTP-ABH plate integrated with DVAs at the center of the ABH feature (VTP-ABH-DVA). In Section 3, a finite element (FE) simulation model was developed and verified against the experimental results. This model was then used in Section 4 to investigate the vibration control effectiveness versus the added masses of the proposed approach of combining DVAs and ABH features together. It was demonstrated that, in the frequency range where mechanical vibration is usually of important concerns (e.g., 0–800 Hz), the concept of combining the ABH and DVA together always leads to the lowest response level under the design constraint of the total structural mass being equal.

### 2. EXPERIMENTAL INVESTIGATION

To evaluate the vibration control effectiveness of the combined ABH-DVA concept, it is started with experimental investigations on vibration characteristics of different plate structures. As explained in Table 1, four types of plate structures were prepared for the experiments. They were referred to as UTP (i.e., the baseline case), VTP-ABH, VTP-ABH-Damp and VTP-ABH-DVA.

#### 2.1. Experimental Setup

In this experimental study, the baseline UTP structure was a rectangular steel panel with a uniform thickness of 4.5 mm. Its length and width were 399.0 by 198.0 mm, respectively. As illustrated in Fig. 1(a), for the variable thickness plate, the UTP structure was modified to have two circular indentations forming the ABH feature whose thickness was variable, hence the VTP-ABH structure. The two circular indentations were positioned symmetrically in the plate with a centre-to-centre distance of 149.0 mm. The thickness outside the circular area was 4.5 mm, same as the UTP case. Within the circular area, the thickness profile of the indentation followed the mathematical form described in Eq. (1), which was necessary to form the ABH feature.

\[
h(r) = r^n + h_{\text{min}}. \tag{1}\]

In Eq. (1), \(r\) was the radial distance measured from the center of the circle; \(h\) was the thickness at the radial location of
Figure 1. Schematic of (a) top-view of the VTP-ABH structure; (b) side view of the VTP-ABH structure; (c) cross-section of the VTP-ABH-DVA structure where the ABH indentation was filled with damping material; (d) cross-section of the VTP-ABH-DVA structure where DVA mass was attached at the center of the ABH indentation through a viscoelastic element; and (e) cross-section of the VTP-ABH-DVA-2 structure where DVA mass was attached at the center where the ABH indentation was filled with damping material; (d) cross-section of the VTP-ABH-DVA-2 structure where DVA mass was attached at the center of the ABH indentation through a damping element and a spring element.

\( r; \epsilon \) and \( n \) were constants; and \( h_{\text{min}} \) was the minimum thickness at the centre of the circular area. As widely discussed in the literature, although having a non-zero minimum thickness at the centre of the ABH, indentation would degrade the ABH effectiveness. This non-zero thickness was required to give sufficient rigidity in actual structures. This minimum non-zero thickness was set to be 0.85 mm in the experimental structure. The radius of the circular indentation area was 57.0 mm. Given the above parameters and assuming a taper power coefficient of \( n = 2.2 \), it was calculated that the constant \( \epsilon \) of the thickness profile was 5.0e-4.

For the VTP-ABH-Damp structure, the ABH indentation areas were filled with asphalt damping material, which resulted in a uniform thickness of 4.5 mm when measured everywhere on the plate. This is illustrated in Fig. 1(c). For the VTP-ABH-DVA structure used in the experiment and illustrated in Fig. 1(d), one DVA was attached to each ABH indentation by connecting a steel block (i.e., DVA mass) to the center of the indentation area through an asphalt damping layer acting as both damping and stiffness for the DVA. The total additional mass added by each DVA, including the damping element, was 37.5 g. This arrangement was made to ensure that the total structural mass was identical for the VTP-ABH-DVA case and the VTP-ABH-Damp case in the experiment. Finally, Fig. 1(e) shows that the VTP-ABH-DVA structure considered in the simulation model in Section 3 where the DVA mass was attached to the structure through a stand-alone damping element and a spring element.

Table 2 summarizes parameter values that were associated with the experimental structures as well as those used in the simulation model, including material properties of different plate structures, dimensions of the ABH features and parameters of the damping layers and DVAs. In the Table, \( \rho \) was material density, \( E \) was Young’s modulus, \( \nu \) was Poisson’s ratio, \( \eta \) was material loss factor, \( m \) was mass and \( k \) is stiffness. Subscript “d” denoted damping layer and “D” denoted DVA. Subscript “1” and “2” denoted the first and second ABH indentation areas, respectively.

The test setup is illustrated in Fig. 2. The rectangular shaped plate structure under test was supported on a test rig with the two short edges clamped and the two long edges left free. An electromagnetic shaker was positioned underneath the plate applying a point force excitation at the geometrical center of the plate structure. The driving point was marked as Point “I” in the figure. The vibration response of the plate was measured by an accelerometer attached at a point marked as “O”. Point “O” was located along the longitudinal centerline of the plate and 30 mm away from the lower edge. The transfer function between the acceleration response at point “O” and the excitation force at point “I” (i.e., the acceleration response per unit force or acceleration transmissibility) was then obtained.

2.2. Experimental Results

Using the experimental setup illustrated in Fig. 2, vibration responses of the UTP, VTP-ABH, VTP-ABH-Damp and VTP-ABH-DVA structures under harmonic point force excitations were measured in the frequency range up to 3000 Hz.

Figure 3 shows the measured acceleration transmissibility of the UTP, VTP-ABH and VTP-ABH-Damp cases. Comparing to the baseline UTP case, when adding two ABH indentations, the responses of the plates were substantially the same at low frequencies (e.g., \(< 500 \) Hz). Above 500 Hz, resonances of the VTP-ABH case were shifted to lower frequencies with slightly reduced amplitudes. Meanwhile, when the ABH indentations were filled with damping materials, the vibration response of the VTP-ABH-Damp case was observed to be significantly attenuated at high frequencies (e.g., \(> 1400 \) Hz). In detail, an attenuation about 10 dB was observed around the resonance peaks of the VTP-ABH case at 1400 and 2700 Hz. This result demonstrated the benefit of using damping layer to attenuate the vibration response of a plate structure embedded with ABH at mid-high frequencies, which was consistent with findings obtained by previous studies.19, 20
Table 2. Parameters for both the numerical model and the experimental study.

<table>
<thead>
<tr>
<th>Parameters of the plate structures</th>
<th>Density (kg/m$^3$)</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
<th>Loss factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>7850</td>
<td>196</td>
<td>0.3</td>
<td>0.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters of the ABH Profile</th>
<th>$n$</th>
<th>$\epsilon$</th>
<th>Thickness at the ABH center point, $h_0$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.2</td>
<td>5.0E-04</td>
<td>0.85</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters of the damping layer</th>
<th>Density (kg/m$^3$)</th>
<th>Young’s modulus (GPa)</th>
<th>Structural Damping</th>
<th>Mass of the added damping layer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1922</td>
<td>1.64</td>
<td>0.55</td>
<td>$m_d = m_d1 = m_d2 = 3.75E - 02$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters of DVAs</th>
<th>DVA mass $m_D$(kg)</th>
<th>Loss factor of the DVA damping</th>
<th>$\eta_D = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4. Comparison of the acceleration transmissibility of the UTP, VTP-ABH and VTP-ABH-DVA cases obtained in the experiment.

In Fig. 4, the experimental results of the VTP-ABH and the VTP-ABH-DVA cases are compared. For completeness, the UTP case was included again as a baseline. Figure 4 shows that adding two DVAs significantly attenuated the vibration level between 1750 and 2000 Hz. Outside this narrow frequency range, the responses of the VTP-ABH and VTP-ABH-DVA cases were similar. Based on the DVA parameters listed in Table 2, it was calculated that the tuning frequency of both DVAs used in the experiment was approximately 1824 Hz, which was believed to be responsible for the dramatic vibration reduction around this frequency.

As shown in Fig. 5, another interesting comparison can be made between the results of the VTP-ABH-DVA case and the VTP-ABH-Damp case to show the effect of adding damping versus DVA to the ABH feature. It was seen that the measured acceleration response level for the VTP-ABH-DVA case was nearly 20 dB lower than the VTP-ABH-Damp case around the DVA tuning frequency. Although the VTP-ABH-Damp case resulted in lower response level at high frequencies (e.g., > 2000 Hz), the vibration attenuation effect obtained by adding damping diminished with decreasing frequency. Based on the results shown in Figs. 3, 4, 5, both the VTP-ABH-Damp and VTP-ABH-DVA structures led to lower vibration response levels at various frequency regions compared to the baseline UTP structure, yet the mass of the experimental VTP-ABH-Damp structure and the VTP-ABH-DVA structure was 8.24% less than that of the UTP structure. Thus, when properly tuned, the concept of combining the ABH feature and DVA can be a promising lightweight approach for structural vibration control at low-mid frequencies. This conclusion will be further validated using simulation models in Sections 3 and 4.

3.3 NUMERICAL MODEL DEVELOPMENT

3.1 Model Description

In the previous section, the vibration control effectiveness of the ABH concept combined with both the damping layer and DVA was demonstrated experimentally. It was also shown that the approach of adding DVA together with the ABH feature could effectively attenuate resonance peaks. This is of particular interest for vibration control at low-mid frequencies where neither the ABH nor the damping layer method is very effective when used alone or in combination. To further investigate the characteristics of the approach combining the ABH feature with DVA, especially its vibration control effectiveness in lightweight structures, a finite element (FE) simulation model was developed.

To conveniently compare with experimental results obtained in Section 2, the FE model was first setup to simulate the four experimental cases listed in Table 1. In the model, plate dimensions, materials and boundary conditions were kept identical with the corresponding experimental cases (with an exception of the thickness for the UTP structure). Figures 1(c) and 1(e) illustrate how the ABH indentations, damping layer and DVAs were configured in the model. In addition to the four experimental cases, two additional cases were included in the simulation. They were the uniform thickness plate which was fully covered with a 0.494 mm thick damping layer (UTP-
Damp), and the uniform thickness plate was attached with two DVAs (UTP-DVA). These two cases were added to compare with the corresponding VTP cases (i.e., VTP-ABH-Damp and VTP-ABH-DVA). For fair comparison, the locations where the DVAs were attached on the UTP were the same as in the VTP-ABH-DVA case.

One goal of the numerical simulation was to study the vibration control effectiveness of various approaches versus the total mass of the structure. For this purpose, the thickness of the UTP in the simulation model was chosen to be 4.0 mm (instead of 4.5 mm used in the experiment). With 4.0 mm thickness, the total mass of the UTP and VTP-ABH cases were equal and weighed 2.48 kg. Similarly, the total mass of the additional damping layers or DVAs added to the primary structure were also identical resulting in a total mass of 2.56 kg for UTP-Damp, UTP-DVA, VTP-ABH-Damp and VTP-ABH-DVA cases. Detailed parameters for the experimental and simulation cases can be found in Table 2. The point excitation force in the model was applied at the geometrical center of the plate, which was also identical to the arrangement in the experiments.

Figure 6 illustrates the mesh of the FE model developed in HYPERMESH software. Planar structures, including the uniform thickness plate and the added damping layer, were modeled with shell elements. As shown in Fig. 6(a), for the UTP-Damp case, rigid connection elements were used between the damping layer and the primary plate structure to ensure the same degree-of-freedom (DOF) at the interface. For the cases involved with variable thickness plate (VTP-ABH), the variable thickness profile described by the power-law in Eq. (1) can be easily accommodated using tetrahedral elements as illustrated in Fig. 6(b). The damping layer (light-gray elements) filled in the ABH indentation shared the same set of nodes with the VTP-ABH structure (dark-gray elements) at the interface to ensure the same DOF. The CBUSH element was used to simulate DVA. Mesh sizes were refined to improve the accuracy of calculation up to 3000 Hz. This model was solved using the commercial FE program NASTRAN.

Slightly different from the experimental study which only measured the vibration response at a single point, another straightforward and arguably more meaningful measure for comparing the vibration control effectiveness of different approaches is the average acceleration response amplitude ($A_s$) across the entire plate surface. The calculation of $A_s$ is given in Eq. (2), which was conveniently obtained with the FE model.

$$A_s(f) = 20\log_{10} \left[ \frac{1}{n} \sum_{i=1}^{n} |a_i(f)| \right]. \tag{2}$$

In Eq. (2), $f$ is frequency, $a_i$ was the acceleration amplitude at the $i$-th node on the surface of the primary structure in the FE model, $i = 1, 2, 3, \ldots, n$, and $n$ were the total number of nodes.

3.2. Model Validation

To verify the feasibility of the model, the UTP case (with 4.5 mm thickness) and VTP-ABH case presented in Section 2 were simulated with the FE model and compared to the experimental results.

Figure 7(a) compares the acceleration transmissibility of the UTP case. Figure 7 shows that the numerical curve matched the experimental result very well in the frequency range up to 3000 Hz. Figure 7(b) compares the simulation and experimental results for the VTP-ABH case. This time, the two curves reasonably matched with each other below 1700 Hz. Above 1700 Hz, some discrepancies in the resonance locations were observed. This could be due to the geometry differences in the VTP-ABH plates used in the simulation model and the experiment. A thickness profile of the ABH indentation following Eq. (1) was perfectly accommodated in the FE model, while this was not a trivial task in practice due to dimension variations and uncertainties when machining the actual VTP-ABH plate. Furthermore, the boundary condition may not be perfectly clamped (for the two short sides as shown in Fig. 2 in the experiment either, which could also contribute to the discrepancies between the experimental and simulation curves. Nevertheless, the overall response trend presented by the simulation and experimental curves for the VTP-ABH case matched reasonably well. Therefore, it was concluded that results in Fig. 7 demonstrate the feasibility of the FE simulation model.

4. SIMULATION RESULTS AND DISCUSSIONS

Using the FE model developed in Section 3, the average acceleration response amplitude of various plate structures as defined in Eq. (2) can be easily calculated and compared.

4.1. Vibration Control of Uniform Thickness Plate with Damping Layer and DVA

To demonstrate the advantages of the variable thickness plate embedded with ABH indentations, it is necessary to understand the vibration responses of a corresponding uniform
thickness plate when integrated with different vibration control approaches.

Figure 8 plots the simulated average acceleration transmissibility levels of the UTP, the UTP-Damp structure covered with 0.494 mm thick damping layer and the UTP-DVA structure attached with two DVAs. For the UTP-DVA case, the two DVAs were tuned to the first two resonances at 132 and 648 Hz. Comparing to the UTP case, it is seen that adding a free damping layer on the surface of the plate resulted in some attenuation at resonance peak locations, particularly at higher frequencies. For example, about 2.3 dB attenuation is observed at the third response peak around 750 Hz. Below this frequency, there is hardly any attenuation by simply adding the damping layer. On the other hand, when DVAs were attached to the UTP targeting at vibration suppression of the first two resonance modes, approximately 16.0- and 19.5-dB reductions occurred consequently at their tuning frequencies.

4.2. Vibration Control of Plate with Two ABH-DVA Pairs

Figure 9 compares the simulated average acceleration responses for the VTP-ABH, VTP-ABH-Damp and VTP-ABH-DVA cases. Note that Fig. 9 investigates the case where the VTP was embedded with two ABH-DVA pairs. This is the same as the experimental configuration shown in Fig. 1. It is seen that adding either additional damping layer or DVA to the existing ABH structure led to more attenuation in vibration responses. Comparing to the VTP-ABH case, when filling the two ABH indentations with damping materials, the response peak levels were reduced. In detail, the first five peaks except for the third one presented about 6 dB reduction, while a nearly 15.5 dB attenuation was achieved around 2850 Hz (i.e., the sixth resonance peak of the VTP-ABH case). Per design, two DVAs were attached at the center of the two circular ABH indentations (see Fig. 1) with natural frequencies tuned at 164 and 764 Hz. Around these two frequencies, the VTP-ABH-DVA curve in Fig. 9 shows significant attenuation of 18.5 dB and 11.5 dB over the VTP-ABH case and is also much lower than the VTP-ABH-Damp case.

In typical mechanical structures like car bodies and machine enclosures, vibrations at the low to medium frequency range (e.g., < 1000 Hz) is generally of more concern in practice. Furthermore, as lightweight design is often preferred, the amount of mass addition to the primary structure associated with the vibration control approach is usually also an important factor to consider. Figure 10 summarizes the total masses of various plate structures and their overall average acceleration response values calculated within the 0–800 Hz frequency range. Both simulation results and experimental results (if available from Section 2) are included for comparison. It can be seen that, in general, the response level is lower when the UTP is modified to have the ABH feature (VTP-ABH) while keeping the same total mass value. Both the simulation and experimental results indicate that the VTP-ABH-DVA case resulted in the lowest overall response level among all cases including the VTP-ABH-Damp case. For the experimental case,
although 8.24% less in the total mass, the variable thickness plate integrated with both ABH and DVA led to an overall response level of 7.8 dB lower than that of the UTP case within 800 Hz. In this simulation, the VTP-ABH-DVA case resulted in 7.0 dB more attenuation than the UTP-DVA case with the same amount of mass addition. In summary, results in Fig. 10 directly demonstrate the potential of combining DVAs and ABH effects together as an effective lightweight vibration control approach. It may be particularly attractive to apply this approach in the low-mid frequency range where adding damping is less effective.

### 4.3. Vibration Control of Plate with More Than Two ABH-DVA Pairs

Existing research has demonstrated that adding more ABH features to a plate can improve the vibration control effectiveness.\(^\text{19}\) Therefore, it is of interest to study the plate response when more than two ABH-DVA pairs are embedded. This can be accomplished using the FE model. Figure 11 displays a plate symmetrically embedded with two, four, and six ABH features. For convenience, they are referred to as VTP-ABH2, VTP-ABH4 and VTP-ABH6 in the following discussion. When DVAs are added, the corresponding plate structures are named VTP-ABH2-DVA2, VTP-ABH4-DVA4 and VTP-ABH6-DVA6. The dimensions of each ABH indentation are the same as in Fig. 1. However, to accommodate up to six ABH features, the size of the primary plate was enlarged to be 400 by 300 mm with a thickness of 4.5 mm. The plate material was again assumed to be steel. The mass of each DVA and the loss factor of the DVA damping material were 0.15 kg and 0.3, respectively. The DVA tuning frequency was adjusted by changing its stiffness in the simulation model.

With multiple DVAs integrated in the structure, each DVA can be tuned to suppress one individual resonance of the primary structure (i.e., VTP-ABH2, VTP-ABH4 and VTP-ABH6). As listed in Table 3, for each case, DVAs were tuned to attenuate lower order resonances starting with the first mode of the primary structure.

Figure 12 compares the simulated average acceleration responses of the variable thickness plates with 2, 4 and 6 ABH-DVA pairs. The uniform thickness plate with the larger dimension of 400 by 300 by 4.5 mm was added as the base-line. This simulation case is referred to as UTP-Large to differentiate from the previous UTP case. As expected, whenever a DVA is added, certain level of vibration attenuation is observed around the tuning frequency. For example, compared with the UTP-Large plate, the first two resonance peaks are about 12.8 and 5.6 dB lower for the VTP-ABH2-DVA2 case. For the VTP-ABH4-DVA4 and VTP-ABH6-DVA6 cases, the largest attenuation seems to occur at the third and sixth DVA tuning frequencies, respectively. On the other hand, a direct comparison of the responses of plates with various number of ABH-DVA pairs in Fig. 12(d) shows that increasing the number of ABH-DVA pairs may not always lead to lower vibration response levels.

Note that, to show the vibration control effectiveness versus the total mass of the structure, all cases in Fig. 12 have the same total mass of 4.23 kg. Figure 13 compares the overall average acceleration response values calculated over frequency ranges of 0–300, 0–500 Hz, 0–800 Hz and 0–3000 Hz. Table 4 lists the differences in the average acceleration response levels of various VTP-ABH-DVA structures compared to the baseline UTP-Large case in different frequency ranges. A positive value indicates more attenuation. It is seen that, for the low-mid frequency ranges (e.g., 0–300, 0–500 and 0–800 Hz), the cases of the VTP-ABH attached with DVAs always result in lower overall response levels. For example, within 0–300 Hz, the overall response level of the VTP-ABH2-DVA2 case is 6.2 dB lower than the baseline UPT-Large case. For the 0–500 and 0–800 Hz ranges, lower response levels are obtained by the cases with 4 or 6 DVA-ABH pairs. Since all cases have identical total structure mass, this comparison indicates that the VTP-ABH plate combined with DVAs has better vibration control performance, particularly at low-mid frequencies.

On the other hand, it is also interesting to notice that this approach may not be a good candidate for high frequency vibration control as the UTP-Large case presents the lowest response level in the 0–3000 Hz range. Based on Fig. 12, this observation may be explained by two effects. First, the VTP case has lower structural rigidity because of the embedded ABH features, which leads to more modes occurring below 3000 Hz. Second, targeting DVAs to lower order modes sometimes increases the responses at high frequencies due to the interaction between the DVAs and the primary structure.

### 5. SUMMARY AND CONCLUSIONS

Though the ABH and ABH integrated with damping have been demonstrated as promising methods for structural vibration and noise control, they are not effective as low frequen-
Table 3. Tuning frequencies of DVAs.

<table>
<thead>
<tr>
<th>Plate type</th>
<th>VTP-ABH2-DVA2</th>
<th>VTP-ABH4-DVA4</th>
<th>VTP-ABH6-DVA6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>160</td>
<td>420</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>388</td>
<td>864</td>
<td>1120</td>
</tr>
<tr>
<td></td>
<td>148</td>
<td>368</td>
<td>804</td>
</tr>
<tr>
<td></td>
<td>1088</td>
<td>1384</td>
<td>2044</td>
</tr>
</tbody>
</table>

Figure 12. Comparison of area-average acceleration transmissibility across entire plate surface in the frequency range of 0–3 kHz: (a) UTP-Large vs. VTP-ABH2-DVA2; (b) UTP-Large vs. VTP-ABH4-DVA4; (c) UTP-Large vs. VTP-ABH6-DVA6; (d) VTP-ABH2-DVA2 vs. VTP-ABH4-DVA4 vs VTP-ABH6-DVA6.

Figure 13. The overall average acceleration response values of different plate structures with the same total mass of 4.23 kg.

This study proposes a concept of combining DVAs and the ABH features to further improve the vibration control effectiveness of plate structures primarily aiming at low-mid frequencies.

Using a two-side-clamped and two-side-free steel plate structure, it was demonstrated experimentally that, with 8.24% less weight, the VTP plate integrated with two ABH-DVA pairs resulted in 7.8 dB reduction in the vibration response level compared to the uniform thickness plate in the frequency range up to 800 Hz. With the help of the FE model, plates with different ABH-DVA scenarios were simulated. It was shown that, under the constraint of the same structural mass, embedding ABH-DVA pairs in the plate structure generally led to lower response levels at low-mid frequencies. Overall, considering that most structural excitations are within 1000 Hz, the experimental and numerical results in this study suggested that the concept of combining ABH and DVA together is beneficial for vibration and noise control of structures where lightweight design is desirable.

The numerical simulation of the current study assumed that ABH-DVA pairs were symmetrically distributed on the plate structure and each DVA was tuned to a different frequency. Although the results demonstrated the effectiveness of this concept, it was also shown that adding more ABH-DVA pairs did not always lead to better vibration control performance. Depending on the frequency range of interest, different numbers of ABH-DVA pairs led to different vibration attenuation levels. For future work, it is thus worthwhile to continue investigating the influences of ABH-DVA locations and DVA tuning strategies on the control effectiveness in various frequency bands.

6. ACKNOWLEDGEMENTS

This study was funded by the following projects: The National Natural Science Foundation of China under grant No. 51375066, 61573078, 61175101 and 11472072.
Table 4. Attenuation in the average acceleration response level achieved by plate structures with 2, 4 and 6 ABH features combined with DVAs compared with the baseline UTP-Large case (positive values indicate more attenuation).

<table>
<thead>
<tr>
<th>Attenuation in the average acceleration response levels vs. the UTP-Large case (dB)</th>
<th>VTP-ABH2-DVA2</th>
<th>VTP-ABH4-DVA4</th>
<th>VTP-ABH6-DVA6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–300 Hz</td>
<td>6.2</td>
<td>4.3</td>
<td>1.0</td>
</tr>
<tr>
<td>0–500 Hz</td>
<td>2.4</td>
<td>5.0</td>
<td>4.7</td>
</tr>
<tr>
<td>0–800 Hz</td>
<td>2.6</td>
<td>4.9</td>
<td>4.3</td>
</tr>
<tr>
<td>0–3000 Hz</td>
<td>-0.2</td>
<td>-0.1</td>
<td>-0.9</td>
</tr>
</tbody>
</table>

REFERENCES


Two-Temperature Thermoelastic Damping of a Gold Nano-Beam Resonator with Variable Young’s Modulus

Hamdy M. Youssef
Faculty of Education, Mathematics Department, Alexandria University, Alexandria, Egypt.

Alaa A. El-Bary

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This paper deals with the thermoelastic damping (Q-factor) of a gold nano-beam resonator and is based on two-temperature thermoelasticity models. An explicit formula of the Q-factor has been derived when Young’s modulus is variable as a function of the room temperature. The length of the beam and Young’s modulus has been studied with comparison being made between the Biot model and the Lord-Shulman model (L-S). The numerical results show that the values of beam length, the relaxation time parameter, and the two-temperature parameter have a strong influence on the thermoelastic damping quality factor.

1. INTRODUCTION

The governing equations of the coupled thermoelasticity theory (Biot model) consist of the equation of motion, which is a hyperbolic partial differential equation, and of the energy conservation equation, which is parabolic based on the classical Fourier’s law of heat conduction. This model of heat conduction generates the infinite speed of the thermal wave propagation inside the thermelastic medium.1

Lord and Shulman (L-S) introduced the first generalization to thermoelasticity, which is called the “theory of generalized thermoelasticity with one relaxation time,” for an isotropic body.10 Among this theory, a modification law of heat conduction including both the heat flux and its time derivative (Cattaneo’s heat conduction), which is called non-Fourier’s law of heat conduction, replaces the conventional Fourier’s law. The heat equation of this theory is hyperbolic, which fixed the paradox of infinite speeds of propagation inherent in both.3

Youssef introduced a new theory of two-temperature generalized thermoelasticity and proved the general uniqueness theorem for the initially mixed boundary value problems.18 Sharma and Marin studied the effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space. Sharma and Marin showed that the effect of the two temperature parameters have a significant impact on amplitude ratios.14

Many applications, such as mechanical signal processing, scanning probe microscopes, and ultrasonic mass detection became based on micro- and nanoelectromechanical resonators. The most critical parameter of a microresonator is its Q-factor, or quality factor, and it’s closely related to the accuracy of measurement in many applications. A higher the Q-factor leads to both less energy being dissipated during vibrations and an increase in the resonator’s sensitivity. Thus, the study of the energy dissipation mechanism is of great significance for the development and improvement of the design for micro/Nano-electromechanical resonators.5,6

The inherent loss consists of thermoelastic effects and internal friction. The external loss includes support loss, gas damping, and surface loss. Intrinsic material damping due to thermoelastic coupling effect is the mechanism that imposes an upper limit on the achievable Q-factor for micro-electromechanical system (MEMS) resonators. In general, thermoelastic damping, support loss, and surface loss are the three primary sources of energy dissipation in MEMS.5–9

Zener was the first researcher on the thermoelastic damping problem that analyzed the problem by treating the viscoelastic material and obtaining the formula for the Q-factor of thermoelastic damping.9 Lifshitz and Roukes gave a formula of Q-factor for thermoelastic damping based on classical Fourier thermal conduction theory. Their model predicts that there is a peak of thermoelastic damping, which occurs at micrometer scale.5

When the beam height goes up to more than 100 μm or down to nanometer scales, the thermoelastic damping will decrease accordingly. However, the experimental results show that the Q-factor tends to decrease monotonously when the dimensions of microresonators go down to nanometer scale.5

Thermoelastic damping has been studied in many works, some of which use the classical theory of thermoelasticity based on Fourier’s law of heat conduction to calculate the thermoelastic damping.6,8,12–17 In the classical theory of thermoelasticity, the Fourier law of heat conduction is of a parabolic type that offers the infinite speed of propagation for the thermal wave. To eliminate this paradox, several non-classical theories have been formulated to admit the finite speed of thermal wave propagation, which is called the second sound theory for adding a first time derivative of the heat flux into the Fourier’s law of heat conduction.9 Sun et al. analyzed the thermoelastic damping of a beam resonator based on the generalized thermoelastic theory with one relaxation time.15 Sharma and Sharma studied damping in micro-scale circular plate resonators by the L-S theory of generalized thermoelasticity theory.13 Marin represented that the amplitude of the vibrations decays exponentially with the distance to the base. This decay estimate is similar to that of the Saint-Venant type.13
2. VARIABLE YOUNG’S MODULUS

Variable Young’s modulus of some materials was measured between 293 K and 973 K using the impulse excitation method, and by comparing with literature, data was reported. The data was measured with

\[ E(T) = E_0 - bT e^{(-T_0/T)}. \]  

The values of the fitting parameters “\( E_0 \)” and “\( T_0 \)” were related to the Debye temperature and the parameter “\( b \)” represented the harmonic character of the bond.\(^{2,4,5}\)

Farraro and Rex\(^4\) found that no departure from linearity was detected when they studied the dependence of Young’s modulus on the temperature, and determined the linear relation

\[ E(T) = E_0 - E_1 T; \]  

where \( E_0 \) was Young’s modulus in the ordinary case and \( E_1 \) was a constant value of the order \( 10^4 \) and they measured it for pure Nickel, Platinum, and Molybdenum.\(^2,4,5\)

Following this, we considered the fact that Young’s modulus depends on the temperature by using the following function

\[ E(T) \approx E(T_0) = E_0(1 - \gamma T_0); \]  

where \( \gamma = E_1/E_0 \) was a constant value.\(^2,4\)

3. FORMULATION OF NANO-BEAM EQUATIONS BASED ON TWO-TEMPERATURE MODEL

The formula of the two-temperature heat conduction model, which was derived by Youssef, took the form \(^4\):

\[ q_i(r, t) + \tau_0 \frac{\partial q_i(r, t)}{\partial t} = -KT_{ii}^C(r, t); \]  

and

\[ T^C(r, t) - T^D(r, t) = \beta T_{ii}^C(r, t); \]  

where \( q_i \) was the heat flux vector, \( T^C \) was the absolute conductive temperature, \( r = r(x_i) \) the position vector, \( K \) the thermal conductivity of the material, \( \tau_0 \) the relaxation time, \( T^D \) was the absolute dynamical temperature, and \( \beta \) was the two-temperature parameter.

The heat flux, temperature and volumetric strain for a thermoelastic isotropic body had the following relation:\(^5\)

\[ - q_{i,j}(r, t) = \rho C_v \frac{\partial T^D(r, t)}{\partial t} + \frac{T_0 \alpha E(T_0)}{1 - 2\nu} \frac{\partial e(r, t)}{\partial t}; \]  

where \( e = e_{ii} \) was the volumetric strain, \( \alpha \) was the thermal expansion coefficient, \( C_v \) was the specific heat at constant volume, \( E(T_0) \) was Young’s modulus as a function of the room temperature \( T_0 \), \( \nu \) was Poisson’s ratio, and \( \rho \) represented mass density.

Equation (4) gave the following equation

\[ q_{i,j}(r, t) + \tau_0 \frac{\partial q_{i,j}(r, t)}{\partial t} = -KT_{ii}^C(r, t). \]  

When substituting from Eq. (7) into Eq. (6), we obtained

\[ KT_{ii}^C(r, t) = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left[\rho C_v \frac{\partial T^D(r, t)}{\partial t} + \frac{T_0 \alpha E(T_0)}{1 - 2\nu} \frac{\partial e(r, t)}{\partial t}\right]. \]  

We then re-wrote Eq. (5) in the form

\[ \theta(r, t) = \varphi(r, t) - \beta \varphi_{ii}(r, t); \]  

where \( \varphi(r, t) = (T^C(r, t) - T_0) \) and \( \theta(r, t) = (T^D(r, t) - T_0) \). Hence, Eq. (8) took the form

\[ K \varphi_{ii}(r, t) = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left[\rho C_v \frac{\partial \theta(r, t)}{\partial t} + \frac{T_0 \alpha E(T_0)}{1 - 2\nu} \frac{\partial e(r, t)}{\partial t}\right]. \]  

When substituting from Eq. (9) into Eq. (10), we got

\[ \varphi_{ii} + \frac{\beta \rho C_v}{K} \left(\frac{\partial \varphi_{ii}}{\partial t} + \tau_0 \frac{\partial^2 \varphi_{ii}}{\partial t^2}) + \frac{T_0 \alpha E(T_0)}{K(1 - 2\nu)} \left(\frac{\partial e}{\partial t} + \tau_0 \frac{\partial^2 e}{\partial t^2}\right) = \right. \]  

4. THE RECTANGULAR NANO-BEAM RESONATOR

We considered the small flexural vibrations of a thin elastic beam of length \( L \) and a rectangular cross-section of dimensions \( h \). We took the x-axis along the axis of the beam, the y-axis along the thickness and the z-axis along the width direction. Thus, the strain tensor and volumetric strain took the forms:7–9

\[ e_{xx} = -\frac{\partial^2 w}{\partial x^2}; \]  

\[ e_{yy} = e_{zz} = \nu g \frac{\partial^2 w}{\partial x^2} + (1 + \nu)\alpha \theta; \]  

and

\[ e = e_{xx} + e_{yy} + e_{zz} = 2(1 + \nu)\alpha \theta - (1 - 2\nu)g \frac{\partial^2 w}{\partial x^2}; \]  

where \( w \) was the deflection of the beam.

The beam was free of stress, deformation and was kept at a uniform temperature \( T_0 \) in a state of equilibrium. The beam underwent flexural vibrations of small amplitudes about the x-axis in the x-y plane such that the deflection was consistent with the linear Euler-Bernoulli beam theory.

For this case, the equation of motion with thermoelastic coupling for the beam was given by7–9,17

\[ \frac{\partial^2 w}{\partial t^2} + \frac{E(T_0) I}{\rho A} \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{E(T_0) \alpha I_T}{\rho A} \frac{\partial^2 w}{\partial x^2} = 0; \right. \]  

\[ A = h \times b \] is the area of the cross-section \( I \) and \( I_T \) are the moment of inertia and thermal moment of the beam, respectively, which are given by7–9

\[ I = \int_A y^2 \, dy \, dz; \]  

and

\[ I_T = \int_A y \theta \, dy \, dz. \]
An Euler-Bernoulli beam Eq. (11), by using Eq. (13), took the form

\[ \varphi_{xx} + \frac{\beta}{\chi} \left( \frac{\partial \varphi_{xx}}{\partial t} + \tau_0 \frac{\partial^2 \varphi_{xx}}{\partial t^2} \right) + \frac{\Delta E \varepsilon^*}{\alpha} \left( \frac{\partial^3 w}{\partial x^2 \partial t} + \frac{\tau_0}{\chi} \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + \frac{1}{\chi} \left( \frac{\partial^2 \varphi}{\partial t^2} + \tau_0 \frac{\partial^2 \varphi}{\partial t^2} \right) + \frac{2 \Delta E \varepsilon^* (1 + \nu)}{\chi (1 - 2\nu)} \left( \frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} \right) = 0 \]  

(17)

where \( \chi = K / \rho C_v \) was thermal diffusivity of the material and \( \Delta E = T_0 E_0 \alpha^2 / \rho C_v \) was the relaxation strength of Young’s modulus.

Because the temperature gradients in the plane of the cross-section along the \( y \)-direction were much larger than those along the \( x \)-direction and that no gradients existed in the \( z \)-direction, we replaced \( \varphi_{xx} \) with \( \frac{\partial^2 \varphi}{\partial y^2} \). Also, \( \Delta E = T_0 E_0 \alpha^2 / \rho C_v < 10^{-6} \) then, Eq. (17) was simplified to

\[ \frac{\partial^2 \varphi}{\partial y^2} + \frac{\beta}{\chi} \frac{\partial \varphi}{\partial y} + \frac{\tau_0}{\chi} \frac{\partial^4 \varphi}{\partial y^2 \partial t^2} = \frac{1}{\chi} \left( \frac{\partial^2 \varphi}{\partial t^2} + \tau_0 \frac{\partial^2 \varphi}{\partial t^2} \right) - \frac{\Delta E \varepsilon^*}{\chi \alpha} \left( \frac{\partial^3 w}{\partial x^2 \partial t} + \tau_0 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right). \]  

(18)

To know the effect of thermoelastic coupling on the harmonic vibrations of the beam resonator, we assumed that:

\[ w(x, t) = W(x) e^{i \omega t}; \]  

(19a)

\[ \varphi(x, y, t) = \varphi(x, y) e^{i \omega t}; \]  

(19b)

\[ \theta(x, y, t) = \Theta(x, y) e^{i \omega t}. \]  

(19c)

By substituting the expressions in Eqs. (19) into Eq. (18), we obtained

\[ \left( 1 + \frac{\beta i \omega}{\chi} - \frac{\beta \tau_0 \omega^2}{\chi} \right) \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\chi} \left( i \omega - \tau_0 \omega^2 \right) \left( \frac{\phi - \Delta E \varepsilon^* y \frac{\partial^2 W}{\partial x^2}}{2} \right). \]  

(20)

The last Eq. (20) was written as a differential equation in the form

\[ \frac{\partial^2 \phi}{\partial y^2} - \frac{(i \omega - \tau_0 \omega^2)}{\chi + \beta i \omega - \beta \tau_0 \omega^2} \phi = \left( \frac{\Delta E \varepsilon^* y}{\alpha (\chi + \beta i \omega - \beta \tau_0 \omega^2)} \right) \frac{\partial^2 W}{\partial x^2}. \]  

(21)

Hence, the solution of Eq. (21) took the form

\[ \phi = A_1 \cos(ky) + A_2 \sin(ky) + \frac{\Delta E \varepsilon^* y}{\alpha} \frac{\partial^2 W}{\partial x^2}; \]  

(22)

where

\[ k = \sqrt{\frac{\omega}{\chi} \sqrt{a_1 - i a_2}} = \frac{\xi \eta}{h} i \xi \eta a_2; \]  

(23a)

\[ \xi = h \sqrt{\frac{\omega}{2 \chi}}; \]  

(23b)

\[ \eta = \sqrt{a_1 + a_2^2 + a_2^2}; \]  

(23c)

\[ a_1 = \frac{\omega \left[ \tau_0 (\chi - \omega^2 \tau_0 \beta) - \beta \right]}{(\chi - \omega^2 \tau_0 \beta)^2 + \omega^2 \beta^2}; \]  

(23d)

\[ a_2 = \frac{\chi}{(\chi - \omega^2 \tau_0 \beta)^2 + \omega^2 \beta^2}; \]  

(23e)

We considered that the boundaries of the beam were adiabatic [6], that is,

\[ \frac{\partial \phi}{\partial y} = 0 \text{ at } y = \pm h/2. \]  

(24)

The temperature distribution across the thickness had been obtained as:

\[ \phi = \frac{\Delta E \varepsilon^*}{\alpha} \frac{\partial^2 W}{\partial x^2} \left( y - \frac{\sin(ky)}{k \cos(kh/2)} \right). \]  

(25)

According to the medium dimension, the moment of inertia and the thermal moment took the forms

\[ I = \int_A y^2 dy dz = \int_0^{h/2} \int_{-h/2}^{h/2} y^2 dy dz = \frac{bh^3}{12}; \]  

(26)

and

\[ I_T = \int_A y \theta dy dz = e^{i \omega t} \int_0^{h/2} \int_{-h/2}^{h/2} y \theta dy dz. \]  

(27)

From Eq. (5) and Eq. (19), we determined

\[ \Theta = \phi - \beta \frac{\partial \phi}{\partial y}; \]  

(28)

which, by using Eq. (25), gave the thermal moment as follows:

\[ I_T = e^{i \omega t} \frac{bh^3 \Delta E \varepsilon^*}{12 \alpha} \left[ 1 + \frac{24 (1 + \beta k^2)}{h^4 k^4} \left( \frac{hk}{2} - \tan \left( \frac{hk}{2} \right) \right) \frac{\partial^2 W}{\partial x^2} \right]. \]  

(29)

By substituting Eq. (19) and Eq. (29) into Eq. (14), we obtained

\[ \omega^2 W = \frac{E_0 E^* I}{\rho A} \left[ 1 + \Delta E \varepsilon^* \left( 1 + f(\omega) \right) \right] \frac{\partial^2 W}{\partial x^2}; \]  

(31)

Alternately, we simplified the last equation to be in the form

\[ \omega^2 W = \frac{E_0 E^* I}{\rho A} \left[ 1 + \Delta E \varepsilon^* \left( 1 + f(\omega) \right) \right] \frac{\partial^2 W}{\partial x^2}; \]  

(31)

where the complex function \( f(\omega) \) took the form

\[ f(\omega) = (k(\omega)) = \frac{24 (1 + \beta k^2)}{h^4 k^4} \left( \frac{hk}{2} - \tan \left( \frac{hk}{2} \right) \right). \]  

(32)

From Eq. (31), we drove the vibration frequency in the presence of thermoelastic coupling, the thermal relaxation time and the two-temperature parameter as

\[ \omega = \omega_0 \sqrt{1 + \Delta E \varepsilon^* (1 + f(\omega))}; \]  

(33)

where \( \omega_0 \) was the isothermal value of frequency given by

\[ \omega_0 = \frac{q_0^2}{12 \rho} \]  

(34)
and

\[ q_n \{4.73, 7.853, 10.996, \ldots \}, \quad n = 1, 2, 3, \ldots \] (35)

Since \( \Delta_E < 1 \) we expanded the right-hand side of Eq. (33) by using the Taylor series for only the first order and obtained

\[
\omega = \omega_0 \left[ 1 + \frac{\Delta_E E^*}{2} (1 + f(\omega)) \right].
\] (36)

Because thermoelastic damping is feeble, that is, we replaced \( f(\omega) \) with \( f(\omega_0) \). The frequency relation Eq. (36) took the form

\[
\omega = \omega_0 \left[ 1 + \frac{\Delta_E E^*}{2} (1 + f(\omega_0)) \right].
\] (37)

The amount of thermoelastic damping was expressed regarding the inverse of the quality factor.\(^{7,9}\)

\[
Q^{-1} = 2 \left| \frac{Im(\omega)}{Re(\omega)} \right|.
\] (38)

After separating the real and imaginary parts of \( \omega \), the thermoelastic damping of a beam resonator in the context of two-temperature parameters was then given by Eq. (39) (see on the top of the next page).

5. NUMERICAL RESULTS AND DISCUSSIONS

The relationships between the variations of the Q-factor of the beam height \( h \), dimensionless variable \( \xi \) different values of the two-temperature parameter \( \beta = \{0.0, 10^{-10}\} \) and the thermal relaxation time \( \tau_0 = \{0.0, 10^{-10}\} \) of a microbeam resonator, which is made of Gold (Au) and clamped at two ends.

Material properties of Gold (Au) are listed as follows \(^{[14]}\):

\[
K = 318 \text{W/(mK)}; \quad (40a)
\]

\[
\alpha = 14.2 \times 10^{-6} \text{K}^{-1}; \quad (40b)
\]

\[
T_0 = 300 \text{K}; \quad (40c)
\]

\[
\rho = 1930 \text{kg/m}^3; \quad (40d)
\]

\[
C_r = 1301 \text{(kg K)}; \quad (40e)
\]

\[
E = 180 \text{GPa}; \quad (40f)
\]

\[
\gamma = 1 \times 10^{-4}; \quad (40g)
\]

\[
\nu = 0.44. \quad (40h)
\]

Figures 1–4 represent the Q-factor concerning the full range of beam height \( h (10^{-9} \leq h \leq 10^{-4}) \) based on two different models of thermoelasticity; coupled thermoelasticity (Biot model) and generalized thermoelasticity with one relaxation time (L-S model) in the context of the two-temperature thermoelasticity model (Youssef model). Figure 1 represents the results based on Biot’s model \((\tau_0 = 0.0)\) with different values for Young’s modulus parameter \( \gamma \) where the solid line represents the case of constant Young’s modulus, \( \gamma = 0 \), and the dashed line represents the case of variable Young’s modulus, \( \gamma \neq 0 \), model with one temperature type \( \beta = 0.0 \) we find that \( \gamma \) has a significant effect on the Q-factor of the nanobeam resonator. In Fig. 2, the same situation happened when the results based on L-S model \((\tau_0 = 10^{-10})\). In the two models, when Young’s modulus is variable, the Q-factor decreases up to the peak point, then the Q-factor increases. Figure 3 and 4 compare between Biot model and L-S model based on two-temperature type respectively. We found that, in the two cases where Young’s modulus is variable, the Q-factor increases. Figures 5–8 represent the Q-factor with respect to the wide range of the dimensionless variable \( \xi (1 \leq \xi \leq 10) \) based on two different models of thermoelasticity; coupled thermoelasticity (Biot model) and generalized thermoelasticity with one relaxation time (L-S model) in the context of the two-temperature thermoelasticity model (Youssef’s model). Figures 5 and 6 represent Biot model \((\tau_0 = 0.0)\) and L-S model \((\tau_0 = 10^{-10})\) with one temperature type \( \beta = 0.0 \) respectively. We found that the variability of Young’s modulus has very small effects on the Q-factor distribution with respect to the dimensionless variable \( \xi \). Figures 7 and 8 compare between Biot model \((\tau_0 = 0.0)\) and L-S model \((\tau_0 = 10^{-10})\) based on two-temperature type \( \beta = 10^{-10} \) respectively. We found that the variability of Young’s modulus has significant effects on the Q-factor distribution where it increases its value.
\[ Q^{-1} = \frac{24E_0E^*\alpha^2T_0}{C'\nu} \left[ \frac{\alpha^2}{\xi^2 \left( \eta^2 + \frac{a^2}{\eta^2} \right)^2} \right] = \frac{24E_0E^*\alpha^2T_0a}{C'\nu h^2} \left[ \frac{\alpha^2}{\xi^2 \left( \eta^2 + \frac{a^2}{\eta^2} \right)^2} \cos(\eta\xi) + \cosh \left( \frac{\xi a}{\eta} \right) \right], \]  

(39)

Figure 3. The Q-factor based on two-temperature model of Biot.

6. CONCLUSION

Young’s modulus parameter \( \gamma \) has significant effects on the Q-factor of the nanobeam resonator in the context of the Biot and L-S models of one temperature and also of two-temperatures. The variability of Young’s modulus has minimal effects on the Q-factor distribution concerning the dimensionless variable \( \xi \).

REFERENCES


Figure 5. The Q-factor based on one temperature model for Biot.

Figure 6. The Q-factor based on one temperature model for L-S.

Figure 7. The Q-factor based on two-temperature model for Biot.

Figure 8. The Q-factor based on two-temperature model for L-S.


Modal Analysis of Mistuned Turbine Blade Packet Due to Combined Blade and Lacing Wire Damage

Mangesh S. Kotambkar

Department of Mechanical Engineering, Visvesvaraya National Institute of Technology, Nagpur, India-440010.

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The turbine disk blade system is a cyclic symmetric structure, initially tuned with all its blades perfectly identical in geometry and material properties; similarly interconnecting lacing wires are of equal stiffness. The cyclic symmetry of the bladed disks gets destroyed due to small differences in material properties or geometric variation between individual blades or lacing wires causing mistuning. Although mistuning is typically small, it can have a drastic effect on the dynamic response of the system. In particular, mistuning can also cause vibration localization for a few blades and the associated concentration of vibration energy can lead to an increase in blade amplitude and stress levels. Numerical simulations are performed with the characteristic equations of the simplified continuum model. Two different damage severity indices are included in the model to study the combined effect of cracked blades and damaged lacing wires on the natural frequencies of grouped blades. This study highlights the characteristic changes in the sub modal frequencies under combined damage in a stand still position. Although the major cause of mistuning is blade damage, lacing wire damage is more frequent and often acts as a precursor to blade damage and thus the present study focuses on mistuning due to combined damage.

1. INTRODUCTION

The effect of mistuning on turbo machine blade vibration in a grouped blade-disk system has been a widely researched area for the last two to three decades. Research in this area has gained importance, essentially due to the critical applications of turbo machines in aero engines and power plants. It has been found that even a small amount of mistuning can lead to a stress build up through mode localization under forced vibration. During the operational life stages of the system, mistuning is caused due to the presence of blade to blade vibration in geometry, manufacturing tolerances or the evolution of cracks in the blade and / or damage in the lacing wire.

Prohl was first to study the dynamics of blade groups using lumped parameter modeling of blades with shroud rings attached to the blade tips.1 In 1924, Campbell wrote a breakthrough paper. This is considered to be the first work to explain mode localization in bladed disk assemblies. Experimental results were used to support the phenomenon explained theoretically by the presence of travelling and standing waves.2 Weaver and Prohl used the energy method and presented the modal characteristics of a packet of blades. They observed that the blade disk system had more modes and frequencies than the single blade.3 Deak and Baird analyzed a blade packet interconnected by lacing wire, using a coupled model with two bending and one torsional mode. They also studied the effect of root flexibility and centrifugal stiffening.4 Montoya developed the equations of motion to calculate the dynamic stiffness matrix for the coupled bending and torsional vibration of a twisted blade.5 A model was formulated using the variational method by Rao, in which governing differential equations were derived from an energy integral using Hamilton’s principle.6 Huang developed a computational procedure for calculating the free vibration of rotationally cyclic structures with various types of connecting elements using the transfer matrix method.7 Ewins and Imregun used method of substructure synthesis with receptance coupling.8 Mercadal et al. studied the issues that arise in blade resonance identification using Non-contacting Stress Monitoring Systems (NSMS) when blade resonances have slight variation causing mistuning and are dynamically coupled.9 Grossi et al. used the calculus of variations to obtain the equations of motion and natural boundary conditions at the intermediate elastic constraints like lacing wire connections.10,11 Wang et al. investigated the minimum stiffness of additional support that raises the natural frequency of a beam to its upper limit for different boundary conditions.12 Petreski presented the results of the investigation of the dynamic behavior, i.e. the natural frequencies and mode shape changes for a group of two, three and five blades as a result of changes made with the lacing wire.13 In recent work, Lim et
al. have proposed a modeling method for the modal analysis of a rotating multi-packet blade system. The effects of angular speed, disc flexibility, shroud flexibility, shroud location, disc radius, thickness and width taper ratios of the beam cross section, number of packets and total number of blades on the modal characteristics of the system are investigated.\textsuperscript{14}

Initial investigations and research works were mainly focused on tuned blade packets in which all the blades were considered to be identical to each other. The effect of mistuning in a blade group was first investigated by Ewins.\textsuperscript{15} In the next year, Ewins studied the mistuning effect and found the response of the system to be different from that of tuned system. He attributed the change to changed mode shapes.\textsuperscript{16} Subsequently, Ewins specifically dealt with completely shrouded or unshrouded bladed disk constructions. Various dimensional variations were found to be causing 2 to 3 % variation in 1-F frequency from the tuned frequency.\textsuperscript{17}

Singh and Ewins presented a probabilistic analysis of results by considering a random arrangement of blades on the disk and using the Monte Carlo simulation. For a mistuned bladed disk system, the vibration energy was found to be unevenly distributed among all blades.\textsuperscript{18, 19} Wei and Pierre studied mistuning caused by geometric variations and reported that even a small mistuning in such cases can lead to vibration localization and amplification of stress amplitudes in the blades.\textsuperscript{20}

Kuang and Huang considered both the free and forced response analysis of rotating, shrouded bladed disks by modeling each blade as a Euler-Bernoulli beam. They considered crack effect as local disorder of the system. They found out that the vibration response of a bladed disk changes significantly due to the existence of a crack and caused mode localization.\textsuperscript{21} In another recent work, Kuang and Huang analyzed the stability of a rotating bladed disk using Euler-Bernoulli beam models in conjunction with Galerkin’s method for formulating the equations of motion, which they solved with perturbation techniques.\textsuperscript{22}

Fang et al. investigated the vibration localization of bladed disks due to cracks for various parameters including internal coupling factor, crack severity, engine order of excitation, and the number of blades.\textsuperscript{23} Marinescu et al. have proposed a novel method applied to a finite element model of an industrial blisk.\textsuperscript{24} Recently, Hai long Xu et al. have studied the vibration characteristics of mistuned rotating blades using the coupled lumped parameter model. Numerical analysis has been done to study the effects of crack depth and its location on frequency, amplitude and vibration localization parameters.\textsuperscript{25}

Most of the researchers have considered blade damage as the primary source of mistuning. However, during the operational life of the turbine, lacing wire damage is more frequent than blade damage. Though damage in a lacing wire does not require the immediate shutdown of the turbo machine, if it goes undetected, such damages invariably induce stress localization in the blades and lead to its failure. Thus, the present work investigates the effect of combined damages, i.e. crack in one of the blades and damage in one of the lacing wires, on sub modal frequencies of the group. Damage manifests itself by loss of cyclic symmetry in the blade group leading to characteristic changes in the modal properties. In the beginning, the entire group of blades is considered tuned and later on mistuning is introduced through a damage severity parameter representing damage in the lacing wire and non-dimensional crack flexibility at the section of crack in the blade. The natural frequencies of the mistuned system are computed, and the perturbations are correlated with the damage.

2. OUTLINE OF THE INVESTIGATION

Shroud rings connect adjacent blades at the blade tips and are generally used for high pressure stages of the turbine. The turbine in these stages is dynamically robust, as the blades are of short lengths having high natural frequencies. On the other hand, low pressure stages have slender long blades which undergo comparatively much higher flexural deformations. The current work was carried out for low pressure steam turbines and therefore the blades were represented by a Euler-Bernoulli beam. The groups of blades formed with lacing wire connections in the turbine blade disk system were called blade packets, is shown in Fig. 1.

Each blade packet possessed a symmetrical configuration within the packet as well as for the whole disk. This enabled modeling and limited dynamic analysis to only one blade packet with appropriate boundary conditions at the interface. However, for a mistuned system, analysis had to be done with an entire model for which the FEM approach was mostly used. In this study, if an analytical approach was attempted, it would be highly complex if the entire disk bladed model were to be considered. Therefore, analysis was confined to a packet of three blades with uniform cross sections under both tuned and mistuned conditions.

In the present study the analytical model of blade group was suitably modified from those used by Chatterjee and Kotambkar.\textsuperscript{26, 30} Chatterjee et al. studied mistuning induced by lacing wire damage and its identification had been elaborated for both the simplified and actual industrial turbine blade packet.\textsuperscript{26} Kotambkar and Chatterjee presented blade damage induced mistuning.\textsuperscript{30} Here, combined damage induced mistuning is focused upon.

This study focuses on the characteristic changes of modal parameter, i.e. sub mode natural frequencies represented by $\beta L$ (Non dimensional angular frequency) of various flexural modes of mistuned packet of blades with uniform cross sec-
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The mistuning induced by lacing wire damage combined with blade damage was studied with various parameters like relative stiffness ratio \( \lambda \) of lacing wire and blade, lacing wire damage severity \( \alpha \), local flexibility coefficient due to crack in blade \( c_f \), Non dimensional cracked section flexibility coefficient \( \gamma \), non-dimensional crack location parameter \( \mu \) and crack severity parameter \( \phi \).

A crack in a blade was modeled as a torsional spring whereas damage in a lacing wire was modeled by an open coil helical spring. The non-dimensional crack flexibility at the cracked section of the blade was obtained from fracture mechanics using the strain energy density function. The effect of blade crack severity and its location combined with damage severity of single lacing wire on one of the modal parameters, i.e. the natural frequency of the packet of blades was investigated.

3. TUNED PACKET OF BLADES WITH LACING WIRE

The tuned packet of three blades, which was connected with the lacing wire at the tip of the blades, was initially assumed to have identical material properties and geometry for all its blades and lacing wires that were of equal stiffnesses. The packet of blades possessed additional frequencies compared to the frequency of a single blade. Each of the modes for a single blade was split into three sub modes due to the grouping of three blades. All the blades oscillated in phase in the first sub mode. Outer blades were out of phase and central blades remained neutral in the second sub mode. Outer blades were in phase and the central blades were out of phase in third sub mode. This caused higher deformation in the lacing wire in higher sub modes leading to increased stiffness contribution in the respective sub mode of the grouped blades. Thus higher sub mode frequencies values were greater than the first sub mode frequency of the fundamental mode. However, the stiffness contribution of the lacing wire would be relatively less in sub modes of higher modes of packet as the flexural stiffness of the blades were very high. The formation of analytical model and its detailed analysis of tuned system have been elaborated in.

4. MISTUNING DUE TO CRACKED UPPER BLADE AND DAMAGED LACING WIRE

Lacing wire damage in the form of a partial crack or complete breakage is more frequent during operational life of the turbine and may result in the cracking of the blade due to stress build up. It is assumed that the damage in the lacing wire reduced its axial stiffness, whereas the crack in the blade reduced its bending stiffness. This reduction in stiffnesses perturbs the cyclic symmetry of the blade group, initiating mistuning. The combined damage in the lacing wire and the blade modified the natural frequencies of the blade packet in a characteristic manner. This manner is investigated and presented below. Figs. 2(a-d) show four possible situations of combined damage in blade and lacing wire (partial damage and complete breaking) that were considered in the analysis.

The lacing wire connecting a pair of undamaged blades (middle and lower blade) was damaged, whereas the lacing...
wire connecting the damaged upper blade and the undamaged middle blade was presumed to be undamaged (Fig. 2a). This made spring stiffness \( k_2 \) less than \( k_1 \). In the case of the complete breakage of this lacing wire, \( k_2 \) was reduced to zero (Fig. 2b) thus uncoupling the pair. Similarly, the lacing wire connecting the segment of the cracked upper blade with the middle blade (Fig. 2c) was damaged. The lacing wire connecting the pair of undamaged middle and lower blades was intact. This made the spring stiffness \( k_1 \) less than \( k_2 \). In the case of the complete breakage of this lacing wire, \( k_1 \) was reduced to zero (Fig. 2d). As a crack in upper blade was dynamically the same as a crack in the lower blade, it was not considered additionally.

A lacing wire damage severity factor \( \alpha_i \) is introduced here as

\[
\alpha_i = 1 - \frac{k_i}{k} \tag{1}
\]

for \( i \)th lacing wire with damage.

The blade crack severity ratio, \( \phi = h/H \) which was the crack depth to thickness of the blade was varied from 0 to 0.5 and at different locations varied from 0.05 to 0.5. Accordingly, flexibility \( 'c_f' \) due to the crack and non-dimensional cracked section flexibility, \( \gamma = EIc_f/L \) values were obtained by using following Eqs. (2-3).\(^{27}\)

\[
f(\phi) = 0.6384 - 1.035\phi + 3.7201\phi^2 - 5.1773\phi^3 + 7.553\phi^4 - 7.332\phi^5 + 2.4900\phi^6; \tag{2}
\]

\[
\frac{1}{c_f} = k = \frac{bH^2E}{72\pi n^2I}(f(\phi)) \tag{3}
\]

The analytical model formulation was explained for both the tuned and mistuned packet to study lacing wire damage induced mistuning by Chatterjee,\(^{25}\) while Kotambkar presented blade crack induced mistuning.\(^{29}\) The mistuned model was developed with the assumption that the location of the crack divides the blade into two segments with the torsional spring connection of local flexibility \( 'c_f' \). The cracked blade therefore had eight constants, four in each segment with a separate coordinate system at the ends. However, all the other blades had four constants in their mode shape description.

At the blade roots i.e. at \( x_i = 0 \) the displacement fields \( y_i(x_i,t) \) for the three blades satisfied the forced geometric boundary conditions

\[
y_i(0, t) = 0 \quad \text{and} \quad \frac{\partial y_i(0, t)}{\partial x} = 0. \tag{4a}
\]

At the blade tip of right segment of the cracked blade 1 i.e. at \( x_2 = 0 \), the natural boundary conditions were

Bending moment \( EI\frac{\partial^2 y_2}{\partial x^2} = 0 \) and shear force

\[
\frac{\partial}{\partial x} \left[ EI\frac{\partial^2 y_2}{\partial x^2} \right] = (y_3 \text{ at } x_3 = L - y_2 \text{ at } x_2 = 0). \tag{4b}
\]

Whereas for the un-cracked blades 2 and 3, at the blade tips i.e. at \( x = L \), the boundary conditions were

Bending moment \( EI\frac{\partial^2 y_3}{\partial x^2} = 0 \) for all \( y_i \) where \( j = 3 \) and 4 and shear force for Blad2:

\[
\frac{\partial}{\partial x} \left[ EI\frac{\partial^2 y_3}{\partial x^2} \right] = k_2 (y_3 - y_4) + k_1 (y_3 - y_{2\text{at}x=0}); \tag{5a}
\]

Blade 3:

\[
\frac{\partial}{\partial x} \left[ EI\frac{\partial^2 y_4}{\partial x^2} \right] = k_2 (y_4 - y_3); \tag{5b}
\]

The compatibility conditions for displacement, slope, bending moment and shear force at the location of crack for the blade 1 gave rise to four equations as follows. The presences of cracks gave rise to a sudden jump in slope.\(^{26-28}\)

\[
y_1 \text{ at } x_1 = a = y_2 \text{ at } x_2 = b \tag{6a}
\]

\[
y_1 \text{ at } x_1 = a + EIc_f y_2 \text{ at } x_2 = b = y_2 \text{ at } x_2 = b; \tag{6b}
\]

\[
y_1 \text{ at } x_1 = a = y_2 \text{ at } x_2 = b; \tag{6c}
\]

\[
y_1 \text{ at } x_1 = a = y_2 \text{ at } x_2 = b. \tag{6d}
\]

For the non-trivial solution, the determinant of coefficients were zero. The expansion of the determinant of size 9 \( \times \) 9 lead to the characteristic equation. The roots of the equations were the set of natural frequencies.

\[
\begin{vmatrix}
0 & 0 & -2 \frac{\lambda_i}{(\beta L)^3} & 1 & -1 \\
R_a & S_a & -R_b & V_b & W_b \\
Q_a & Q_a & -R_b & V_b & W_b \\
S_a & P_a & Q_b & -T_b & U_b \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 \frac{\lambda_i}{(\beta L)^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{vmatrix} = 0; \tag{7}
\]

where \( \lambda_i = \frac{EI}{L^3} \) – stiffness ratio between the lacing wire and the blade:

\[
P = \cosh \beta L + \cos \beta L; \quad Q = \sinh \beta L + \sin \beta L; \quad R = \cosh \beta L - \cos \beta L, \quad \text{and} \quad S = \sinh \beta L - \sin \beta L; \quad P_a = \cosh \beta L + \cos \beta L; \quad Q_a = \sinh \beta L + \sin \beta L; \quad R_a = \cosh \beta L - \cos \beta L \mu \quad \text{and} \quad S_a = \sinh \beta L \mu - \sin \beta L \mu; \quad \text{in which} \mu = a/L \text{ is normalized crack location.}
\]

\[
F_b = \cosh \beta L (1 - \mu) + \cos \beta L (1 - \mu); \quad Q_b = \sinh \beta L (1 - \mu) + \sin \beta L (1 - \mu); \quad R_b = \cosh \beta L (1 - \mu) - \cos \beta L (1 - \mu) - \sin \beta L (1 - \mu) - \sin \beta L (1 - \mu); \quad T_b = \cosh \beta L (1 - \mu); \quad U_b = \cosh \beta L (1 - \mu); \quad V_b = \sinh \beta L (1 - \mu); \quad W_b = \sin \beta L (1 - \mu). \tag{8}
\]

For simulation, two levels of lacing wire damage severity, i.e. 0.6 (partial damage) and 1.0 (complete breakage), were considered. At each of these lacing wire damage levels, four
Table 1. Natural frequencies (βL values) due to partially damaged distant lacing wire, α = 0.6 and cracked upper blade at a/L = 0.05

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>βL values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
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<td></td>
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<tr>
<td>III</td>
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</tr>
</tbody>
</table>

Table 2. Natural frequencies (βL values) due to partially damaged distant lacing wire, α = 0.6 and cracked upper blade at a/L = 0.15

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>βL values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
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<tr>
<td>II</td>
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<td>III</td>
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</table>

Table 3. Natural frequencies (βL values) due to partially damaged distant lacing wire, α = 0.6 and cracked upper blade at a/L = 0.3

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>βL values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
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<tr>
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<td>III</td>
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</tbody>
</table>

Table 4. Natural frequencies (βL values) due to combined blade and lacing wire damage

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>βL values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
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<tr>
<td>II</td>
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</tbody>
</table>

Table 5. Natural frequencies (βL values) due to broken distant lacing wire, α = 1.0 and cracked upper blade at a/L = 0.05

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>βL values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
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<tr>
<td>II</td>
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<tr>
<td>II</td>
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<tr>
<td>III</td>
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<td></td>
</tr>
<tr>
<td>III</td>
<td></td>
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</tr>
</tbody>
</table>

Due to lacing wire damage alone, the first sub mode frequency of all the modes is unchanged while higher sub modes frequencies for all the modes are reduced. However, with additional damage to the blade, the frequency of all the sub modes was reduced with the growth of blade damage. The variation of sub modal natural frequencies with crack severity at a particular location accompanied with partially damaged lacing wire, is shown in Figs. 3a-3c.

In the natural frequency for the first sub mode (Sub Mode 1) of all the modes is similar to what was observed with the growth of damage in only the blade. However, in the current situation, a drop in the frequency value for higher sub modes is more due to combined lacing wire damage and blade damage.

From Figs. 3a-3c, it can be seen that the natural frequency of the Sub Mode 1 of all the modes is affected more for the crack location near the root. A drop in natural frequency for the first sub mode is more significant as it can be seen in the figures. However, higher modes (Mode 2 and Mode 3) are better indicators for crack location compared to the fundamental mode especially if the crack location is near the anti-node of the mode. The anti-node for the second mode (Mode 2) is at a/L = 0.5 and for the third mode (Mode 3) it is at a/L = 0.3.

4.2. Cracked Upper Blade and Fully Damaged Distant Lacing Wire

For the completely broken distant lacing wire and damaged blade (Fig. 2b), sub modal frequencies were obtained and presented in Tables 6, 7 and 8. The blade crack severity φ is varied from 0 to 0.5 at different locations a/L= 0.05, 0.15, 0.3 and 0.5.

The breaking of lacing wire caused the separation of the three-blade packet system into two sub systems. Thus, in the
absence of blade damage ($\phi = 0$), the first sub mode natural frequency value appeared twice as indicated in the first column of Tables 5, 6, 7 and 8. The frequency spectrum had only two peaks for each of the modes as opposed to three distinct peaks for tuned system. In the present case, one of the sub systems had a single undamaged (lower) blade and the other had a pair of damaged (upper) and undamaged blade (middle) connected with intact lacing wire. Therefore, the frequency spectrum had three distinct peaks for each of the modes in spite of losing one sub mode frequency due to separation of system. Only one peak of the mistuned system matches with that of the tuned system whereas other two peaks show split with first and second sub mode frequencies.

The frequency peaks of the separated single undamaged blade matched the first sub mode frequency peaks of the tuned system. The separated pair with one damaged blade had both peaks mismatched with the corresponding frequencies of the tuned system.

Figs. 4a-4c clearly show that, with broken lacing wire and undamaged blades there shall be only two peaks in the frequency spectrum that get split into three peaks due to the advent of damage in the blade of the separated sub systems. The
split was widening with crack severity for cracks located near to the root and diminishing with cracks located away from the root for the first mode. However, for higher modes, if the crack location was closer to the anti-node of respective mode, the split widened further. It can also be seen that in all the modes, the second sub mode frequency remained constant after getting converged to the first sub mode frequency, representing separated single blade sub system. Whereas dropping of one of the sub mode frequencies below the first sub mode frequency indicates presence of blade damage in the other separated sub system.

### 4.3. Cracked Upper Blade and Partially \((\alpha = 0.6)\) Damaged Adjacent Lacing Wire

The cracking of upper blade accompanied with damage in the adjacent lacing wire, i.e. the connecting upper blade to middle blade, was considered (Fig. 2c). The lacing wire damage, \(\alpha = 0.6\), was assumed to remain same whereas crack severity \(\phi\) in the blade at a particular location grew from 0 to 0.5. The crack locations were \(a/L = 0.05\), 0.15, 0.3 and 0.5. The sub modal frequencies are given in Tables 9, 10, 11 and 12.

In the absence of a damaged blade, both the cases of damaged distant or adjacent lacing wire are dynamically identical, thus having a similar effect on sub modal frequencies. However lacing wire damage accompanied with blade damage changed the dynamics of these two cases. From Tables 9-12, it was observed that crack in upper blade had more pronounced effect when combined with the damaged adjacent lacing wire than distant lacing wire for Sub Modes I and II whereas the effect was reversed for Sub Mode III. Figs. 5a-5c show that sub modal natural frequency drop appears qualitatively identical to the previous case of combined damage, i.e. the crack in

### Table 9. Natural frequencies \((\beta L)\) values due to partially damaged adjacent lacing wire, \(\alpha = 0.6\) and cracked upper blade at \(a/L = 0.05\)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>(\beta L) values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(\phi = 0)</td>
<td>1.87510, 1.87353, 1.86871, 1.85901, 1.84090, 1.81063</td>
</tr>
<tr>
<td>II</td>
<td>(\phi = 0.1)</td>
<td>1.98440, 1.98181, 1.97498, 1.96475, 1.94204</td>
</tr>
<tr>
<td>III</td>
<td>(\phi = 0.2)</td>
<td>2.24791, 2.24782, 2.24758, 2.24721, 2.24673, 2.24620</td>
</tr>
<tr>
<td>I</td>
<td>(\phi = 0.3)</td>
<td>4.69410, 4.69875, 4.67066, 4.63740, 4.59096, 4.53442</td>
</tr>
<tr>
<td>II</td>
<td>(\phi = 0.4)</td>
<td>4.70177, 4.69698, 4.69723, 4.69695, 4.69685, 4.69680</td>
</tr>
<tr>
<td>III</td>
<td>(\phi = 0.5)</td>
<td>4.72743, 4.72726, 4.72705, 4.72691, 4.72683, 4.72678</td>
</tr>
</tbody>
</table>

### Table 10. Natural frequencies \((\beta L)\) values due to partially damaged adjacent lacing wire, \(\alpha = 0.6\) and cracked upper blade at \(a/L = 0.15\)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>(\beta L) values</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(\phi = 0)</td>
<td>1.87510, 1.87396, 1.87055, 1.86387, 1.85168, 1.83095</td>
</tr>
<tr>
<td>II</td>
<td>(\phi = 0.1)</td>
<td>1.98440, 1.98368, 1.97210, 1.96824, 1.95621, 1.94673</td>
</tr>
<tr>
<td>III</td>
<td>(\phi = 0.2)</td>
<td>2.24791, 2.24785, 2.24768, 2.24742, 2.24707, 2.24664</td>
</tr>
<tr>
<td>I</td>
<td>(\phi = 0.3)</td>
<td>4.69410, 4.69368, 4.69210, 4.68824, 4.68125, 4.67136</td>
</tr>
<tr>
<td>II</td>
<td>(\phi = 0.4)</td>
<td>4.70177, 4.70109, 4.69831, 4.69760, 4.69726</td>
</tr>
<tr>
<td>III</td>
<td>(\phi = 0.5)</td>
<td>4.72743, 4.72740, 4.72733, 4.72725, 4.72714, 4.72706</td>
</tr>
</tbody>
</table>

![Figure 4. Variation of sub modes frequencies \((\beta L)\) values) due to crack severity at different location in upper blade and broken distant lacing wire of (a) Mode 1 (b) Mode 2 (c) Mode 3.](image-url)
Table 11. Natural frequencies ($\beta L$ values) due to partially damaged adjacent lacing wire, $\alpha = 0.6$ and cracked upper blade at $a/L = 0.3$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>$\beta L$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>I</td>
<td>1.87510 1.87448 1.87264 1.86918 1.86312 1.85301</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>1.98440 1.98342 1.98071 1.97618 1.96963 1.96145</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>2.24791 2.24788 2.24768 2.24750 2.24725</td>
</tr>
<tr>
<td></td>
<td>IV</td>
<td>4.69410 4.69359 4.69155 4.68634 4.67683 4.66274</td>
</tr>
<tr>
<td></td>
<td>V</td>
<td>7.85476 7.84624 7.83895 7.83197 7.82420 7.81666</td>
</tr>
</tbody>
</table>

Table 12. Natural frequencies ($\beta L$ values) due to partially damaged adjacent lacing wire, $\alpha = 0.6$ and cracked upper blade at $a/L = 0.5$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>$\beta L$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>I</td>
<td>1.87510 1.87490 1.87430 1.87322 1.87142 1.86857</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>1.98440 1.98412 1.98332 1.98191 1.97970 1.97652</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>2.24791 2.24788 2.24750 2.24725 2.24719 2.24698</td>
</tr>
<tr>
<td></td>
<td>V</td>
<td>7.85476 7.85475 7.85475 7.85473 7.85471 7.85468</td>
</tr>
</tbody>
</table>

Table 13. Natural frequencies ($\beta L$ values) due to broken adjacent lacing wire, $\alpha = 1.0$ and cracked upper blade at $a/L = 0.25$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>$\beta L$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>I</td>
<td>1.87510 1.87048 1.85757 1.83544 1.80196 1.75601</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>1.87510 1.87510 1.87510 1.87510 1.87510 1.87510</td>
</tr>
<tr>
<td></td>
<td>III</td>
<td>2.21350 2.21350 2.21350 2.21350 2.21350 2.21350</td>
</tr>
<tr>
<td></td>
<td>IV</td>
<td>4.69410 4.68642 4.66558 4.63184 4.58504 4.52817</td>
</tr>
<tr>
<td></td>
<td>V</td>
<td>7.85476 7.84660 7.82488 7.79104 7.74669 7.69649</td>
</tr>
</tbody>
</table>

4.4. Cracked Upper Blade and Adjacent Lacing Wire Fully ($\alpha = 1.0$) Damaged

Similarly the set of sub modal frequencies was obtained for completely broken adjacent lacing wire with cracked upper blades and presented in Tables 13, 14, 15 and 16.

Due to the breaking of the adjacent lacing wire, one of the split sub systems had a single damaged blade and the other had a pair of undamaged blades connected with intact lacing wire. This could be inferred from Figs. 6a-6c as in all modes the second and third sub mode converged to tuned sub system and remained constant thereafter representing the separated sub system with pair of undamaged blades. On the other hand the first sub mode frequency reduced with crack growth thus representing another sub system with damaged blade.

In either of the cases of breaking of lacing wire, the first sub mode frequency of all the modes matched with the first tuned sub mode frequency in the spectrum. The match of the
Figure 6. Variation of sub modes frequencies (βL values) due to crack severity at different location in upper blade and broken adjacent lacing wire of (a) Mode 1 (b) Mode 2 (c) Mode 3.

Table 14. Natural frequencies (βL values) due to broken adjacent lacing wire, α = 1.0 and cracked upper blade at a/L = 0.15

<table>
<thead>
<tr>
<th>Mode</th>
<th>Sub Mode</th>
<th>φ = 0</th>
<th>φ = 0.1</th>
<th>φ = 0.2</th>
<th>φ = 0.3</th>
<th>φ = 0.4</th>
<th>φ = 0.5</th>
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<tbody>
<tr>
<td>1</td>
<td>I</td>
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<tr>
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<tr>
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<td>7.85476</td>
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Table 15. Natural frequencies (βL values) due to broken adjacent lacing wire, α = 1.0 and cracked upper blade at a/L = 0.3

<table>
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<th>Sub Mode</th>
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<tr>
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Table 16. Natural frequencies (βL values) due to broken adjacent lacing wire, α = 1.0 and cracked upper blade at a/L = 0.5

<table>
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<th>Sub Mode</th>
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<th>φ = 0.3</th>
<th>φ = 0.4</th>
<th>φ = 0.5</th>
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<td></td>
<td>III</td>
<td>7.86097</td>
<td>7.86097</td>
<td>7.86097</td>
<td>7.86097</td>
<td>7.86097</td>
<td>7.86097</td>
</tr>
</tbody>
</table>

The similar phenomenon can be observed if the lower blade is cracked and one of the lacing wires is broken as the system is dynamically same as the one discussed above and therefore not discussed separately.

5. MISTUNING DUE TO CRACKED MIDDLE BLADE AND DAMAGED LACING WIRE

Due to symmetry of this case with cracked middle blade connected on either side to other blades, damage in any of the lacing wires is dynamically identical and hence only one case of damaged lacing wire is considered. The similar two levels of lacing wire damages are taken i.e. partial damage and com-
peculiar damage. The frequency determinant in Eq. (7) is modified for the cracked middle blade and damaged lacing wire, to get determinant in Eq. (9).

\[
\begin{vmatrix}
    P - R \frac{\lambda_1}{(\beta L)} & Q \\
    S - R \frac{\lambda_2}{(\beta L)} & 0
\end{vmatrix}
= 0
\]

\[
\begin{vmatrix}
    0 & 0 & 0 & 0 & 0 & 2 \frac{\lambda_1}{(\beta L)} \\
    0 & 0 & R_a & S_a & -P_b \\
    0 & 0 & Q_a & R_a & S_a + R_b \gamma \beta L \\
    0 & 0 & P_a & S_a & -R_b \\
    0 & 0 & S_a & P_a & Q_b \\
    0 & 0 & 0 & 0 & 2 \frac{\lambda_2}{(\beta L)}
\end{vmatrix}
= 0
\]

\[
\begin{vmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{vmatrix}
\]

5.1. Cracked Middle Blade and Partially Damaged (\(\alpha = 1.0\)) Lacing Wire

The sub modal frequency values were calculated from the characteristic equation obtained from the frequency determinant (Eq. 9), the results of numerical simulations are presented in Tables 17, 18, 19, and 20.

The crack parameters and lacing wire damage index were varied to find their effect on sub modal frequencies. It is observed from Tables 17-20 and Figs. 7a-7c that sub modal frequencies are dropping with the growth of cracks in a way that is identical to the case of middle blade damage alone as reported by Kotambkar et al.\(^{30}\) except for the small drop in second sub modal frequency here. This is a clear indication of combined damage as otherwise the second sub modal frequency remains unaffected due to cracking in middle blade only.

5.2. Cracked Middle Blade and Fully Damaged (\(\alpha = 1.0\)) Lacing Wire

The two sub systems, due to the breaking of either of the lacing wires and cracked middle blade, are similar. The resulting sub systems in either case have one subsystem with an undamaged blade and another with a pair of damaged and undamaged blades interconnected with intact lacing wire. It is also observed that the current separated sub systems are identical to those obtained due to the breakage of distant lacing wire accompanied with cracks in upper blades (Section 4.2) or lower blades, thus adding to the difficulty in identifying a cracked blade. The modal spectrum thus would not indicate clearly whether split is due to crack in upper blade or middle blade accompanied with complete breakage of lacing wire.

However, in both the cases that involved one of the cracked blades (either upper or middle) combined with partial damage in one of the lacing wires, it’s observed that a drop in the first and second sub mode frequency is more. However, in the third sub mode, the drop is less when the crack is in an upper or lower blade compared to a crack in the middle blade. This is for all the crack locations, crack severity and lacing wire damage severity.
6. CONCLUSIONS

In the simplified model of blades with uniform cross section, flap wise bending modes will be excited first and these modes will dominate the free vibration response. As there is no twist, bending modes are not coupled with torsional modes which will be there in real turbine blade. It has been reported in,\(^\text{(3)}\) that for the blades with twist, first few modes which dominate the response are bending modes only as torsional mode natural frequencies are generally much higher. Hence torsional modes are not considered in the present work.

Cracking in one of the blades of the packet causes a drop in natural frequency for all of the modes and their sub modes. However, damage in a lacing wire causes a drop in frequencies for all of the sub modes except for the first sub mode. Thus, if the modal spectrum shows a mismatch for all of the sub mode frequencies, it implies the presence of a crack in the blade along with the possibility of a partially damaged lacing wire but certainly without a breaking of the lacing wire. A split in all the sub modal frequencies, except a few higher sub modes, indicates a broken lacing wire and a damaged blade. Thus, many damage scenarios could be identified by monitoring changes in the sub mode frequencies of various modes in the modal spectrum.

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Optimized Mounting Positions for Vibratory Machines in Buildings Based on Structure-Borne Sound Power Transmission and Machine Stability

Zhen Wang and Cheuk Ming Mak

Department of Building Services Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong.

Dayi Ou

School of Architecture, Huaqiao University (361021), Xiamen, People’s Republic of China.
State Key Laboratory of Subtropical Architecture Science, South China University of Technology (510640), Guangzhou, People’s Republic of China.

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Optimized mounting positions for isolated vibratory machines in buildings require a minimum transmission of structure-borne sound power from the machine to the floor structure and relative stability of the isolated machine. Previous work by Mak and co-investigators has indicated the importance of using structure-borne sound power to assess vibration isolation and to select the best mounting positions by considering the structure-borne sound power transmission. This paper is a first attempt to utilize both the structure-borne sound power transmission and the rotational velocity (or the stability) of the machine to select the optimized mounting positions for an isolated vibratory machine. The results reveal that a vibratory machine should be symmetrically installed on diagonal lines of the receiving floor structure.

1. INTRODUCTION

Many vibratory machines are installed in buildings, such as chillers, boilers, pumps, air compressors, electric motors, and generators. They transmit structure-borne sound power to the floor structure. The structural acoustics process can be subdivided into four main stages: generation, transmission, propagation, and radiation. Transmission covers the transfer of oscillatory energy from the mechanisms of generation to a (passive) structure. Vibration isolators are therefore used to reduce this transmission of structure-borne sound power from the vibratory machine to the floor structure. The “force transmissibility” method is widely used in the building services industry to evaluate the performance of vibration isolators. The “isolation efficiency” index used in this method is based on a ratio of forces transmitted through a single contact point with and without vibration isolation. Mak and Su indicated that the structure-borne sound power transmission is closer than the transmitted forces to the sound radiation as it considers the interaction and phase difference of motion between the complex vibratory source and the receiving floor structure. They therefore proposed the “power transmissibility” method to evaluate the performance of vibration isolation. In the development of the power transmissibility method, multiple contact structure-borne sound power sources were considered. The determination of the structure-borne sound power transmission from a machine with multiple contact points to the floor structure required source activity, source mobility, and receiver mobility. Mak and Su highlighted the effect of receiver mobility on structure-borne sound power transmission and the performance of vibration isolation. Much previous work has thus focused on receiver mobility. Petersson and Plunt proposed the effective mobility method to calculate structure-borne sound power. Gibbs and co-investigators proposed the reception plate method to calculate structure-borne sound power. Mayr and Gibbs developed an approximation method to predict point and transfer mobility of lightweight, point-connected ribbed plates. Gibbs also investigated the uncertainties resulting from utilizing a simplified method in predicting structure-borne sound power for multi-contact sources in buildings. Differences between predictions of structure-borne sound power transmission with and without terminal cross-coupling have also been investigated.

Zu and Mak proposed a method to determine the best mounting positions for a vibratory machine by considering the transmission of structure-borne sound power from the machine to the floor structure. They suggested that the best mounting positions for an isolated machine produce minimum structure-borne sound power transmission to the floor structure. In fact, source activity, source mobility, and receiver mobility not only affect the structure-borne sound power transmission, but also the stability of the machine. A relatively low structure-borne sound power transmission does not imply a relatively more stable machine or vice versa. The optimized mounting positions should therefore consider both the structure-borne sound power transmission and the stability of the isolated machine. This paper is a first attempt to utilize both the structure-borne sound power transmission and the stability of the machine to
select the optimized mounting positions for an isolated vibratory machine. Yun and Mak proposed using normalized average vibration velocities and rotational velocities to assess the stability of a vibratory machine. Therefore, rotational velocity will be used in this study to assess the stability of the isolated machine with an inertia block. The mobility of the receiving floor is determined by utilizing the finite difference method. Two different boundary conditions, totally simply supported plate and totally fixed supported plate, will be used in this study.

2. THEORY

2.1. Simple Model for the Vibratory Machine and the Inertia Block

A vibratory machine model with four symmetrical contact points \((p_1, p_2, p_3, \text{ and } p_4)\) is shown in Fig. 1.

The diameter of the lower and upper parts of the machine model is \(D_1\) and \(D_2\), respectively. The height of both parts is \(H\). The mass of the lower and upper parts is \(M_{01}\) and \(M_{02}\), respectively. The mass of the vibratory machine model is \(M_0\). It is assumed that the vibratory machine is excited by an eccentric force \(F_0\) at the centre of the vibratory machine. Take the centre of the bottom face of the inertia block as the coordinate origin point. The moments of inertia of the vibratory machine model throughout the coordinate origin point are given by:

\[
I_{yy} = I_{xx} = \frac{1}{12} M_0 (D_1^2 + H^2) + M_{02} (D_2^2 / 4 + 7 \times H^2 / 3).
\]

The vibratory machine is symmetrically placed on an inertia block with four symmetrical contact points \((q_1, q_2, q_3, \text{ and } q_4)\), which is installed on the floor as shown in Fig. 2. The dimensions of the inertia block are \(L_b \times L_b \times H_b\). The mass of the inertia block is \(M_b\) with uniform mass distribution.

Take the centre of the bottom face of the inertia block as the coordinate origin point. The moments of inertia of the symmetrical vibratory machine model with inertia block throughout the coordinate origin point are given by:

\[
I_{yy}' = I_{xx}' = I_{xx} + M_b (L_b^2 / 12 + H_b^2 / 3) + I_{add}.
\]

where 
\[
I_{add} = M_{01} [(H_b + H/2)^3 - H^3] / H + M_{02} [(H_b + 2H/3)^3 - 8H^3] / H.
\]

2.2. Floor Mobility

The mobility of a plate can be determined through analysis. This first solves the equation of the motion of the plate. The general two-dimensional equation of motion for a plate without excitation is given by Cremer, Heckl, and Petersson:

\[
D \nabla^4 v(x,y) - m'' \omega^2 v(x,y) = 0;
\]

where \(\nabla^4\) denotes the biharmonic operator, \(D\) denotes the bending stiffness, \(m''\) denotes the area density, and \(v(x,y)\) denotes the velocity of the point with coordinates \((x,y)\). The general two-dimensional equation can be solved by utilizing the variable separation method. The natural frequencies and mode shapes of a simply supported rectangular plate are given by Fahy and Gardonio:

\[
\omega_r = \sqrt{D / m'' \left( (r_1 \pi / L_x)^2 + (r_2 \pi / L_y)^2 \right)};
\]

\[
\phi_r(x,y) = 2 \sin(r_1 \pi x / L_x) \sin(r_2 \pi y / L_y);
\]

where \(r_1\) and \(r_2\) are the modal indices of the \(r^{th}\) mode, and \(L_x\) and \(L_y\) denote the length and width of the plate.

The general expression of the mobility of the point with coordinates \((x,y)\) on the plate can be expressed as follows:

\[
Y_r(x,y|x,y) = j \omega \sum_{r=1}^{\infty} M_p \phi_r^2(x,y) / \omega^2 (1 + j \eta - \omega^2);
\]

\[
[Y_S] = \begin{pmatrix} 1 + \alpha^2 & 1 - \alpha^2 & 1 - \alpha^2 & 1 \\ 1 & 1 + \alpha^2 & 1 & 1 - \alpha^2 \\ 1 - \alpha^2 & 1 & 1 + \alpha^2 & 1 \\ 1 & 1 - \alpha^2 & 1 & 1 + \alpha^2 \end{pmatrix}.
\]

2.3. Source Mobility

To describe the dynamic characteristics of the multi-point vibratory machine, a \(4 \times 4\) source mobility matrix is expressed as follows:

\[
[Y_S] = \begin{pmatrix} 1 + \alpha^2 & 1 - \alpha^2 & 1 - \alpha^2 & 1 \\ 1 & 1 + \alpha^2 & 1 & 1 - \alpha^2 \\ 1 - \alpha^2 & 1 & 1 + \alpha^2 & 1 \\ 1 & 1 - \alpha^2 & 1 & 1 + \alpha^2 \end{pmatrix}.
\]

where \(\alpha^2 = (M_0 + M_b) \cdot L_b^2 / 4 I_{xx}\).

where \( M_p \) denotes the mass of the plate and \( \eta \) denotes the loss factor of the plate.

The general expression of the transfer mobility on the plate between two points \((x_1, y_1)\) and \((x_2, y_2)\) can be expressed as follows:

\[
Y_r(x_1, y_1|x_2, y_2) = j\omega \sum_{r=1}^{\infty} \frac{\phi_r(x_1, y_1)\phi_r(x_2, y_2)}{M_p[\omega^2(1 + j\eta) - \omega^2]}.
\]  \( (8) \)

In this equation, the natural frequencies \( \omega_r \) and the modal shapes \( \phi_r(x, y) \) need to be solved. For totally simply supported rectangular plates, analytical solutions of natural frequencies and modal shapes can be obtained. However, analytical solutions cannot be obtained for plates with other kinds of boundary conditions and complex geometries. Numerical methods, such as finite element methods and finite difference methods should be utilized to obtain approximate results. The floor model being investigated in this study is assumed to be a flat plate with uniform thickness and density. For the assumption of uniform structure, the finite difference method is quite convenient to be applied and relatively straightforward in understanding and application.

Therefore, the finite difference method was utilized in this study.

### 2.3. Structure-Borne Sound Power and Stability

Assuming that the free velocity vector of the source with inertia block is given by:

\[
[V_S] = V_0[1, 1, 1, 1]^T.
\]  \( (9) \)

Four spring isolators were installed between the inertia block and the receiving floor. The dynamic force transmitted from the machine to the receiving floor system at the four contact points is given by:

\[
[F'] = \left( [Y_S']^H + \left( \frac{j\omega}{K} + \frac{1}{j\omega M_k} \right) [I] + [Y_{Rq}] \right)^{-1} [V_S];
\]  \( (10) \)

where \( K \) denotes the elastic coefficient of each spring isolator, \( M_k \) denotes the mass of each spring isolator, and \( Y_{Rq} \) denotes the matrix of floor mobility. The active power transmitted to the floor through the four spring isolators is given by:

\[
P = \frac{1}{2} \text{Re}([F']^H [Y_{Rq}] [F']).
\]  \( (11) \)

The vibration velocity vector of the isolated machine with inertia block is given by:

\[
[V_0] = \left( \frac{j\omega}{K} + \frac{1}{j\omega M_k} \right) [I] + [Y_{Rq}] [F'].
\]  \( (12) \)

In this research, it is assumed that the rocking motion of the machine is small. The rotational velocity indicating the rocking motion of the isolated machine with inertia block is given by:

\[
R_v = \sqrt{|V_{q_1} - V_{q_2}|^2 + |V_{q_3} - V_{q_4}|^2/L_b}.
\]  \( (13) \)

### 2.4. Objective Functions

The objective functions for optimizing the mounting positions for the machine with the minimum structure-borne sound power transmission and the minimum rotational velocity can be given as follows: Minimize \( g_1 = P \) and Minimize \( g_2 = R_v \), respectively. For the multi-objective optimization problem of requiring minimum structure-borne sound power transmission and minimum rotational velocity, the simplest method is to combine these two objective functions into a weighted objective function, as:

\[
\text{Minimize } G = w_1 \times 10 \times \log(g_1/P_{ref}) + w_2 \times 20 \times \log(g_2/V_{ref})
\]  \( (14) \)

where \( w_1 \) and \( w_2 \) denote the weighting coefficients of the level of structure-borne sound power transmission and the level of rotational velocity, respectively, which express the relative importance of the objectives; \( P_{ref} = 10^{-12} \text{ W} \) and \( V_{ref} = 10^{-9} \text{ m/s} \). It is obvious that when \( G \) is minimized, the optimized mounting position can be obtained.

### 3. ANALYSIS

The parameters for the vibratory machine are \( R_1 = 0.25 \text{ m}, R_2 = 0.125 \text{ m}, H = 0.3 \text{ m}, \) density \( \rho_1 = 7.8 \times 10^3 \text{ kg/m}^3, \) and mass \( M_m = 5.74 \times 10^2 \text{ kg} \). The parameters of the inertia block model are \( L_b = 0.7 \text{ m}, H_b = 0.2 \text{ m}, \) density \( \rho_2 = 2.8 \times 10^3 \text{ kg/m}^3, \) and mass \( M_b = 2.74 \times 10^2 \text{ kg} \). The parameters for the rectangular receiving floor are density \( \rho_3 = 2.8 \times 10^3 \text{ kg/m}^3, \) Young’s modulus \( E = 2.1 \times 10^10 \text{ N/m}^2, \) loss factor \( \eta = 0.02, \) and Poisson’s ratio \( \mu = 0.2. \) The dimensions for the rectangular receiving floor are length \( l_x = 3.5 \text{ m}, \) width \( l_y = 3.5 \text{ m}, \) thickness \( d = 0.24 \text{ m}, \) and mass \( M_f = 8.23 \times 10^3 \text{ kg}. \) The mass of the floor is far greater than the mass of the vibratory machine. The deformation of the floor result from the weight of the machine model is ignored in this research. Assuming that the frequency of the excitation force is 15 Hz, the elastic coefficient of the spring isolator is \( K = 1.88 \times 10^6 \text{ N/m}. \)

As plotted in Fig. 3, the rectangular receiving floor is divided into a grid of 100 equal squares. The numbers in Fig. 3 indicate the sequence number of each grid point. For symmetry, only ten cases were investigated for the symmetrical arrangement of the inertia block and the machine mounted on the receiving floor. Any other cases can find their equivalent installation effects in these ten cases. Positions 1 to 10 are the ten points selected for installation of contact point \( q_1 \) of the inertia block and the machine. Mounting points \( q_1, q_2, q_3, \) and \( q_4 \) on the receiving floor for each case are shown in Table 1.
\[ P = \frac{1}{2} \text{Re} \left\{ \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \left( \left[ Y'_s \right] + \frac{j \omega}{K} \left[ I \right] + \left[ Y_{rq} \right] \right)^{-1} \right\} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}. \] (15)

**Figure 3.** Schematic diagram of the floor model divided into a grid of 100 equal squares.

### 3.1. Determination of Floor Mobility Using the Finite Difference Method

The receiving floor was divided into a grid of 100 × 100 equal squares in the finite difference analysis. The modal frequencies obtained by using 50 × 50 meshes were compared to the solutions obtained by using 100 × 100 meshes. The relative errors were less than 0.1%. This means that 50 × 50 meshes are fine enough to obtain solutions with acceptable relative errors. Assume that the receiving floor is simply supported. To validate the results obtained by the finite difference method, natural frequencies were compared with theoretical analysis solutions. The relative errors of natural frequencies obtained by the finite difference method compared to theoretical analysis solutions were less than 0.1% for frequencies below 2000 Hz.

### 3.2. Structure-Borne Sound Power and Rotational Velocity

According to Eqs. (9), (10), and (11), the structure-borne sound power transmission can be calculated by Eq. (15).

According to Eqs. (9), (10), (12), and (13), the vibration velocity vector of the isolated machine with inertia block is given by:

\[ \begin{bmatrix} V_{q} \end{bmatrix} = V_0 \left( \frac{j \omega}{K} \left[ I \right] + \left[ Y_{rq} \right] \right) \left( \left[ Y'_s \right] + \frac{j \omega}{K} \left[ I \right] + \left[ Y_{rq} \right] \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}. \] (16)

The structure-borne sound power transmitted to the receiving floor and the rotational velocity can be calculated through Eqs. (13), (15), and (16), respectively. The frequency band between 1 Hz to 1000 Hz was investigated.

It is assumed that the free velocity of the machine is 1 m/s in the frequency band being analysed. The structure-borne sound power transmission of each mounting position was then calculated and plotted in Fig. 4. According to Fig. 4, it can be found that the mounting position for the vibratory machine and the inertia block with the minimum structure-borne sound power transmission differs throughout the frequency band. To make the results clearer, for each frequency, the mounting position with the minimum structure-borne sound power transmission was then plotted in Fig. 5a. For the frequency ranges increased from 1 Hz–10 Hz to 1 Hz–1000 Hz (the increment is 10 Hz), the possibility of each mounting position to be the position with the minimum structure-borne sound power transmission was picked out and plotted in Fig. 5b. Referring to Fig. 5a and Fig. 5b, it can be observed that the mounting positions 1, 5, and 10 are the positions with the largest possibility to be the position with
the minimum structure-borne sound power transmission, especially for the frequency band between 1 Hz and 500 Hz. With the increasing of the frequency range being considered between 1 Hz and 500 Hz, the possibility of the mounting position 1 decreased, while the possibility of mounting positions 5 and 10 increased. With the increasing of the frequency range being considered above 600 Hz, the possibility of the mounting positions 1 and 10 increased, while the possibility of mounting position 5 decreased. The mounting positions 1, 5, and 10 are symmetrical through the diagonal line of the receiving floor. It seems that the optimized mounting positions for a vibratory machine, based on the criteria of the minimum rotational velocity, are symmetrical about the diagonal line of the receiving floor.

The rotational velocity of the vibratory machine with the inertia block for different mounting positions is calculated and plotted in Fig. 6. According to Fig. 6, the curves of rotational velocity are quite complex. Some drop and rise sharply at very narrow frequency band ranges. For each frequency, the mounting position with the minimum rotational velocity was picked out and plotted in Fig. 7a. For each frequency range increased from 1 Hz–10 Hz to 1 Hz–1000 Hz (the increment is 10 Hz), the possibility of each mounting position to be the position with the minimum rotational velocity was calculated and plotted in Fig. 7b. Referring to Fig. 7a and Fig. 7b, it can be observed that the mounting positions 1, 2, 3, 6, 7, and 9 have very small possibilities (less than 5%) to be the mounting position with the minimum rotational velocity. The mounting positions 5, 8, and 10 are the positions with the largest possibility to be the mounting position with the minimum rotational velocity. The mounting positions 5, 8, and 10 are symmetrical through the diagonal line of the receiving floor. It seems that the optimized mounting positions for a vibratory machine, based on the criteria of the minimum rotational velocity, are symmetrical about the diagonal line of the receiving floor.

By comparing Fig. 5 with Fig. 7, one can see that minimum rotational velocity (that is, a relatively stable machine) does not imply minimum transmission of structure-borne sound power from the machine to the floor or vice versa. To make it clearer, the correlation between rotational velocity and structure-borne sound power transmission was considered. For each frequency, we sorted different mounting positions by structure-borne sound power transmission value and rotational velocity value in descending order, respectively. For each frequency, the correlation coefficient of rank order between these two kinds of criteria was then calculated and plotted in Fig. 8. Referring to Fig. 8, it can be found that less structure-borne sound power transmission was not strongly associated with smaller rotational velocity. All these phenomena indicate that optimized mounting positions for an isolated vibratory machine are determined by both structure-borne sound power transmission and the rotational velocity (or stability) of the machine as minimum structure-borne sound power transmission does not imply a stable machine (low rotational velocity) or vice versa.

The receiving floor with totally fixed boundary conditions is also analysed in this study. The structure-borne sound power transmission and rotational velocity of the symmetrical vibratory machine model at frequencies between 1 Hz and 1000 Hz were analysed. The structure-borne sound power transmission of the vibratory machine at each mounting position is shown in Fig. 9. For the case with totally fixed boundary conditions, the
The mounting position with the minimum structure-borne sound power transmission for each frequency was picked out and plotted in Fig. 10a. For the frequency range increased from 1 Hz–10 Hz to 1 Hz–1000 Hz (the increment is 10 Hz), the possibility of each mounting position to be the position with the minimum structure-borne sound power transmission was calculated and plotted in Fig. 10b. Referring to Fig. 10a and Fig. 10b, it can be observed that the mounting positions 1, 7, 8, and 10 are the positions with the largest possibility to be the position with the minimum structure-borne sound power transmission. Just like the case with totally simply supported conditions, these mounting positions are symmetrical through the diagonal line of the receiving floor.

The rotational velocity of the vibratory machine with the inertia block for different mounting positions is calculated and plotted in Fig. 11. For each frequency, the mounting position with the minimum rotational velocity was picked out and plotted in Fig. 12a. For each frequency range increased from 1 Hz–10 Hz to 1 Hz–1000 Hz (the increment is 10 Hz), the possibility of each mounting position to be the position with the minimum rotational velocity was calculated and plotted in Fig. 12b. Referring to Fig. 12a and Fig. 12b, it can be observed that the mounting positions 1, 7, 8, and 10 are the position with the largest possibility to be the mounting position with the minimum rotational velocity. The mounting positions 1, 8, and 10 are symmetrical through the diagonal line of the receiving floor.

The curves in Fig. 9 present some minor differences from the curves in Fig. 4, but they have almost the same rule. The mounting positions of the inertia block and the machine with the minimum structure-borne sound power transmission are on diagonal lines of the receiving floor. It can be seen that the curves in Fig. 6 and Fig. 11 have similar trends and position 10 seems to be the optimized choice because the normalized rotational velocity is small (that is, the machine is relatively stable). The analysed results reveal that the structure-borne sound power transmission from the machine to the floor and the stability of the machine are not significantly different for the receiving floor with totally fixed boundary conditions and the receiving floor with totally simple boundary conditions. It can be concluded that the optimized mounting positions for
3.3. Multi-objective Optimization

According to Eq. (14), the optimized mounting position of the symmetrical machine model can be found. The two weighting coefficients $w_1$ and $w_2$ can be decided by the users in real cases. If the structure-borne sound power transmission was concerned more, the weighted coefficient $w_1$ should be larger; if the stability was concerned more, the weighting coefficient $w_2$ should be larger. In this research, $w_1 = 1$ and $w_2 = 0.5$.

For the case of simply supported boundary condition, the values of $G$ and the optimum mounting positions for the symmetrical machine model and the inertial block were shown in Fig. 13a and Fig. 13b, respectively. Referring to Fig. 13a and Fig. 13b, it can be observed that the mounting positions 1, 5, 8, and 10 are the positions with the largest possibility to be the optimum mounting position. The mounting positions 1, 5, 8, and 10 are symmetrical through the diagonal line of the receiving floor.

For the case of fixed boundary condition, the values of $G$ and the optimum mounting positions for the symmetrical machine model and the inertial block were shown in Fig. 14a and Fig. 14b, respectively. Referring to Fig. 14a and Fig. 14b, it can be observed that the mounting positions 1, 8, and 10 are the positions with the largest possibility to be the optimum mounting position. The mounting positions 1, 8, and 10 are symmetrical through the diagonal line of the receiving floor.

4. CONCLUSION

A symmetrical machine model with an inertia block has been used to study the structure-borne sound power transmission from the machine to the floor and the rotational velocity of the machine. Two different boundary conditions, totally simply supported plate and the totally fixed supported plate, have been used. To calculate the structure-borne sound power transmission and the rotational velocity for a totally fixed supported plate, the mobility of the receiving floor system was obtained using the finite difference method. This is a first attempt to utilize both the structure-borne sound power transmission and the rotational velocity (or the stability) of the machine to select the optimized mounting positions for an isolated vibratory machine. The results reveal that, for optimized mounting positions, the vibratory machine should be symmetrically installed on diagonal lines of the receiving floor structure.

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Figure 12. The mounting positions with the minimum rotational velocity: (a) the selected mounting position for each frequency; (b) the possibility of each mounting position to be the one with the minimum rotational velocity.


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Figure 14. The floor with totally fixed boundary condition: (a) values of the weighted objective function $G$ of the symmetrical machine model and the inertial block for different mounting positions; (b) the optimized mounting position for each frequency.
Achieving More Stringent Levels of Comfort via an Adaptive Fuzzy Controller Optimized by the Gravitational Search Algorithm for a Half-Body Car Model

Mohammad Javanbakht and Mohammad Javad Mahmoodabadi
Department of Mechanical Engineering, Sirjan University of Technology, Sirjan, Iran.

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An optimal adaptive fuzzy controller is designed to achieve more stringent levels of comfort for a half-body car model. This aim will be fulfilled by reducing road disturbances and decreasing the acceleration of the body. The proposed controller consists of two adaptive fuzzy controllers with two fuzzy systems. Each one has two inputs, one output and twenty five linguistic fuzzy IF-THEN rules. Every input has five Gaussian membership functions and uses the product inference engine, singleton fuzzifier and the centre average defuzzifier. In order to determine the optimal parameters for the Adaptive Fuzzy Controller (AFC), the Gravitational Search Algorithm (GSA) is applied. The relative displacement between spring mass and tire, along with the acceleration of the body, are the two objective functions being applied in the optimization algorithm. The results illustrate the superiority of the proposed optimal adaptive fuzzy controller in comparison with traditional controllers.

1. INTRODUCTION

Suspension is one of the essential elements of vehicles which are connected to the road through tire and also is one of the attractive subjects for researchers. For this reason, over the decades, many researchers have studied road excitation and its effects on vehicle comfort and road holding capability.\textsuperscript{1–3} In fact, vehicle suspension systems have been designed to get more comfort, delightful riding, and to protect the chassis from the irregularity of roads. The importance of this issue has been proven for all car companies that feel that comfort is a criterion for customers who are choosing their cars. In other words, up to now, the ability of comfortable driving for car companies has been an important topic. Additionally, vibrations from road excitation are harmful for the car body. Therefore, car companies solved this problem by designing the suspension system, and the competition between companies has lead to the invention of many new suspension systems.

At first, the passive suspension systems involving a spring and a damper, mounted in parallel, as well as another spring, connected with them in series, were considered. In these systems, the dampers are devised for absorbing vibration energy and do not require any extra power. Tamboli and Joshi have designed a passive suspension system for a vehicle that takes random road excitations into consideration.\textsuperscript{4} Bagheri et al. have formulated an optimum design problem for the vehicle suspension system of a quarter-car model and a new multi-objective genetic algorithm is used for the Pareto optimization of this vehicle vibration model while taking the two conflicting objective functions into consideration.\textsuperscript{5} Mahmoodabadi et al. have proposed a multi-objective hybrid algorithm for the Pareto optimum design of a five-degree-of-freedom (DOF) vehicle vibration model that uses a combination of Particle Swarm Optimization (PSO) and the Genetic Algorithm (GA), based on the fuzzy probabilities.\textsuperscript{6} Moreover, Hu et al. have investigated mixed ride comfort and tire grip performance optimization with equal suspension relative displacement between the sprung mass and the tire.\textsuperscript{7} Furthermore, Hanafi et al. have developed a neuro model to represent the nonlinear model of a quarter-car passive suspension system.\textsuperscript{8}

A semi-active suspension system is used to improve both the stability and good driving characteristics through changes in the stabilizer. This system can extirpate the roll and pitch motions and the deviations caused by braking. Since the system is designed to operate at low frequencies, the hardware used is simpler and less costly, while its power consumption is far lower than an active suspension system. Semi-active systems have no additional output to suppress the external disturbance sensed by electronic devices. The suspension system can change to hard (stiff) or soft so that it can timely and effectively eliminate the vibrations of the vehicle body, thereby improving ride comfort. Crolla has offered a semi-active suspension model for improving the performance of the vehicle.\textsuperscript{9} Dhananjay Rao and Kumar have designed a linear quadratic regulator for the quarter-car semi-active suspension system.\textsuperscript{10} Furthermore, Guo et al. have proposed a fuzzy proportional-integral-derivative (PID) hybrid control strategy based on generic algorithm optimization for a vehicle semi-active suspension system.\textsuperscript{11}

Active suspension is another model that was made for creating more free hands in suspension system design, and to provide great handling and pleasant riding. Such systems have the ability of saving, producing and damping energy and can change their own properties regarding the car conditions. Practical considerations of the realization of active suspensions in real-world applications include choosing appropriate actuators that can fit into the suspension packaging space and satisfy the practical power and bandwidth requirements, as well as choosing available measurements for feedback control. Bouazara has studied the influence of the suspension system parameters on the vibration of the vehicle for two, five and eight-DOF models.\textsuperscript{12} Hrovat has used a three-dimensional vibration model instead of the five-DOF model to get more exact results.\textsuperscript{13} After that, Bouazara and Richard studied three types of suspension systems (active, semi-active and passive) for the
eight-DOF vibration model. In a similar work, Bouazara and Richard combined all of the performance criteria to form an objective function for a single objective optimization process. Gáspár et al. proposed the mixed synthesis for active suspension problems and have presented a full-body car model to handle the uncertain components. Wang et al. (2005) have designed a full-body active suspension system and the $H_\infty$ controller to simultaneously improve the vehicle ride comfort and steady-state handling performance. Du and Zhang have also considered a $H_\infty$ control problem for active vehicle suspension systems with actuator time delays for a quarter-car model. Priyandoko et al. studied the active suspension systems of vehicles by using skyhook adaptive neuro active force control. Further, Fayad designed a PID controller including hydraulic dynamics for a quarter-car model of a passenger car to improve both the comfort of the ride and road holding ability. Li et al. investigated the problem of sampled-data $H_\infty$ control for uncertain vehicle suspension systems. In the same year, Li et al. also designed output-feedback $H_\infty$ control for a class of active quarter-car suspension systems with control delay. Peng et al. offered a new hierarchical control strategy for active hydro pneumatic suspension systems and noticed the dynamic characteristics of the actuator. Cheng et al. studied the design of a robust $H_\infty$ controller for the series active variable geometry suspension of the road vehicle. Furthermore, Sun et al. used active suspensions with performance constraints and actuator saturation for vibration isolation. Bello et al. designed two loop (inner and outer) PID controllers for the four-DOF suspension system. Li et al. successfully controlled a quarter-car model using an adaptive inverter that was installed in parallel with a spring and designed a damper based on a state-feedback $H_2$ controller for an active suspension system. Finally, Zhao et al. recorded the unintentional vibrations of a moving vehicle caused by road roughness and presented an adaptive neural network approach, mixed with linear quadratic regulator control, for the quarter-car active suspension systems to stabilize the image captured area of the camera.

One of the successful and powerful methodologies for the active suspension systems, in order to damp the vehicle vibrations caused by the road roughness, is the fuzzy controller. Ro et al. developed an active suspension system based on the fuzzy logic for a quarter-car model. Yester and Mcfall utilized the fuzzy logic to design an automotive suspension system associated with active elements to make an effective controller. Cherry and Jones illustrated the application of fuzzy logic techniques by controlling a continuously variable damping automotive suspension system, based on a multi-body model of a passenger car. Huang and Chao proposed a grey predictor fuzzy control scheme in order to remove the tire deformation and improve the control performance. Cao et al. used an interval fuzzy controller for half-body suspension to resolve the nonlinear control problems of active systems. Furthermore, Cao et al. extended the fuzzy logic controller based on interval fuzzy membership functions for a vehicle active suspension system. In the next year, Changizi and Rouhani studied the fuzzy logic technique to control a continuously damping automotive suspension system and compared the PID and fuzzy logic controllers for a quarter-car suspension system. Li et al. designed suspension systems with actuator delay and focused on the reliable fuzzy $H_\infty$ controller. Al-dair and Wang investigated an artificial intelligent neuro fuzzy technique to design a robust controller and used electromagnetic actuators to convert the vibration energy that arises from the rough road into useful electrical energy so as to reduce the energy consumption by the active suspension systems. Shehata et al. demonstrated the application of the fuzzy logic technique to design a controller for an active vehicle suspension system and improved its performance by altering the number and arrangement of rules set and the universe of discourse.

The design of a traditional fuzzy controller fully depends on the experience of an operator to establish the fuzzy rules set. Hence, self-tuning algorithms and adaptation laws can be applied to adjust fuzzy parameters and improve the control performance based upon a specified performance index. By using the learning mechanism, Rao and Prahlad designed a tunable fuzzy controller for an active suspension system. Huang and Chen proposed a free adaptive sliding controller to suppress the position oscillation of the sprung mass as a response to road surface variations; in addition to introducing a fuzzy scheme with online learning ability to improve the control performance. A case study based on a quarter-car active suspension model demonstrated that the type-2 fuzzy controller significantly outperforms conventional fuzzy controllers. Aldair and Wang proposed a field programmable gate array to build an adaptive neuro fuzzy inference system for controlling a full vehicle nonlinear active suspension system with hydraulic actuators. Soleymani et al. applied an adaptive fuzzy controller for a full-body model suspension system and have designed two separate fuzzy controllers for the front and rear suspension systems. Li et al. designed an adaptive sliding-mode control for nonlinear active suspension systems via the Takagi-Sugeno fuzzy approach for a half-body suspension system. Moreover, Shin and Seung-Bok introduced a new adaptive fuzzy controller and presented its application for the vibration control of a vehicle seat installed with a magneto-rheological damper.

Since many industrial plants are often burdened with problems such as high order, time delays and nonlinearities, it has been quite difficult to properly tune the gains of fuzzy controllers. Over the years, several heuristic methods, such as evolutionary algorithms, have been proposed to properly adjust fuzzy controller parameters. Du and Zhang considered Genetic Algorithm (GA) for the minimization of relative displacement between the spring mass and tire, along with the body acceleration for a quarter-car model. Shirahtati et al. developed a full-car model for passive and active suspension systems, and designed a linear-quadratic controller for the active suspension system. Then, Shirahtati et al. employed the GA to minimize the passenger bounce, passenger acceleration and tire displacement, as well as comparing results to those obtained by the Simulated Annealing (SA) algorithm. Chiu et al. developed the adaptive fuzzy PID-controllers for a quarter-car model by using the Particle Swarm Optimization (PSO) algorithm. Their results effectively reduced the suspension displacement, the body acceleration and the beating distance between tire and ground.

Here, a novel optimal adaptive fuzzy controller was proposed to control the vibrations of a vehicle system with five degrees of freedom. This model had two suspension systems that each contained an independent closed loop controller. Moreover, every controller comprised two fuzzy systems. Each controller had two inputs, one output and twenty-five linguistic fuzzy if-then rules. Additionally, each input had five Gaussian membership functions and used the product inference engine, singleton fuzzifier and the centre average defuzzifier. Further-
more, the Gravitational Search Algorithm (GSA)\textsuperscript{28} was used for minimization of a weighting function of the relative displacement and the body acceleration as the objective function. Fifty design variables were utilized for optimization and the algorithm acquired the optimal values for the fuzzy systems’ parameters with regard to the objective function. The results show the superiority of the proposed optimal AFC in comparison with other approaches.

The rest of this paper is organized as follows. Section 2 explains the vehicle dynamic formulations and irregularities mathematical model. Section 3 recalls the adaptive fuzzy controller. Section 4 presents the GSA optimization and illustrates the design variables as well as the cost function. Section 5 shows the results and confirms the superiority of the proposed approach. Section 6 concludes the paper.

\section{Vehicle Dynamical Model}

A half-body car vibration model with active suspension extracted from is illustrated in Figure 1. The variables $Z_1, Z_2, Z_{s1}, Z_{s2}, Z_c, Z_s,$ and $\theta$ represented respectively: the vertical displacements of: front tire, rear tire, front part of the vehicle body, rear part of the vehicle body, the seat, and the gravitational centre of the vehicle body; together with the rotational movement of the vehicle body’s gravitational centre ($\theta$). By employing the Newton’s second law, the linear differential Eqs. of motion for the half-body car model were written as follows:\textsuperscript{12}

\begin{align}
Z_{ps} &= Z_s - r\theta; \quad (1) \\
Z_{s1} &= Z_s - l_1\theta; \quad (2) \\
Z_{s2} &= Z_s + l_2\theta; \quad (3)
\end{align}

\begin{align*}
F_{ss} &= K_{ss}(Z_c - Z_{ps}) + C_{ss}(\dot{Z}_c - \dot{Z}_{ps}); \quad (4) \\
F_{s1} &= K_{s1}(Z_{s1} - Z_1) + C_{s1}(\dot{Z}_{s1} - \dot{Z}_1); \quad (5) \\
F_{s2} &= K_{s2}(Z_{s2} - Z_2) + C_{s2}(\dot{Z}_{s2} - \dot{Z}_2); \quad (6) \\
M_s\ddot{Z}_s &= -F_{ss} - F_{s1} - F_{s2} + F_{ss} - u_1 + u_2; \quad (8) \\
I_s\ddot{\theta} &= l_1F_{s1} - l_2F_{s2} - rF_{ss} + l_1u_1 + l_2u_2; \quad (9) \\
M_1\ddot{Z}_1 &= F_{s1} - K_{p1}(Z_1 - Z_{p1}) + u_1; \quad (10) \\
M_2\ddot{Z}_2 &= F_{s2} - K_{p2}(Z_2 - Z_{p2}) - u_2. \quad (11)
\end{align*}

In Eqs. (1) to (11), the first and second derivatives of $Z_c, Z_s, Z_{s1},$ and $Z_{s2}$ denoted the vertical velocity and vertical acceleration of the seat, the gravitational centre of the vehicle body, the front part of the vehicle body and the rear part of the vehicle body, respectively. Further, variables $\theta, \dot{Z}_1,$ and $\dot{Z}_2$ represented the rotational acceleration of the body, vertical acceleration of the front tire and the vertical acceleration of the rear tire, respectively. The input values of the fixed parameters are given in Table 1.\textsuperscript{12} Bouazara considered the disturbance of the five-DOF system as two consecutive sinusoidal waves (double bumps).\textsuperscript{12} At first, the front tire passed through the bump, and then the rear tire. Due to the constant velocity of the vehicle, which equalled $v = 20 \frac{m}{s},$ the delay time was calculated by $\Delta t = \frac{(l_1+l_2)}{v},$ $Z_{p1},$ and $Z_{p2}$ indicated the bumps of the road imposed on the front and rear tire, respectively (Fig. 1). The bump Eqs. were regarded as the postponement for both front and rear tires, shown in Eqs. (12) and (13) and depicted in Fig. 2.

\begin{align*}
\begin{cases}
Z_{p1}(t) = 0.05 \sin(2\pi t) & \text{if } 0.5 < t < 2.5 \\
Z_{p1}(t) = 0 & \text{else}
\end{cases}
\end{align*} \quad (12)

\begin{align*}
\begin{cases}
Z_{p2}(t) = 0.05 \sin(2\pi(t - 0.1407)) & \text{if } 0.5 < t < 2.5 \\
Z_{p2}(t) = 0 & \text{else}
\end{cases}
\end{align*} \quad (13)

where the amplitude, frequency and period of the vibration were considered to be 0.05 m, 1 Hz, and 1 s, respectively.

\section{Adaptive Fuzzy Controller}

In this section, the fuzzy controllers were designed by using fuzzy systems and membership functions. Since the coefficients of the fuzzy systems change with error, adaptive rules were implemented to match them with the system states. Each

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The half-body car model of an active suspension system.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The considered excitation as the double bumps.}
\end{figure}
tire had a separate control signal in the considered half-body car model. Suppose that the plant was shown by the following differential Eqs.:\(^{(58)}\)

\[
\dot{Y} = F(Y) + G(Y)u;
\]

(14)

where \(F(Y)\) and \(G(Y)\) were two unknown functions; and \(Y\) was the state vector. Moreover, \(Y\) was the derivative of the state vector, which can be replaced by \(Z_{s1}, Z_{s2}, Z_{t1}\) and \(Z_{t2}\) for front tire and \(Z_{s2}, Z_{t2}\) for the rear tire. Further, \(u = u(t)\) referred to the input of the plant or the feedback controller, based on the fuzzy systems and adaptation laws. In other words, it represented the external input force caused by an actuator to reach the desired force value.

In order to describe the input-output behavior of \(F\) and \(G\) for the front and rear suspension systems, a set of fuzzy IF-THEN rules were collected. In other words, the unknown \(F\) and \(G\) were replaced by fuzzy systems constructed from the below fuzzy IF-THEN rules:

\[
\text{IF} \; e \; \text{is} \; F_1^r \; \text{and} \; \dot{e} \; \text{is} \; F_2^r, \; \text{THEN} \; F(Y) = f_{m1};
\]

(15)

which describes \(F(Y)\), and

\[
\text{IF} \; e \; \text{is} \; G_1^r \; \text{and} \; \dot{e} \; \text{is} \; G_2^r, \; \text{THEN} \; G(Y) = g_{m1};
\]

(16)

which described \(G(Y)\). In Eqs. (15) and (16), \(e\) represented the system error vector. Note that \(F_1^r, F_2^r, f_{m1}, G_1^r, G_2^r\) and \(g_{m1}\) were fuzzy sets; and \(r, n=1,2, \ldots, 5\).

Here, the fuzzy systems of \(F(e, \dot{e})\) and \(G(e, \dot{e})\) were constructed for the front and rear tire, based on the product inference engine, singleton fuzzifier and the centre average defuzzifier via the following Eqs.:

\[
\begin{align*}
F_{\text{Front}}((e, \dot{e})|f_{i1+i2}) &= \frac{\sum_{i=1}^{5} \sum_{i=2}^{5} f_{i1+i2}(\mu_{A1}^{i1}(e_1) \mu_{A2}^{i1}(e_1))}{\sum_{i=1}^{5} \sum_{i=2}^{5} f_{i1+i2}(\mu_{A1}^{i1}(e_1))}; \quad (17) \\
G_{\text{Front}}((e, \dot{e})|g_{i1+i2}) &= \frac{\sum_{i=1}^{5} \sum_{i=2}^{5} g_{i1+i2}(\mu_{A1}^{i1}(e_1) \mu_{A2}^{i1}(e_1))}{\sum_{i=1}^{5} \sum_{i=2}^{5} g_{i1+i2}(\mu_{A1}^{i1}(e_1))} \\
F_{\text{Rear}}((e, \dot{e})|f_{i1+i2}) &= \frac{\sum_{i=1}^{5} \sum_{i=2}^{5} f_{i1+i2}(\mu_{A1}^{i1}(e_2) \mu_{A2}^{i1}(e_2))}{\sum_{i=1}^{5} \sum_{i=2}^{5} f_{i1+i2}(\mu_{A1}^{i1}(e_2))}; \quad (18) \\
G_{\text{Rear}}((e, \dot{e})|g_{i1+i2}) &= \frac{\sum_{i=1}^{5} \sum_{i=2}^{5} g_{i1+i2}(\mu_{A1}^{i1}(e_2) \mu_{A2}^{i1}(e_2))}{\sum_{i=1}^{5} \sum_{i=2}^{5} g_{i1+i2}(\mu_{A1}^{i1}(e_2))}
\end{align*}
\]

where, \(f_{i1+i2}\) and \(g_{i1+i2}\) were the output membership functions of two fuzzy systems in either of the suspension systems, shown in Figs. 3(a) and 3(b) respectively. The errors \(e_1\) and \(e_2\) in the above equations were computed according to Eqs. (19) and (20).

\[
e_1(t) = x_{d1} - (Z_{s1}(t) - Z_{t1}(t)); \quad (19)
\]

\[
e_2(t) = x_{d2} - (Z_{s2}(t) - Z_{t2}(t)). \quad (20)
\]

The reference value \(x_{d}^1 (i = 1, 2)\) was called the desired set point, and its value was taken as zero. Therefore, the errors were shown as \((Z_{s1}(t) - Z_{t1}(t))\) and \((Z_{s2}(t) - Z_{t2}(t))\) for the front and rear suspension systems, respectively. In fact, \((Z_{s1}(t) - Z_{t1}(t))\) and \((Z_{s2}(t) - Z_{t2}(t))\) denoted respectively, the relative displacement between sprung mass and front tire, and the relative displacement between sprung mass and the rear tire. Based on the reported researches by Bouazara, reducing the relative displacement between the sprung mass and tire leads to achieve more comfort for passengers.\(^{(12)}\)

The five Gaussian membership functions \(\mu_{A1}^p\) and \(\mu_{A2}^p\) \((p = 1, 2, \ldots, 5)\) were given as follows. It is notable that Eqs. (24) to (25) articulated the first input membership functions of the front tire, while Eqs. (26) to (30) elucidated its second input membership functions. Further, Eqs. (31) to (35) expressed the first input membership functions of the rear tire, while Eqs. (36) to (40) explained its second input membership functions.

\[
\mu_{F1}^1(e_1) = \exp\left[-\left(\frac{e_1 + 1.8}{4}\right)^2\right]; \quad (21)
\]

\[
\mu_{F2}^1(e_1) = \exp\left[-\left(\frac{e_1 + 0.9}{4}\right)^2\right]; \quad (22)
\]

\[
\mu_{F1}^2(e_1) = \exp\left[-\left(\frac{e_1 + 0.9}{4}\right)^2\right]; \quad (23)
\]

\[
\mu_{F2}^1(e_1) = \exp\left[-\left(\frac{e_1 - 0.9}{4}\right)^2\right]; \quad (24)
\]

\[
\mu_{F1}^2(e_1) = \exp\left[-\left(\frac{e_1 - 1.8}{4}\right)^2\right]; \quad (25)
\]

\[
\mu_{F2}^2(e_1) = \exp\left[-\left(\frac{\dot{e}_1 + 1.8}{4}\right)^2\right]; \quad (26)
\]
Figure 3. Linguistic fuzzy IF-THEN rules for (a) the fuzzy system $F(e, \dot{e})$ and (b) the fuzzy system $G(e, \dot{e})$.

\[
\mu_{F_1}(e_2) = \exp\left[-\left(\frac{e_2 - 0.9}{4}\right)^2\right]; \quad (34)
\]

\[
\mu_{F_1}(e_2) = \exp\left[-\left(\frac{e_2 - 1.8}{4}\right)^2\right]; \quad (35)
\]

\[
\mu_{F_1}(\dot{e}_2) = \exp\left[-\left(\frac{\dot{e}_2 + 1.8}{4}\right)^2\right]; \quad (36)
\]

\[
\mu_{F_1}(\dot{e}_2) = \exp\left[-\left(\frac{\dot{e}_2 + 0.9}{4}\right)^2\right]; \quad (37)
\]

\[
\mu_{F_2}(\dot{e}_2) = \exp\left[-\left(\frac{\dot{e}_2 + 0.0}{4}\right)^2\right]; \quad (38)
\]

\[
\mu_{F_2}(\dot{e}_2) = \exp\left[-\left(\frac{\dot{e}_2 - 0.9}{4}\right)^2\right]; \quad (39)
\]

The mentioned membership functions for both fuzzy systems were the same as each other. For the second fuzzy system, $G_1^e$ and $G_2^e$ were used instead of $F_1^e$ and $F_2^e$, respectively. These five membership functions, for each input, covered the range of error and error derivative variations as well.

On the other hand, the output membership functions of the fuzzy systems significantly depended on the errors. Thus, to adapt these design variables, the Lyapunov synthesis approach was applied. As a matter of fact, by minimization of the Lyapunov function, the output membership functions of the fuzzy systems were obtained.\(^{38}\)

\[
\left\{ \begin{array}{l}
f_{i_1i_2} = -0.5e^T \beta_{i_1i_2} (e, \dot{e}) \\ g_{i_1i_2} = -0.01e^T \beta_{i_1i_2} (e, \dot{e}) \\
\end{array} \right.
\]

and

\[
\beta_{i_1i_2} (e, \dot{e}) = \frac{\mu_{A_1^i} (e) \mu_{A_2^j} (\dot{e})}{\sum_{i=1}^{5} \sum_{j=1}^{5} (\mu_{A_1^i} (e) \mu_{A_2^j} (\dot{e}))}; \quad (42)
\]

where, $i_1, i_2 = 1, 2, \ldots, 5$. Then, the adaptive fuzzy systems were constructed as follows:

\[
\left\{ \begin{array}{l}
F ((e, \dot{e})|f_{i_1i_2}) = f^T \beta (e, \dot{e}) \\ G ((e, \dot{e})|g_{i_1i_2}) = g^T \beta (e, \dot{e}) \\
\end{array} \right.
\]

(43)

The control efforts $u_1$ and $u_2$ were designed as follows.

\[
\dot{x} = f(x) \rightarrow u_1 = x_d - \dot{x}_1 - F_F ((e_1, \dot{e}_1)|f_{i_1i_2}) = \frac{G_F (e_1, \dot{e}_1)|g_{i_1i_2}}{G_F ((e_1, \dot{e}_1)|g_{i_1i_2})} \dot{Z}_{i_1}(t) - \dot{Z}_{i_1}(t) - F_F ((e_1, \dot{e}_1)|f_{i_1i_2}); \quad (44)
\]

\[
\dot{x} = f(x) \rightarrow u_2 = x_d - \dot{x}_2 - F_R ((e_2, \dot{e}_2)|f_{i_1i_2}) = \frac{G_R (e_2, \dot{e}_2)|g_{i_1i_2}}{G_R ((e_2, \dot{e}_2)|g_{i_1i_2})} \dot{Z}_{i_2}(t) - \dot{Z}_{i_2}(t) - F_R ((e_2, \dot{e}_2)|f_{i_1i_2}); \quad (45)
\]
4. GRAVITATIONAL SEARCH ALGORITHM (GSA)

The GSA was based on the Newtonian laws of gravitation and motion in which masses are considered the agents and affect each other through the gravity force. In this algorithm, the heavy masses had slower motions than light masses and attracted them until an optimum solution was presented.

In this algorithm, the gravity force of every agent was calculated by using the following equation:

\[ F_{ij}^d(t) = G_c(t) \frac{M_i M_j}{R_{ij}^d(t) + \varepsilon} \left( x^d_i(t) - x^d_j(t) \right) ; \]  

(46)

where, \( M_a \) and \( M_p \) represented the active and passive gravitational masses of the agents \( i \) and \( j \), respectively; \( \varepsilon \) was a small constant and takes the proposed value 2.2204 \times 10^{-16}; \( x^d_i(t) - x^d_j(t) \) referred to the distance vector between the two agents; \( d \) was an integer between 1 and the number of design variables; \( R_{ij}^d(t) \) was the Euclidian distance between agents \( i \) and \( j \); and finally, \( G_c(t) \) indicated the gravitational constant at time \( t \) being calculated as follows:

\[ G_c(t) = G_c(t_0) e^{-\frac{t}{\tau}} ; \]  

(47)

where, \( G_c(t_0) \) was the value of the gravitational constant at the first cosmic quantum-interval of time \( t_0 \); \( \alpha \) is set to 20; and \( T \) was the total number of iterations. Then, the acceleration, velocity and, at last, the position of any agent were calculated via the following equations:

\[ a^d_i(t) = \frac{F_{ij}^d(t)}{M_i} ; \]  

(48)

\[ v^d_i(t + 1) = \text{Rand}_i \times v^d_i(t) + a^d_i(t) ; \]  

(49)

\[ x^d_i(t + 1) = x^d_i(t) + v^d_i(t + 1) ; \]  

(50)

while, \( M_i \) denoted the inertial mass of agent \( i \); and \( \text{Rand}_i \) was a uniform random number in the interval \([0,1] \).

The values of masses were calculated based on the fitness evaluation with the assumption of the gravitational and inertial masses being equal. They were computed by the following equations:

\[ M_a = M_p = M = M_i, \quad i = 1, 2, \ldots, N ; \]  

(51)

\[ m_i(t) = \frac{\text{fit}_i(t) - \text{worst}(t)}{\text{best}(t) - \text{worst}(t)} ; \]  

(52)

\[ M_i(t) = \frac{m_i(t)}{\sum_{k=1}^{N} m_k(t)} ; \]  

(53)

while, \( \text{fit}_i(t) \) referred to the fitness value of agents \( i \) at time \( t \). Further, worst(\( t \)) and best(\( t \)) were defined as the maximum and minimum fitness for a minimization problem, respectively.

5. PROPOSED STRATEGY, RESULTS AND DISCUSSION

The block diagram of the proposed optimal Adaptive Fuzzy Controller (AFC) is shown in Figure 4 and the system errors of front and rear tire outputs, together with the desired inputs are respectively given in Eqs. (54) to (56).

Front tire output: \[ \begin{aligned} Z_{x1}(t) & - Z_1(t) \end{aligned} \]  

(54)

Rear tire output: \[ \begin{aligned} Z_{x2}(t) & - Z_2(t) \end{aligned} \]  

(55)

The desired input of both tires: \[ \begin{aligned} x_d = 0 \end{aligned} \]  

(56)

The main purpose of the inner control loops is to generate an effective adaptive controller. In order to reach a better performance, the optimization loops have been monitored and the best design variables are found. For the considered suspension system problem, the outputs are defined as the relative displacement between sprung mass and tires, which should converge to zero.

In the following, the gravitational search algorithm was applied for obtaining the optimal parameters of the AFC. The \( f_{1i,j} \) and \( g_{1i,j} \) of the front and rear tires were assumed as the design variables. In the current study, the initial population was 100 and the maximum iteration was 150. The relative displacement between the spring mass and the front and rear tires, along with the body and seat accelerations were considered to be four objective functions. Therefore, the fitness function was the weighting summation of these objectives as follows:

\[ z = w_1 \times f_1 + w_2 \times f_2 + w_3 \times f_3 + w_4 \times f_4 ; \]  

(57)

where, \( f_1, f_2, f_3 \) and \( f_4 \) were, respectively, the integrals of absolute relative displacement between the sprung mass and the front and rear tires as well as the absolute body and seat accelerations, altogether defined below:

\[ F_1 = \int |Z_{x1}(t) - Z_1(t)| \text{dt} ; \]  

(58)

\[ F_2 = \int |Z_{x2}(t) - Z_2(t)| \text{dt} ; \]  

(59)

\[ F_3 = \int \frac{1}{M_2} \left( -F_{s1} - F_{s2} + F_{sa} - u_1 + u_2 \right) \text{dt} ; \]  

(60)

\[ F_4 = \int \frac{-F_{sa}}{M_e} \text{dt} . \]  

(61)

Moreover, \( w_1, w_2, w_3, \) and \( w_4 \) were the weight coefficients of the fitness function which are supposed to adjust the effect of each objective on the fitness value and are equal to 10, 10, 1 and 1, respectively. Tables 2, 3, 4 and 5 show the obtained values of the decision variables corresponding to \( F_{front} \), \( G_{front} \), \( F_{rear} \), and \( G_{rear} \), respectively.

The relative displacement between sprung mass and the two tires for the passive model, the PID controller-based active model and the proposed optimal AFC are shown in Figs. 5 and 6. According to these figures, the proposed optimal AFC has exhibited less relative displacement between sprung mass and the front and rear tires as well as the absolute body and seat accelerations. In fact, these results confirm the superiority of the optimal AFC in comparison with both the active model developed by Mahmoudabadi and Mortazavi Yazdi, and the passive model developed by Bouazara. As it is observable, the proposed optimal AFC can display smaller settling time, overshoot and undershoot in comparison with those of other approaches.
Figure 4. The block diagram of the proposed optimal adaptive fuzzy controller for a five-DOF vehicle suspension system.

Table 2. The optimum design variables of the front tire in the fuzzy system $F_{\text{front}} ((e, \dot{e})|f_{1,12})$.

<table>
<thead>
<tr>
<th>$g_{15}$</th>
<th>$g_{55}$</th>
<th>$g_{45}$</th>
<th>$f_{45}$</th>
<th>$f_{55}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-13258.5705</td>
<td>-2430.51524</td>
<td>129.157423</td>
<td>3119.25117</td>
<td>10379.7795</td>
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<tr>
<td>-11691.9778</td>
<td>-4339.70188</td>
<td>74.2374755</td>
<td>4783.83743</td>
<td>10679.3768</td>
</tr>
<tr>
<td>-13196.4044</td>
<td>-2880.22074</td>
<td>-319.208363</td>
<td>3143.02864</td>
<td>11002.0956</td>
</tr>
<tr>
<td>-10190.2224</td>
<td>-4803.17138</td>
<td>281.067472</td>
<td>5033.51824</td>
<td>11734.9347</td>
</tr>
<tr>
<td>-10640.9218</td>
<td>-2601.12162</td>
<td>90.8897046</td>
<td>4368.83215</td>
<td>11488.9341</td>
</tr>
</tbody>
</table>

Table 3. The optimum design variables of the front tire in the fuzzy system $G_{\text{front}} ((e, \dot{e})|g_{1,12})$.

<table>
<thead>
<tr>
<th>$g_{15}$</th>
<th>$g_{25}$</th>
<th>$g_{35}$</th>
<th>$g_{45}$</th>
<th>$g_{55}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4.21594255</td>
<td>-1.54091229</td>
<td>-4.28359843</td>
<td>0.642398422</td>
<td>-6.39669582</td>
</tr>
<tr>
<td>-1.39683132</td>
<td>-4.35494416</td>
<td>7.84455379</td>
<td>-1.80539822</td>
<td>7.779073432</td>
</tr>
<tr>
<td>-0.25091811</td>
<td>-1.83804355</td>
<td>-1.65327663</td>
<td>-2.04109238</td>
<td>2.65644444</td>
</tr>
<tr>
<td>-3.43805308</td>
<td>5.94209975</td>
<td>-4.77355678</td>
<td>1.57575871</td>
<td>-3.053240745</td>
</tr>
</tbody>
</table>

Table 4. The optimum design variables of the rear tire in the fuzzy system $F_{\text{rear}} ((e, \dot{e})|f_{1,12})$.

<table>
<thead>
<tr>
<th>$f_{15}$</th>
<th>$f_{25}$</th>
<th>$f_{35}$</th>
<th>$f_{45}$</th>
<th>$f_{55}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10928.3009</td>
<td>-2404.94394</td>
<td>129.097515</td>
<td>4036.71148</td>
<td>10849.2262</td>
</tr>
<tr>
<td>-11886.0909</td>
<td>-3850.93368</td>
<td>109.33461</td>
<td>4856.08789</td>
<td>10604.6622</td>
</tr>
<tr>
<td>-13512.2490</td>
<td>-4459.58815</td>
<td>-324.965743</td>
<td>2613.91394</td>
<td>11455.89</td>
</tr>
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<td>-10533.0794</td>
<td>-461.70808</td>
<td>292.728841</td>
<td>5084.30976</td>
<td>11020.0369</td>
</tr>
<tr>
<td>-11174.3617</td>
<td>-21458.6896</td>
<td>-177.548031</td>
<td>8274.850225</td>
<td>11404.1775</td>
</tr>
</tbody>
</table>

Table 5. The optimum design variables of the rear tire in the fuzzy system $G_{\text{rear}} ((e, \dot{e})|g_{1,12})$.

<table>
<thead>
<tr>
<th>$g_{15}$</th>
<th>$g_{25}$</th>
<th>$g_{35}$</th>
<th>$g_{45}$</th>
<th>$g_{55}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.92302229</td>
<td>0.514503636</td>
<td>-8.80473832</td>
<td>7.06685744</td>
<td>-6.39685988</td>
</tr>
<tr>
<td>-2.20753243</td>
<td>-3.1547802</td>
<td>-3.14329541</td>
<td>0.09102079</td>
<td>-9.70635995</td>
</tr>
<tr>
<td>-4.13960669</td>
<td>-4.3602366</td>
<td>3.18825063</td>
<td>-6.80449303</td>
<td>5.44905935</td>
</tr>
<tr>
<td>8.71752843</td>
<td>7.36759087</td>
<td>-8.94688002</td>
<td>5.9761861</td>
<td>8.38771547</td>
</tr>
<tr>
<td>7.69435588</td>
<td>8.356605721</td>
<td>2.8104514</td>
<td>-6.87115857</td>
<td>2.98870923</td>
</tr>
</tbody>
</table>
Table 6. Comparison of the objective functions obtained by different approaches.

<table>
<thead>
<tr>
<th>Method</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive model by Bouazara$^{12}$</td>
<td>0.1704</td>
<td>0.0744</td>
<td>6.3805</td>
<td>6.7944</td>
</tr>
<tr>
<td>Active model by Mahmoodabadi and Mortazavi$^4$</td>
<td>0.1069</td>
<td>0.0480</td>
<td>4.0280</td>
<td>4.0981</td>
</tr>
<tr>
<td>Proposed Optimal AFC</td>
<td>0.0726</td>
<td>0.0254</td>
<td>2.3440</td>
<td>3.0016</td>
</tr>
</tbody>
</table>

In Tab. 6, the integrals of absolute relative displacement between the sprung mass and the front and rear tire, as well as the absolute body and seat accelerations ($F_1$, $F_2$, $F_3$ and $F_4$) are expressed. The numerical results of this table confirm that the introduced optimal AFC strategy presents better performance (less objective functions) in comparison with the other models introduced by Bouazara$^{12}$ and Mahmoodabadi and Mortazavi Yazdi.$^4$

6. CONCLUSION

In this paper, an optimal adaptive fuzzy controller based on the gravitational search algorithm has been developed to achieve more stringent levels of comfort for a half-body car model. At the first step, two adaptive fuzzy controllers have
been designed for both front and rear tires hinged on the Gaussian membership functions, the product inference engine, the singleton fuzzifier and the centre average defuzzifier. Each controller comprises two fuzzy systems each of which includes twenty-five IF-THEN fuzzy rules. At the second step, by using the GSA optimization method, the controller gains have been improved to decrease the relative displacement between the sprung mass and the front and rear tires along with the body and seat accelerations. Simulation results indicate that the proposed strategy exhibits most appropriate responses in comparison with the active and passive models reported in the literature. In other words, the active model, based on the optimal adaptive fuzzy controller, demonstrates a superior performance for decreasing the settling time, overshoot and the undershoot. Therefore, the proposed scenario presents a promising and effective model for design of the active suspension systems of vehicles.

**APPENDIX**

The proportional-integral-derivative (PID) controller, in a simple way, can be shown as follows:

\[
\text{PID} = K_p e + K_d \dot{e} + K_i \int e \, dt;
\]  

(62)
where, $K_P$, $K_D$ and $K_I$ are the proportional, derivative and the integral gains, respectively. In this paper, the values of the controller gains for the forward and rear tires have been chosen from Mahmoodabadi and Mortazavi Yazdi (2016) optimized via the multi-objective genetic algorithm and have been brought into Table 7.

REFERENCES


Table 7. Values of the PID controller gains for the forward and rear tires.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Forward tire</th>
<th>Rear tire</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_P$</td>
<td>-9456.8099</td>
<td>5363.1819</td>
</tr>
<tr>
<td>$K_D$</td>
<td>-1244.6119</td>
<td>47.1442</td>
</tr>
<tr>
<td>$K_I$</td>
<td>639.7846</td>
<td>-3281.0305</td>
</tr>
</tbody>
</table>


Pulsation Attenuation Analysis of Double-Chamber Composite Hydraulic Suppressors with Inserted Conical Tubes

Fan Yang and Bin Deng

Department of Mechanical Engineering, Southwest Jiaotong University, Chengdu, China.

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A one-dimensional analytical approach is developed to predict the pulsation attenuation performance of double-chamber compound hydraulic suppressors. The theoretical insertion loss agreed well with experimental results. The need for broadband pressure pulsation attenuation has led to extensive research on the structure improvement. In the present work, the straight-through tube has been replaced by a conical tube and two improved hydraulic attenuator configurations are presented. A parametric study to investigate the effects of different parameters on the research frequencies is included as well. The validity of the models of the improved structures is demonstrated theoretically and experimentally.

1. INTRODUCTION

With the development of hydraulic systems towards higher pressure, vibration and noise have become an urgent problem to be solved. In addition, pressure pulsation is always considered as the main source of these issues. Although there are a number of approaches to attenuate it, installing hydraulic suppressors is considered to be one of the most effective and flexible ways. Currently, there are a variety of attenuators with complicated functions. They can be generally divided into active and passive types. On one hand, because of high design requirements, difficulties, the fact that it’s expensive to manufacture and poor reliability, active pressure pulsation attenuation mainly rests on laboratory research and development phase. On the other hand, passive hydraulic suppressors could be further classified into three forms, resonant type, expansion chamber and interference type. Given that other two forms’ disadvantages, inconvenient movement and bulk mass, expansion chamber types are flexible ways to avoid above problems. Thus, they are used more widely. Incidentally, there are many different kinds of theories and methods aimed at muffler design, for instance, the transfer matrix method, which is widely used for one-dimensional (1D) plane waves. In order to appreciate the limitations of this approach, it is imperative to consider general three-dimensional (3D) wave propagation in tubes. Therefore, a 3D analytical approach, the finite element method (FEM) and boundary element method (BEM) are proposed.

At the very beginning, two double-chamber compound hydraulic suppressors were proposed in this paper. In addition, a 1D analytical approach was developed to predict their pulsation attenuation performance. Then, the need for broadband pressure pulsation attenuation has led to extensive research on the improvement of the structure. Therefore, the straight-through tubes, which were used in these compound hydraulic suppressors, were replaced by conical tubes. After that, the validity of the models of the improved structures was demonstrated theoretically and experimentally. Finally, a parametric study to investigate the effects of different parameters on the research frequencies was also included.

2. THEORY OF MODELLING

Before the mathematical modelling, two assumptions should be put forward:

• In the ideal case of a rigid-walled tube with sufficiently small cross dimensions filled with a stationary ideal (non-viscous) fluid, small-amplitude waves travel as plane waves.
• The variation of pressure pulsation with time is harmonic.

When there was a pressure pulsation disturbance, the state variables in this paper were represented as,

\[ \tilde{p} = P_0 + p; \]
\[ \tilde{u} = U_0 + u; \]
\[ \tilde{\rho} = \rho_0 + \rho. \]

Currently, multi-chamber reactive gas silencers are widely used in medium and high power diesel engines, besides, good middle and low frequency noise elimination effect can be obtained. Thanks to the inspiration from the successful application of these structures in mufflers, two double-chamber compound hydraulic suppressor configurations were put forward in this paper.

Based on the above hypothesis, the Helmholtz equation was obtained,

\[ \nabla^2 p(x, y, z) + k^2 p(x, y, z) = 0; \]
\[ p(x, t) = Ae^{i(\omega t-kx)} + Be^{i(\omega t+kx)}; \]
Using Eq. (5) and Eq. (6), the transfer matrix of a uniform tube was written as,

$$
\begin{bmatrix}
 p_r \\
 v_r
\end{bmatrix} = \begin{bmatrix}
 \cos(k_0 l_r) & j Y_r \sin(k_0 l_r) \\
 j Y_r \sin(k_0 l_r) & \cos(k_0 l_r)
\end{bmatrix} \begin{bmatrix}
 p_{r-1} \\
 v_{r-1}
\end{bmatrix}
$$

Lumped shunt element was represented as,

$$
\begin{bmatrix}
 p_r \\
 v_r
\end{bmatrix} = \begin{bmatrix}
 1 & 0 \\
 1/ Z_r & 1
\end{bmatrix} \begin{bmatrix}
 p_{r-1} \\
 v_{r-1}
\end{bmatrix}
$$

For perforated pipe units, the entire perforation section was divided into several sub-segments. Consider two control volumes, each of length $d x$, as shown in Fig. 2.

For simplicity, the tube and the chamber were assumed to be circular. Integrating the equations of mass continuity and dynamical equilibrium over the finite control volume,

$$
\frac{\partial}{\partial t} \iiint_{\Omega} \rho_i \hat{\rho}_i dV + \oint_{\Gamma} \rho_i \hat{u}_i \cdot \hat{n} dS = 0; \quad i = 1, 2;
$$

$$
\iiint_{\Omega} \frac{\partial \rho_i}{\partial t} dV + \frac{1}{2} \oint_{\Gamma} \hat{\rho}_i \hat{u}_i \hat{n} dS + \frac{1}{2} \iiint_{\Omega} \hat{\rho}_i \hat{d} \hat{u}_i dS = 0; \quad i = 1, 2.
$$

Applying the divergence theorem yields,

$$
\frac{\partial \rho_i}{\partial t} + \rho_0 \frac{\partial u_i}{\partial x} + \rho_0 f_i = 0; \quad i = 1, 2;
$$

where:

$$
\frac{\partial u_i}{\partial t} + \frac{\partial p_i}{\partial x} = 0; \quad i = 1, 2;
$$

(13)

For uniform perforations in a duct of constant cross section, at any position along the perforate,

$$
(p_1 - p_2)/u_1 = \rho_0 c_\xi_p.
$$

(16)

Note, the perforate impedance for quiescent medium,

$$
\xi_p = \left[ 6 \times 10^{-3} + j k_0 (t + 0.75 d_e) \right] / \phi;
$$

(17)

was used in this paper. Employing energy equation and substituting by Eqs. (14), (15) and (16), respectively, in Eq. (12) yields:

$$
\frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} + \rho_0 \frac{\partial u_1}{\partial x} + \frac{4}{d_1} \frac{p_1 - p_2}{c_\xi_p} = 0;
$$

(18a)

$$
\frac{1}{c^2} \frac{\partial^2 p_2}{\partial t^2} + \rho_0 \frac{\partial u_2}{\partial x} - \frac{4}{d_2^2 - d_{1e}^2} \frac{p_1 - p_2}{c_\xi_p} = 0.
$$

(18b)

Combining Eqs. (13) and (18) and subtracting $u_1$ yields:

$$
\frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_2}{\partial t^2} + \frac{4}{c_\xi_p d_1^2} \left( \frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t} \right) = 0;
$$

(19a)

$$
\frac{1}{c^2} \frac{\partial^2 p_2}{\partial t^2} - \frac{\partial^2 p_2}{\partial t^2} - \frac{4}{c_\xi_p (d_2^2 - d_{1e}^2)} \left( \frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t} \right) = 0.
$$

(19b)
Substituting for the time harmonic motion:

\[ p(x, t) = p(x)e^{j\omega t}; \]  

(20)

in Eq. (19) yielded,

\[
\begin{align*}
\frac{\partial^2 p_1}{\partial x^2} + a_1 \frac{\partial p_1}{\partial x} + a_2 p_1 + a_3 \frac{\partial p_2}{\partial x} + a_4 p_2 &= 0; \\
\frac{\partial^2 p_2}{\partial x^2} + a_5 \frac{\partial p_1}{\partial x} + a_6 p_1 + a_7 \frac{\partial p_2}{\partial x} + a_8 p_2 &= 0;
\end{align*}
\]

(21a, 21b)

where:

\[
\begin{align*}
\alpha_1 &= \alpha_3 = \alpha_5 = \alpha_7 = 0; \\
\alpha_2 &= k^2 - \frac{4jk}{d_1^2 c_p}, \\
\alpha_4 &= \frac{4jk}{d_1^2 c_p}, \\
\alpha_6 &= \frac{4jkd_2}{(d_2^2 - d_2^2 c_p)}, \\
\alpha_8 &= k^2 - \frac{4jkd_2}{(d_2^2 - d_2^2 c_p)}.
\end{align*}
\]

(22a, 22b, 22c, 22d, 22e)

Equation (21) was decoupled\(^\text{18}\) and solved in the following equation. Let:

\[
y_1 = p_1', \ y_2 = p_2', \ y_3 = p_1, \ y_4 = p_2.
\]

(23)

Substituting Eq. (23) in Eq. (21) then yields:

\[
\{ y' \} = [B]\{ y \};
\]

(24)

where:

\[
\{ y \} = [y_1, y_2, y_3, y_4]^T;
\]

\[
[B] = \begin{bmatrix}
0 & 0 & -\alpha_2 & -\alpha_4 \\
0 & 0 & -\alpha_6 & -\alpha_8 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

(25, 26)

Let:

\[
\{ y \} = [\psi] [\phi].
\]

(27)

Substituting Eq. (27) in Eq. (24) gave:

\[
\{ \phi' \} = [\psi]^{-1} [B][\psi] [\phi] = [A][\phi];
\]

(28)

where \([A]\) was a diagonal matrix composed of the eigenvalues \(\lambda\) of the matrix \([B]\). Substituting the solution of Eq. (28),

\[
\{ \phi \} = [C_1 e^{\lambda_1 x}, C_2 e^{\lambda_2 x}, C_3 e^{\lambda_3 x}, C_4 e^{\lambda_4 x}]^T;
\]

(29)

in Eq. (27) yields:

\[
\{ y \} = [\psi][C_1 e^{\lambda_1 x}, C_2 e^{\lambda_2 x}, C_3 e^{\lambda_3 x}, C_4 e^{\lambda_4 x}]^T.
\]

(30)

Substituting Eq. (30) in Eq. (23), using Eq. (13), and then rearranging the equation gave:

\[
\{ p \} = [Y(x)][C];
\]

(31)

where:

\[
\{ p \} = [p_1, \rho_0 c u_1, p_2, \rho_0 c u_3]^T;
\]

(32)

\[
\{ C \} = [C_1, C_2, C_3, C_4]^T;
\]

(33)

and:

\[
[Y(x)] = \begin{bmatrix}
\psi_{31} e^{\lambda_{11} x} & \psi_{32} e^{\lambda_{12} x} & \psi_{33} e^{\lambda_{13} x} & \psi_{34} e^{\lambda_{14} x} \\
-\psi_{11} e^{\lambda_{11} x} & -\psi_{12} e^{\lambda_{12} x} & -\psi_{13} e^{\lambda_{13} x} & -\psi_{14} e^{\lambda_{14} x} \\
\psi_{41} e^{\lambda_{21} x} & \psi_{42} e^{\lambda_{22} x} & \psi_{43} e^{\lambda_{23} x} & \psi_{44} e^{\lambda_{24} x} \\
-\psi_{21} e^{\lambda_{21} x} & -\psi_{22} e^{\lambda_{22} x} & -\psi_{23} e^{\lambda_{23} x} & -\psi_{24} e^{\lambda_{24} x}
\end{bmatrix}.
\]

(34)

The following equations started with handling boundary conditions.

\[
\rho_0 c u_2(0)/p_2(0) = -j \tan(k l_2);
\]

(35a)

\[
\rho_0 c u_2(l_1)/p_2(l_1) = j \tan(k l_1).
\]

(35b)

Insertion loss (IL) is then evaluated by,\(^8,9\)

\[
IL = 20 \log \left| \frac{Z_0 A_{11} + A_{12} + Z_{n+1} A_{21} + Z_{n+1} A_{22}}{Z_0 A'_{11} + A'_{12} + Z_{n+1} A'_{21} + Z_{n+1} A'_{22}} \right|.
\]

(36)

Figures 3 and 4. Schematic of a DTETC. \(D_1 = D_2 = 0.0386 m, D = 0.068 m, l_1 = l_2 = 0.074 m, I = 0.175 m, l_1 = l_2 = 1/2, l_2 = 1/4.\)

3. RESULTS AND DISCUSSION

A prototype double-tuned extended-tube expansion chamber (DTETC) used as a contrast was fabricated as shown in Fig. 3.\(^8\)

A schematic of the test rig can be seen in Fig. 4. Flow was provided to the hydraulic system from a 9-piston axial piston pump driven by a variable frequency driver (VFD). The
data from each sensor was collected by a data acquisition card (DAQ) mounted inside of a PC. A back-pressure valve was located downstream of the pressure gauge. This valve was used to load the system to a given static pressure, 13 MPa. The hydraulic oil used in this paper had a density, $866 \text{ kg/m}^3$ and sound speed, 1400 m/s. Due to the fact that the frequency bands of most servo valves and servo systems are below 2000 Hz, the pressure pulsation we were interested in was limited to it. In other words, a signal that is higher than 2000 Hz was be considered in this paper, for it does not interfere with the servo systems. As seen in Fig. 5, the overall matching is reasonably good for the interested frequency domain. The extended tubes of composite structure 1 and 2 were fabricated as shown in Fig. 6. Figures 7 and 8 compare the theoretical predictions and experimental results for composite structures 1 and 2. The experimental results agree reasonably well with the theoretical results among the research frequency band. Only a plane wave would propagate (all higher modes, even if present, being cut-off, that is attenuated exponentially).8

Figure 5. Comparison of theoretical IL and experimental measurement of DTETC.

Figure 6. The experimental devices of extended tubes for the composite structures. 1 - Inlet pipe of structure 1; 2 - Outlet pipe of structure 2; 3 - Outlet pipe of structure 1; 4 - Inlet pipe of structure 2.

Figure 7. IL of the composite structure 1: theory versus experiments.

Figure 8. IL of the composite structure 2: theory versus experiments.

Figure 9 compares the theoretical IL of DTETC and the composite structures.

As shown in Fig. 9, these two composite configurations can improve the IL performance of high frequencies (>1700 Hz), compared with the IL characteristics of DTETC. At intermediate frequencies (1000 Hz~1700 Hz), the IL is greater for the DTETC, particularly the trough for composite structure 2 approximately occurs at 1700 Hz. At low frequencies (<1000 Hz), the effect, however, is marginal.

4. STRUCTURE IMPROVEMENT AND DISCUSSION

In order to improve the IL performance of two composite configurations in mid-frequency domain (1000 Hz~1700 Hz), through the use of structure optimization, broad-spectrum composite configurations were put forward. The improved structures as shown in Fig. 10 show the use of conical tubes. The transfer matrix of conical tubes is defined as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

where:

$$T_{11} = \frac{d_1}{d_0} \cos(k_0l) - \frac{\eta}{k_0d_0} \sin(k_0l);$$

$$T_{12} = \frac{d_1}{d_0} \sin(k_0l) + \frac{\eta}{k_0d_0} \cos(k_0l);$$

$$T_{21} = \frac{\eta}{k_0d_0} \sin(k_0l);$$

$$T_{22} = \frac{\eta}{k_0d_0} \cos(k_0l).$$
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Figure 9. Comparison of theoretical IL of DTETC and composite structures. (a) DTETC versus composite structure 1. (b) DTETC versus composite structure 2.

The conical tubes were fabricated as shown in Fig. 12. The theoretical predictions and experimental results for the fabricated prototypes are compared in Fig. 13. The experimental results agree reasonably well with the theoretical results in the whole research frequency band. Discrepancy between theory and experiments may be attributed to the fact that only a rough estimate of the physical dimensions of the hydraulic suppressors has been used in the computations.

4.1. Parametric Study

The variations of the parameters which affect the performance of the improved composite structures are studied in this section. The sensitivity of the improved composite structures to variations in minor diameter is shown in Figs. 14 to 15.

The prototype hydraulic suppressor was used as the baseline case in all the comparisons that follow (see solid lines in Figs. 14 to 23). Figure 14 shows that the effect of the minor diameter of the conical tube is marginal until about 800 Hz. At

\[ T_{12} = jY(0) \frac{d_0}{d_l} \sin(k_0 l); \]  (39)

\[ T_{21} = \left\{ \frac{j}{Y(0)} \frac{d_0}{d_l} \left( 1 + \frac{\eta^2}{k_0^2} \right) \sin(k_0 l) - \frac{j}{k_0 d_0} Y(0) \frac{d_l}{d_0} \left( 1 - \frac{d_0}{d_l} \right) \cos(k_0 l) \right\}; \]  (40)

\[ T_{22} = \frac{\eta}{k_0 d_1} \sin(k_0 l) + \frac{d_0}{d_l} \cos(k_0 l). \]  (41)

Figure 10. Schematic diagrams of improved composite structures. (a) Improved composite structure 1 (ICS1). \( d_2 = 26.6 \text{ mm}, L_a = 20 \text{ mm}, L = 43.75 \text{ mm}, d_h = 2 \text{ mm}, Q = 1 \%. \) (b) Improved composite structure 2 (ICS2). \( d_2 = 26.6 \text{ mm}, L_a = 40 \text{ mm}, L = 21.875 \text{ mm}, d_h = 2 \text{ mm}, Q = 1 \% \).

Figure 11. A schematic diagram of conical tube section.

Figure 12. Experimental devices of conical tubes. 1 - Conical tube of ICS2; 2 - Conical tube of ICS1.
Figure 13. IL of the improved composite structures: theory versus experiments. (a) IL of ICS1 theory versus experiments. (b) IL of ICS2 theory versus experiments.

Figure 14. Effect of the minor diameter on the IL performance of ICS1. At high frequencies (>1800 Hz), the effect is marginal too. As for Fig. 15, decreasing the minor diameter shifts the peaks and trough in the IL to lower frequencies. These frequency limits depend, to a large extent, on the geometry of the hydraulic suppressor.

Figure 15. Effect of the minor diameter on the IL performance of ICS2.

Figure 16. Effect of the extended length of the conical tube on the performance of ICS1.

Figure 17. Effect of the extended length of the conical tube on the performance of ICS2.

Figures 18 and 19 show the effect of the extended length of the straight-through tubes on the IL performance. As can be seen from Fig. 18, the effect is minimal at frequencies (below 700 Hz), however, at frequencies between 700 Hz and 1800 Hz the IL is greater for the ICS1 with shorter extended length of straight-through tube. At higher frequencies (>1800 Hz),
the effect is just the opposite. As noted in Fig. 19, the effect, however, is minimal below 800 Hz, and at frequencies beyond it, increasing the extended length of straight-through tube shifts the peak and trough in the IL to lower frequencies.

The effect of the porosity of the perforated tube is shown in Figs. 20 and 21. As can be seen in these figures, the effect is minimal at frequencies (below 700 Hz). At porosities greater than 2.3% the IL is marginal where the simple expansion chamber behavior dominates. When the porosities are less than 1%, decreasing the porosities shifts the peaks in the IL to lower frequencies.

The effect of increase in the hole diameter of the perforated tube is shown in Figs. 22 and 23. As can be seen in these figures, the effect is
5. CONCLUSIONS

Owing to the inspiration from the successful application of multi-chamber reactive mufflers in medium and high-power diesel engines, two double-chamber composite hydraulic suppressors and their improved configurations are put forward in this paper. A one-dimensional approach is then presented to analyse them theoretically. The results compare reasonably well with experiments. The extended length of the conical tube is shown to have only a marginal effect among the whole research frequencies for both two improved composite structures. However, other parameters have different effects on IL performance. As for perforated tube used for the hydraulic suppressor, the fact that the hole diameter should not be too small and porosity should not be too large is confirmed in this paper. That is to say, microperforated panel, which is widely used in the muffler design, is not applicable for hydraulic suppressors.

REFERENCES


Effects of Elliptical Ring Electrodes on Shear Vibrations of Quartz Crystal Plates

Rongxing Wu
Piezoelectric Device Laboratory, School of Mechanical Engineering and Mechanics, Ningbo University, Ningbo, Zhejiang 315211, China.
Department of Architectural Engineering, Ningbo Polytechnic, Ningbo, Zhejiang 315800, China.

Ji Wang and Jianke Du
Piezoelectric Device Laboratory, School of Mechanical Engineering and Mechanics, Ningbo University, Ningbo, Zhejiang 315211, China.

Jiashi Yang
Piezoelectric Device Laboratory, School of Mechanical Engineering and Mechanics, Ningbo University, Ningbo, Zhejiang 315211, China.
Department of Mechanical and Materials Engineering, the University of Nebraska-Lincoln, Lincoln, NE 68588-0526, U.S.A.

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A theoretical analysis is performed on the thickness-shear vibrations of an AT-cut quartz piezoelectric crystal plate with elliptical ring electrodes. The scalar differential equation by Tiersten and Smythe is used. An analytical solution is obtained. Numerical results from the solution show that the thickness-shear mode of interest may be trapped by the ring electrodes and can have a convex, concave, or nearly flat vibration distribution near the plate center, which is fundamentally important when the plate is used as an acoustic wave mass sensor. The vibration distribution is found to be sensitive to both the geometric and physical parameters of the electrodes. Therefore, a careful design is needed to realize the desired trapped mode with suitable center convexity for sensor application.

1. INTRODUCTION

Crystal plates can be used to make acoustic wave resonators as frequency standards for both timekeeping and frequency operations with broad applications. They have been under sustained study experimentally, numerically and theoretically (see the review and, more recently). Many crystal resonators operate with the thickness-shear (TSh) mode of an AT-cut quartz plate. The resonant frequencies of a crystal plate can be affected by many effects such as a temperature change, stress or strain, a surface mass layer, or contact with a fluid, etc. Therefore, the detection of frequency shifts in a crystal plate can be used as the basis for making various acoustic wave sensors.

This paper is concerned with mass sensors based on quartz plates vibrating in TSh modes. They are called quartz crystal microbalances (QCMs). In QCMs, the inertia of an additional thin mass layer lowers the resonant frequencies of the plate. The sensitivity of QCMs was given by the well-known Sauerbrey equation, which was based on pure TSh modes without in-plane mode variations. Pure TSh modes can only exist in an unbounded and uniform plate with an additional mass layer that is also uniform and unbounded. However, in real QCMs of finite plates, the operating modes in fact have slow in-plane variations due to the plate boundaries. These in-plane mode variations cause deviation from the frequency prediction by the Sauerbrey equation. In-plane mode variations also arise when the additional mass layers or electrodes cover the crystal plates partially in the central region. Typically, the vibration of the operating mode in a QCM has a bell-shape distribution similar to the normal distribution in statistics, large near the plate center where the additional mass layer or the electrode is, and decays away from the mass layer or electrode edge.

An additional mass layer on a plate can be viewed as a continuous distribution of mass points. In the ideal case of an unbounded crystal plate in which the TSh mode is uniform in the plane of the plate, the frequency response of the plate to a point mass is independent of the location of the mass. However, in a finite plate with in-plane mode variations, the frequency response of the plate to a point mass depends on the location of the mass. Near the plate center where the vibration amplitude is relatively large, so is the frequency shift or sensitivity caused by the point mass there. Qualitatively, the sensitivity distribution roughly follows the vibration amplitude distribution, large near the plate center and small near the plate edges. Ideally, for mass sensor application, a nearly uniform vibration distribution with little in-plane mode variation is desired. In QCMs, researchers have made use of circular and rectangular ring electrodes to achieve nearly flat or uniform vibration distribution in resonators.

It has been known that, because of the in-plane material anisotropy of a quartz plate, the optimal shape of an AT-cut quartz plate device or its electrodes is not exactly circular and instead is almost elliptical, with the major axis exceeding the minor axis by approximately 26%. Existing theoretical re-
sults on elliptical quartz plate devices are for the application of the plates as resonators providing frequency standards rather than mass sensors.\textsuperscript{26–32} Therefore, little consideration has been given to the mode center convexity. In this paper we studied the TSh vibration of an AT-cut quartz plate with elliptical ring electrodes for mass sensor applications. Our main interest was the convexity of the vibration distribution near the plate center and determining whether or not there were any modes near the plate center that were flat.

2. GOVERNING EQUATIONS

Consider the AT-cut quartz plate in Fig. 1. For TSh motions, the dominating displacement component was $u_1(x_1, x_2, t)$. The operating TSh modes for quartz resonators were antisymmetric about the plate middle plane at $x_2 = 0$. These modes can be excited by an electric field along the plate thickness through the piezoelectric constant $c_{33}$. Tiersen and Smythe derived a single scalar differential equation for TSh modes in AT-cut quartz plates.\textsuperscript{33–36} The scalar equation is relatively simple and accurate, and has been widely used in analyzing quartz resonators.\textsuperscript{37–42} We used the scalar equation in the present paper and summarize the relevant expressions below. The $n$th-order TSh displacement $u_1^n(x_1, x_2, t)$ antisymmetric about $x_2 = 0$ was defined by

$$u_1^n(x_1, x_2, x_3, t) = \sum_{n=1,3,5,...}^\infty u_1^n(x_1, x_3, t) \sin \frac{n\pi}{2h}x_2. \quad (1)$$

$n = 1$ represented the fundamental family of modes. $n > 1$ were for the overtone modes. For electroded and unelectroded plates, in time-harmonic motions with a frequency $\omega$, the scalar equation governing $u_1^n$ assumes slightly different forms and is given by

$$M_n \frac{\partial^2 u_1^n}{\partial x_1^2} + c_{33} \frac{\partial u_1^n}{\partial x_3^2} + \rho \left( \omega^2 - \omega_\infty^2 \right) u_1^n = 0; \quad (2)$$

for an electroded plate and

$$M_n \frac{\partial^2 u_1^n}{\partial x_1^2} + c_{33} \frac{\partial u_1^n}{\partial x_3^2} + \rho \left( \omega^2 - \omega_\infty^2 \right) u_1^n = 0; \quad (3)$$

for an unelectroded plate, respectively. In Eqs. (2) and (3), $M_n$ depends on the usual elastic constants $c_{00}$, the piezoelectric constants $c_{ip}$, and the dielectric constants $\varepsilon_{ij}$ of AT-cut quartz:

$$M_n = c_{11} + (c_{12} + c_{06})r + 4\left(\varepsilon_{06} - \varepsilon_{00}\right)\varepsilon_{cc22} + c_{12}\frac{\omega n\pi}{2h} \cot \left(\frac{\omega n\pi}{2h}\right); \quad (4)$$

$$\varepsilon_{66} = \varepsilon_{66} + \frac{\varepsilon_{26}^2}{\varepsilon_{22}^2}; \quad (5a)$$

$$\kappa = \left(\frac{\varepsilon_{66}}{\varepsilon_{22}}\right)^{1/2}; \quad (5b)$$

$$r = c_{12} + c_{06}; \quad (5c)$$

$$\varepsilon_{66} = \frac{\varepsilon_{66}}{\varepsilon_{66} + \varepsilon_{26} - 2R}; \quad (6a)$$

$$\varepsilon_{26}^2 = \frac{\varepsilon_{26}^2}{\varepsilon_{66} + \varepsilon_{26}}; \quad R = \frac{2\rho h'}{\rho h}; \quad (6b)$$

$$\omega_\infty^2 = \frac{n^2\pi^2}{4h^2} \varepsilon_{66}^2; \quad (7a)$$

$$\omega_\infty^2 = \frac{n^2\pi^2}{4h^2} \varepsilon_{66}^2; \quad (7b)$$

where $\rho$ represented the mass density. The thickness and the density of the electrodes were $2h'$ and $\rho'$. $\omega_\infty$ and $\omega_\infty$ were the TSh resonant frequencies for unbounded, electroded and unelectroded plates, respectively. Since the electrode inertia lowers the resonant frequencies and that the shorting of the electrodes reduced the piezoelectric stiffening effect, we have $\omega_\infty > \omega_\infty$.

3. ANALYTICAL SOLUTION

For the applications we are interested in, it was sufficient to consider a special class of elliptical electrodes whose semi-major and semi-minor axes satisfied

$$a_1/b_1 = a_2/b_2 = \lambda. \quad (8)$$

The value of $\lambda$ was determined later. It was easy to verify that when Eq. (8) was satisfied, the elliptical electrode boundaries in Fig. 1 became circles with radii $b_1$ and $b_2$ in the transformed ($\xi_1, \xi_2$) plane defined by

$$x_1 = \lambda \xi_1; \quad (9a)$$

$$x_2 = \xi_2. \quad (9b)$$

Under Eq. (9), the scalar equations in Eqs. (2) and (3) became

$$M_n \frac{\partial^2 u_1^n}{\partial \xi_1^2} + \frac{\partial u_1^n}{\partial \xi_2^2} + \frac{\rho}{c_{55}} \left( \omega^2 - \omega_\infty^2 \right) u_1^n = 0; \quad (10a)$$

$$M_n \frac{\partial^2 u_1^n}{\partial \xi_1^2} + \frac{\partial u_1^n}{\partial \xi_2^2} + \frac{\rho}{c_{55}} \left( \omega^2 - \omega_\infty^2 \right) u_1^n = 0. \quad (10b)$$

We choose $\lambda$ such that

$$\frac{M_n}{c_{55} \lambda^2} = 1. \quad (11)$$

Then Eq. (10) became

$$\nabla^2 u_1^n + \alpha^2 u_1^n = 0; \quad (12a)$$

$$\nabla^2 u_1^n - \beta^2 u_1^n = 0; \quad (12b)$$

Figure 1. Plan view of an unbounded quartz plate with elliptical ring electrodes. The plate thickness is $2h$ along $x_2$ which is determined from $x_3$ and $x_1$ by the right-hand rule. There are two identical electrodes at the plate top and bottom.
where
\[
\nabla^2 = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_3^2};
\]
\[
\alpha^2 = \frac{\rho}{c_{55}}(\omega^2 - \omega_\infty^2);
\]
\[
\beta^2 = \frac{\rho}{c_{55}}(\omega_\infty^2 - \omega^2).
\]

The frequency range of interest was
\[
\omega_\infty^2 < \omega^2 < \omega_\infty^2.
\]

Therefore, \(\alpha^2 > 0\) and \(\beta^2 > 0\). In polar coordinates defined by
\[
\xi_1 = r \cos \theta;
\]
\[
\xi_3 = r \sin \theta.
\]

Eq. (12) takes the following form
\[
\frac{\partial^2 u^n}{\partial r^2} + \frac{1}{r} \frac{\partial u^n}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u^n}{\partial \theta^2} + \alpha^2 u^n = 0;
\]
\[
\frac{\partial^2 u^n}{\partial r^2} + \frac{1}{r} \frac{\partial u^n}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u^n}{\partial \theta^2} - \beta^2 u^n = 0.
\]

For solutions of Eq. (17), we let
\[
u^n(r, \theta) = \begin{cases} u^n(r) \cos m\theta \\ u^n(r) \sin m\theta \end{cases};
\]

where \(m = 0, 1, 2, \ldots\). The substitution of Eqs. (18) into (17) lead to the following ordinary differential equations for \(u^n(r)\)
\[
\frac{d^2 u^n}{dr^2} + \frac{1}{r} \frac{du^n}{dr} + \left(\frac{\alpha^2 - \frac{m^2}{r^2}}{r^2}\right) u^n = 0;
\]
\[
\frac{d^2 u^n}{dr^2} + \frac{1}{r} \frac{du^n}{dr} - \left(\frac{\beta^2 + \frac{m^2}{r^2}}{r^2}\right) u^n = 0.
\]

The general solutions to Eq. (19) are
\[
\begin{align*}
A I_m(\beta r) & \quad r < b_1 \\
B J_m(\alpha r) + C Y_m(\alpha r) & \quad b_1 < r < b_2 \\
D K_m(\beta r) & \quad b_2 < r
\end{align*}
\]

where \(A, B, C\) and \(D\) were undetermined constants. \(J_m\) and \(Y_m\) were Bessel functions of the first and second kind. \(I_m\) and \(K_m\) were modified Bessel functions of the first and second kind. At \(b_1\) and \(b_2\), \(u^n\) and \(du^n/dr\) must be continuous. The substitution of Eq. (20) into these continuity conditions yields
\[
\begin{align*}
A I_m(\beta b_1) = B J_m(\alpha b_1) + C Y_m(\alpha b_1); \\
A \beta I'_m(\beta b_1) = B \alpha J'_m(\alpha b_1) + C \alpha Y'_m(\alpha b_1); \\
B J_m(\alpha b_2) + C Y_m(\alpha b_2) = D K_m(\beta b_2); \\
B \alpha J'_m(\alpha b_2) + C \alpha Y'_m(\alpha b_2) = D \beta K'_m(\beta b_2);
\end{align*}
\]

For comparison, consider the case of a full elliptical electrode with \(b_1 = 0\) mm and \(b_2 = 5\) mm first. This is the classical trapped-energy quartz plate resonator. The device operating TSh mode of interest is shown in Fig. 2. Both a three-dimensional view and a contour view of the distribution of the TSh displacement distribution in the plane of the plate are shown. The vibration is mainly in the electroded central region and decays rapidly outside the electrodes. This is the so-called energy trapping by the electrodes. The vibration decays essentially zero sufficiently far away from the center where device mounting can be designed without affecting the operation of the device. In the electroded central area, the vibration is nonuniform and convex, with the maximum displacement at the center of the electrodes. When the plate is used as an acoustic wave sensor, this nonuniformity of displacement distribution causes deviation of the sensitivity from the theoretical prediction based on the uniform vibration distribution from an unbounded plate model.

For the case of a ring electrode we choose \(b_1 = 0.8\) mm and all of the other parameters were the same as those used for

\[
J'_0 = -J_1; \\
Y'_0 = -Y_1; \\
I'_0 = I_1; \\
K'_0 = -K_1.
\]

4. NUMERICAL RESULTS AND DISCUSSION

As a numerical example, consider a typical AT-cut quartz plate whose half thickness is given by \(h = 0.343915\) mm. The electrode density and thickness together are given by the mass ratio \(R = 0.0102\). These determine that \(\lambda = 1.2641\). \(b_1\) and \(b_2\) will be chosen differently in various figures below. \(a_1 = \lambda b_1\), \(a_2 = \lambda b_2\). For comparison, consider the case of a full elliptical electrode with \(b_1 = 0\) mm and \(b_2 = 5\) mm first. This is the classical trapped-energy quartz plate resonator. The device operating TSh mode of interest is shown in Fig. 2. Both a three-dimensional view and a contour view of the distribution of the TSh displacement distribution in the plane of the plate are shown. The vibration is mainly in the electroded central region and decays rapidly outside the electrodes. This is the so-called energy trapping by the electrodes. The vibration decays essentially zero sufficiently far away from the center where device mounting can be designed without affecting the operation of the device. In the electroded central area, the vibration is nonuniform and convex, with the maximum displacement at the center of the electrodes. When the plate is used as an acoustic wave sensor, this nonuniformity of displacement distribution causes deviation of the sensitivity from the theoretical prediction based on the uniform vibration distribution from an unbounded plate model.
plotting Fig. 2. The operating TSh mode is shown in Fig. 3. The most important difference between Fig. 3 and Fig. 2 is that the mode in Fig. 3 has a concave central region. Therefore, it can be expected that, for some value of the mode in Fig. 3, the operating mode when the central part of the mode may be essentially flat, i.e., neither convex nor concave.

We noted that the frequency of the mode in Fig. 3 was a little higher than that in Fig. 2 because the ring electrodes had less inertia. As the electrode edges, and therefore the mode center becomes more concave. As $R$ increase, the frequency becomes lower as expected because of more electrode inertia.

5. CONCLUSIONS

An AT-cut quartz plate with elliptical ring electrodes can support thickness-shear vibrations modes trapped by the electrodes. The convexity of the vibration distribution near the plate center may vary and it is sensitive to the geometric and physical parameters of the electrodes. With proper design, the vibration distribution can be made essentially flat near the plate center for mass sensor application. The analysis in the present paper is simple and effective for the design of these devices.

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The Key Role of Headrest Optimization in Driver Comfort

Hamid Gheibollahi and Masoud Masih-Tehrani

Vehicle Dynamical System Research Lab, School of Automotive Engineering, Iran University of Science and Technology, Tehran, Iran.

Mohammadmehdi Niroobakhsh

Civil & Mechanical Engineering Department, University of Missouri-Kansas City, Kansas City, USA.

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In this study, adding a headrest to the conventional vehicle driver seat is investigated to improve the driver comfort and decrease the driver damages. For this purpose, a conventional biomechanical human body model of whole-body vibrations is provided and modified by adding a head degree of freedom to the body model and a headrest to the seat model. The basic model is in the sitting posture, lumped parameters and has nine DOFs for the human body, on contrary to the proposed model which has ten DOFs. The new human body DOF is the twisting motion of the head and neck. This new DOF is generated because of headrest adding to the driver’s seat. To determine the head discomforts, the Seat to Head (STH) indexes are studied in two directions: horizontal and vertical. The Genetic Algorithm (GA) is used to optimize the STH in different directions. The optimization variables are stiffness and damping parameters of the driver’s seat which are 12 for the basic model and are 16 for a new seat. The integer programming is used for time reduction. The results show that new seat (equipped by headrest) has very better STH in both directions.

NOMENCLATURE

\( c_{1v}, c_{1h} \) Upper leg vertical and horizontal dampers,
\( c_{2v}, c_{2h} \) Pelvic vertical and horizontal dampers,
\( c_{4v}, c_{4h} \) Back horizontal and vertical dampers,
\( c_{21} \sim c_{54} \) The respective dampers between body segments,
\( C \) Damping matrix,
\( f \) Force vector,
\( F \) Complex Fourier transform of the forces,
\( k_{1v}, k_{1h} \) Upper leg vertical and horizontal springs,
\( k_{2v}, k_{2h} \) Pelvic vertical and horizontal springs,
\( k_{14}, k_{1h} \) Back horizontal and vertical springs,
\( k_{21} \sim k_{54} \) The respective springs between body segments,
\( K \) Stiffness matrix,
\( l \) Distance from headrest to the neck joint,
\( m_1 \) Mass of Upper Leg (left + right),
\( m_2 \) Mass of Pelvic,
\( m_3 \) Mass of Viscera (Soft abdominal body parts),
\( m_4 \) Mass of upper Torso (Including hands),
\( m_5 \) Mass of head and neck,
\( M \) Mass matrix,
\( STH \) Head to seat vibration ratio (vertical),
\( STH_x \) Head to seat vibration ratio (Horizontal),
\( STH_{RMS} \) Root mean square of STH,
\( w_1 \) Transferability weighting coefficients of horizontal vibrations,
\( w_2 \) Transferability weighting coefficients of horizontal vibrations,
\( x \) Complex transfer response vector,
\( X \) Complex Fourier transform of the variables,
\( X_0 \) Seat input excitation in the vertical direction,
\( X_8 \) Back horizontal frequency response,
\( X_9 \) Head vertical frequency response,
\( X_v \) Backrest horizontal excitation,
\( \Theta \) Head twist angle,
\( \omega \) Excitation frequency.

1. INTRODUCTION

The experience of whole-body vibration in daily life is common to most people. It happens when a person is affected by a vibrating surface and thus, all parts of the body that may even be far from the main vibration source are exposed to the vibration. Whole-body vibration at frequencies from 1 to 100 Hz for humans is understandable. Backbone damage caused by long-term vibrations occurs in the frequency range of 4 to 12 Hz. Feeling terrible in the digestive system is a result of being exposed to whole-body vibrations for long periods of time. This inconvenient feeling in the stomach occurs at frequencies between 4 to 5 Hz. This is the resonance range of the stomach. The cardiovascular system can be affected by long-term vibrations at frequencies below 20 Hz. Fast and deep breathing, in addition to increased heart rate, are the results of these vibrations. The resonance frequency for the head and neck is variable from 4 to 13 Hz. Many studies are performed to improve driver comfort with headrest optimization.

Biomechanical studies of body vibration and its damage are conducted on humans, animals, and dummies. These studies on humans date back to 1918, when Hamilton investigated the effect of vibrations on limestone mine workers. In 1984, Alem determined a standard for these damages by performing the axial impact test on nineteen human corpses to study the mechanical properties of the head, neck and spine. In 1998,
in an effort by Boileau overall biodynamical human body response values facing different workplaces were specified from various published data.\(^9\) In 2000, Yoganandan studied the biomechanical body responses of a man and four women in crashes applied to the rear of the body and evaluated neck injury risks.\(^10\) In 2005, Mansfield pointed out in his book that, for whole-body vibrations, people are more sensitive to frequencies below 20 Hz.\(^11\) In 2008, Nelisse and Patra designed two dummies to assess the vibration isolation effectiveness of suspension seats.\(^12\) In 2010, Bovenzi conducted some tests on 202 male drivers. His goal was to address injuries and back pains caused by long distance driving.\(^13\) In 2013, Thamsuwan and his colleagues studied whole-body vibrations of bus drivers with different floor heights of buses and considered their back pain at each height.\(^14\) In 2014, Zhao and his colleagues designed a semi-active control system to control vibrations on the human body by using a four DOFs of the human body model.\(^15\)

Another method in these studies involves the use of biomechanical human body models.\(^16–18\) These models can be classified into lumped-parameter models, multi body models and finite element models.\(^19–22\) In lumped-parameter models, the human body is considered as several concentrated masses that are connected with springs and dampers. Multi body models are composed of several rigid bodies that are connected to each other by either pin connection (two-dimensional) or spherical connection (three-dimensional). For finite element models, it is assumed that the human body contains many finite elements and that the properties of these elements are obtained from experiments on human bodies.

One application of biomedical studies is designing an optimized driver’s seat to reduce body vibrations.\(^23\) Models with this purpose usually consider the optimal parameters for a driver’s seat. However, the headrest and horizontal vibrations applied to the head in long distance traveling is very important.\(^24\) Vibrations caused by the driver’s headrest during long distance travel can cause damage to the upper vertebrae of the spine, head and neck.

In this study, Harsha and his colleague’s model which was introduced in 2014, was chosen as the base model for the human body and driver’s seat.\(^25\) The reason for this selection was that this model contained both vertical and horizontal degrees of freedom simultaneously and a lumped-parameter that is rarely found in other models. Harsha’s model has nine DOFs and vibrations applied to the body in horizontal and vertical directions. However, in his model the effect of input vibrations from the base to the head were not considered and input vibrations were from the seat and backrest of the driver. Also, in Harsha’s model the horizontal DOF of the head was dependent on waist movement and has no independent DOF.\(^25\)

To add the headrest and study passenger comfort, vibrations applied to head were modeled in horizontal and vertical directions and the body had ten DOFs. Then, a biomechanical model of the body and seat was introduced and the governing equations of the base and modified model were derived. The optimization problem to evaluate the passenger’s comfort was extracted and its solution was expressed by a genetic algorithm method. Due to the complexity of the problem and the large number of DOFs (12 optimization variable for the base model and 16 variables for the new model) using this powerful algorithm was an appropriate option. Finally, the optimization results were reviewed and classified.

## 2. MODELING AND METHODOLOGY

In this study, a biomechanical model of whole-body vibration was provided. This model was provided to check head injuries caused by vibrations and finally to design the optimal parameters for the car’s seat. The presented model was in a sitting position, lumped parameter and had ten DOFs. Applied vibrations on model were both vertical and horizontal. The overall structure of the model was obtained from the nine DOFs of Harsha’s model.\(^25\) In Harsha’s model, the body was divided into five concentrated mass that each had two DOFs in horizontal and vertical directions. However, it should be noted that, in Harsha’s model, the horizontal DOF of the head is associated with the horizontal movement of the waist and cannot be considered as an independent DOF. Furthermore, in Harsha’s model the forces that were applied on the body came from the seat and the backrest. Figure 1 shows Harsha’s model with the backrest. In addition to the above forces, the horizontal force applied to the head was also considered. In this way, one rotational DOF was added to the vertical movement of the head that increased DOFs from nine to ten. Figure 2 shows the proposed model in this study. In this paper, the motion equations of the model were extracted and then, by transferring them from time to frequency domain, the vibration transferability parameter from seat to head in the presence of headrest was discussed. Afterward, by defining an objective function of vibration transferability and using a genetic algorithm, seat parameters were optimized.

### 2.1. Governing Equations of Modeling

In general, there are two methods for solving motion equations: solving in time domain and solving in frequency domain. Usually solving in frequency domain is more efficient.
than solving in time domain. Although, for solving in frequency domain, equations must be linear. Transferring from time domain to the frequency domain can be performed by Fourier transform. Equations of motion are written in the general form of Eq. (1):

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f};$$

(1)

where the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $10 \times 10$ and respectively represent mass, damping and stiffness of the system. The matrix $\mathbf{f}$ is $1 \times 10$ and represents the external forces applied to the body by the seat. Using Fourier transform function, Eq. (1) is transmitted from time domain to the frequency domain. Using Fourier transform function, Eq. (1) is transmitted from time domain to the frequency domain.

$$\mathbf{X}(j\omega) = \left[\mathbf{K} - \omega^2\mathbf{M} + j\omega\mathbf{C}\right]^{-1}\mathbf{F}(j\omega);$$

(2)

where $\mathbf{X}(j\omega)$ and $\mathbf{F}(j\omega)$ are complex Fourier transform vectors of $\mathbf{x}$ and $\mathbf{f}$ and $\omega$ is the excitation frequency. Vector $\mathbf{X}(j\omega)$ is the complex frequency response of each of masses that is a function of $\omega$:

$$[X_1(j\omega), X_2(j\omega), ..., X_{10}(j\omega)].$$

(3)

$\mathbf{F}(j\omega)$ includes complex excitation forces which are applied into the body by the seat that is a function of $\omega$.

Vertical vibrations transitivity parameter is defined as a ratio of head output response to seat excitation input as Eq. (4)\textsuperscript{26}:

$$STH = \frac{X_9(j\omega)}{X_0};$$

(4)

where $X_9$ and $X_0$ are respectively the head vertical frequency response and the seat input excitation in vertical direction.

Based on Gan’s studies, the horizontal vibration transmissibility parameter is calculated based on some changes in the model of Eq. (5)\textsuperscript{27}:

$$STH = \frac{l(\Theta(j\omega) + X_8(j\omega))}{X_h};$$

(5)

where $l$ is the distance from headrest to the neck joint, $\Theta$ is the head twist angle, $X_8$ is back horizontal frequency response and $X_h$ is the backrest horizontal excitation.

In this research, the root mean square of these parameters was used to simplify the comparison of the transmissibility parameters. The size of this function was calculated accordance with Eq. (6):

$$STH_{RMS} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} STH_i^2}.$$  

(6)

Tabs. 1-3 show the values of damping, stiffness and mass parameters in Harsha’s model and the proposed model, respectively.

### 2.2. Genetic Algorithm and Optimization

Today, the use of gradual evolution methods for solving optimization problems has been a growing trend. Evolution algorithms have formed according to the simulation of natural evolution. The natural evolution hypothesis is one of the accepted hypotheses by biologists. The genetic algorithm has found a broad application as the most gradual evolution algorithm in unknown search spaces. Evolutionary algorithms are search and optimization methods that are formed based on gradual evolution.\textsuperscript{29}

Genetic algorithms are search algorithms that use the natural genetic principles to solve optimization problems. The preliminary genetic algorithm, which was first proposed by Holland in 1975\textsuperscript{30} and later by Goldberg and others has evolved. It
Table 4. Optimization variables for the proposed model (with headrest).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of variables Variables</td>
<td>16</td>
</tr>
<tr>
<td>Lower bound</td>
<td>${k_{1v}, k_{1h}, k_{2v}, k_{2h}, k_{d}, k_{d_h}, k_{d_v}, c_{1v}, c_{1h}, c_{2v}, c_{2h}, c_{4v}, c_{4h}, c_{5v}, c_{5h}}$</td>
</tr>
<tr>
<td>Upper bound</td>
<td>${1600(N/m) 1(N/m) 1(N/m) 15162(N/m) 90(N/m) 10(N/m) 1(N/m) 3(N/m) 400000(N/m) 150(N/m) 150000(N/m) 400000(N/m) 33(N/m) 15(N/m) 30(N/m) 50(N/m)}$</td>
</tr>
</tbody>
</table>

Table 5. Optimization properties for the base model (without headrest).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of variables Variables</td>
<td>12</td>
</tr>
<tr>
<td>Lower bound</td>
<td>${k_{1v}, k_{1h}, k_{2v}, k_{2h}, k_{d}, k_{d_h}, c_{1v}, c_{1h}, c_{2v}, c_{2h}, c_{4v}, c_{4h}, c_{5v}, c_{5h}}$</td>
</tr>
<tr>
<td>Upper bound</td>
<td>${160000(N/m) 1(N/m) 1516200(N/m) 9000(N/m) 172000(N/m) 230000(N/m) 150000(N/m) 400000(N/m) 1050(Ns/m) 140(Ns/m) 470(Ns/m) 150(Ns/m) 3300(Ns/m) 1500(Ns/m) 3000(Ns/m) 5000(Ns/m)}$</td>
</tr>
</tbody>
</table>

coefficients.

$$y = w_1 \text{RMS}(STH_x) + w_2 \text{RMS}(STH); \quad (7)$$

$$y = w_1 \text{max}(STH_x) + w_2 \text{max}(STH); \quad (8)$$

where $w_1$ and $w_2$ were transferability weighting coefficients of vertical and horizontal vibrations so that their sum was equal to one and each of them was smaller than one. Given that the human body has the highest vibration sensitivity in the frequency range of 4–8 Hz in vertical vibrations and the frequency range of 1–2 Hz in horizontal vibrations,\textsuperscript{31} in calculating all of these functions, the filtered value of these vibrations was measured in the listed intervals.

3. CHARTS AND RESULTS

According to the mentioned objective functions, vibration optimization was performed for Harsha’s model and the proposed model. In the above equations, $w_1$ and $w_2$ were considered equal to 0.5. If the objective function is Eq. (7), Fig. 3 compares the transferability of horizontal and vertical vibrations in Harsha’s model and the optimized one.

As it is shown in Fig. 3, the maximum transferability of horizontal and vertical vibrations in Harsha’s optimized model is reduced significantly compared to the Harsha’s model. In Fig. 4, vibration transferability is optimized in the proposed model, and also reduction of maximum vibration is quite evident in that. In Fig. 5, the optimal amount of horizontal and vertical vibrations in Harsha’s model and the proposed model is compared. Based on the results of the graph, it is found that
Figure 3. Optimization of horizontal and vertical vibrations transmission in Harsha’s model with Eq. (7).

Figure 4. Optimization of horizontal and vertical vibrations transmission in the presented model with Eq. (7).

Figure 5. Compare optimization of horizontal and vertical vibrations transmission in the presented model by Harsha’s model with Eq. (7).

Table 7. The primary and optimal of seat stiffness and damping parameters for Harsha’s model in Eq. (7).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Harsha’s model</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{1v}$</td>
<td>N/m</td>
<td>1600</td>
<td>116486</td>
</tr>
<tr>
<td>$k_{1h}$</td>
<td>N/m</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>$k_{2v}$</td>
<td>N/m</td>
<td>151625</td>
<td>115930</td>
</tr>
<tr>
<td>$k_{2h}$</td>
<td>N/m</td>
<td>905</td>
<td>4184</td>
</tr>
<tr>
<td>$k_{4v}$</td>
<td>N/m</td>
<td>17200</td>
<td>54390</td>
</tr>
<tr>
<td>$k_{4h}$</td>
<td>N/m</td>
<td>2300</td>
<td>10543</td>
</tr>
<tr>
<td>$c_{1v}$</td>
<td>Ns/m</td>
<td>104.35</td>
<td>396</td>
</tr>
<tr>
<td>$c_{1h}$</td>
<td>Ns/m</td>
<td>14</td>
<td>36</td>
</tr>
<tr>
<td>$c_{2v}$</td>
<td>Ns/m</td>
<td>47</td>
<td>323</td>
</tr>
<tr>
<td>$c_{2h}$</td>
<td>Ns/m</td>
<td>15</td>
<td>123</td>
</tr>
<tr>
<td>$c_{4v}$</td>
<td>Ns/m</td>
<td>324.5</td>
<td>2751</td>
</tr>
<tr>
<td>$c_{4h}$</td>
<td>Ns/m</td>
<td>154</td>
<td>1317</td>
</tr>
</tbody>
</table>

As seen in Fig 6, the maximum value of the vertical and horizontal vibration transferability in the optimized Harsha model has been reduced compared to the original Harsha model. In Fig 7, the vibration transferability in the proposed model and its optimized model is observed. It is evident that the value of the maximum vibration transferability parameter in both horizontal and vertical directions is reduced. In Fig. 8, the optimum value of horizontal and vertical vibrations is compared with both Harsha’s model and the proposed model concerning the new objective function. According to the results of Fig. 8, it is found that the root means square for vertical and horizontal vibrations transferability in Harsha’s optimized model are respectively 0.89 and 0.78. The values of these parameters, which have been reduced, for the optimized proposed model are respectively 0.93 and 0.37. Also, the maximum vertical and horizontal vibrations transferability for Harsha’s optimized model are respectively 1.79 and 1.20, while for the optimized proposed model have been calculated respectively 1.29 and 0.70.

In this section, both the basic and optimized values stiffness and damping parameters for the Harsha model and the proposed model were provided respectively. These values had been obtained for the objective function of Eq. (8). Tables 9 and 10 show the value of these parameters for the Harsha’s
Table 8. The basic and optimal of seat stiffness and damping parameters for presented model in Eq. (7).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Presented model</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{1v}$</td>
<td>N/m</td>
<td>1600</td>
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</tr>
<tr>
<td>$k_{1h}$</td>
<td>N/m</td>
<td>15</td>
<td>103</td>
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<td>$k_{2v}$</td>
<td>N/m</td>
<td>151625</td>
<td>686639</td>
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<tr>
<td>$k_{2h}$</td>
<td>N/m</td>
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<td>5240</td>
</tr>
<tr>
<td>$k_{4v}$</td>
<td>N/m</td>
<td>17200</td>
<td>14991</td>
</tr>
<tr>
<td>$k_{4h}$</td>
<td>N/m</td>
<td>2300</td>
<td>230</td>
</tr>
<tr>
<td>$k_{5v}$</td>
<td>N/m</td>
<td>15000</td>
<td>2661</td>
</tr>
<tr>
<td>$k_{5h}$</td>
<td>N/m</td>
<td>4000</td>
<td>418</td>
</tr>
<tr>
<td>$c_{1v}$</td>
<td>Ns/m</td>
<td>104.35</td>
<td>766</td>
</tr>
<tr>
<td>$c_{1h}$</td>
<td>Ns/m</td>
<td>14</td>
<td>89</td>
</tr>
<tr>
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<td>Ns/m</td>
<td>47</td>
<td>279</td>
</tr>
<tr>
<td>$c_{2h}$</td>
<td>Ns/m</td>
<td>15</td>
<td>57</td>
</tr>
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<td>$c_{4v}$</td>
<td>Ns/m</td>
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<td>2191</td>
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<tr>
<td>$c_{4h}$</td>
<td>Ns/m</td>
<td>154</td>
<td>16</td>
</tr>
<tr>
<td>$c_{5v}$</td>
<td>Ns/m</td>
<td>300</td>
<td>2726</td>
</tr>
<tr>
<td>$c_{5h}$</td>
<td>Ns/m</td>
<td>500</td>
<td>510</td>
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</table>

Table 9. The primary and optimal of seat stiffness and damping parameters for Harsha’s model in Eq. (8).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Harsha’s model</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{1v}$</td>
<td>N/m</td>
<td>1600</td>
<td>11701</td>
</tr>
<tr>
<td>$k_{1h}$</td>
<td>N/m</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>$k_{2v}$</td>
<td>N/m</td>
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<td>15412</td>
</tr>
<tr>
<td>$k_{2h}$</td>
<td>N/m</td>
<td>905</td>
<td>8083</td>
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<tr>
<td>$k_{4v}$</td>
<td>N/m</td>
<td>17200</td>
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<td>$k_{4h}$</td>
<td>N/m</td>
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<td>22970</td>
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<td>104.35</td>
<td>14</td>
</tr>
<tr>
<td>$c_{1h}$</td>
<td>Ns/m</td>
<td>14</td>
<td>2</td>
</tr>
<tr>
<td>$c_{2v}$</td>
<td>Ns/m</td>
<td>47</td>
<td>465</td>
</tr>
<tr>
<td>$c_{2h}$</td>
<td>Ns/m</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>$c_{4v}$</td>
<td>Ns/m</td>
<td>324.5</td>
<td>2050</td>
</tr>
<tr>
<td>$c_{4h}$</td>
<td>Ns/m</td>
<td>154</td>
<td>1478</td>
</tr>
</tbody>
</table>

Figure 6. Optimization of horizontal and vertical vibrations transmission in Harsha’s model with Eq. (8).

Figure 7. Optimization of horizontal and vertical vibrations transmission in the presented model with Eq. (8).

Table 11 shows the comparison of adding a headrest to different optimization scenarios. The first column shows the objective functions while two functions are mixed (vertical and horizontal directions) with two different weighting factor couples and other functions are clear (just vertical or just horizontal direction). The third column is the base model (without headrest), and the fourth column is the headrest equipped model. The results show the significant improvement in different objective functions, by adding the headrest.

4. CONCLUSION

In this article, the superiority of adding a headrest to the vehicle’s seat has been investigated to improve the driver comfort. For this purpose, a biomedical model of whole-body vibration together with the seat’s horizontal and vertical vibrations has been introduced to assess the damage caused by vibrations and optimize the vehicle’s seat parameters. This model is in the sitting posture which is the lumped parameter model, and it had ten degrees of freedom. In the basic model, the head has independently no degree of freedom and swings with the waist horizontally. In the new model, considering the backrest, the torsional movements of the head and neck are also considered. However, the base model has nine degrees of freedom and the headrest and horizontal force into the head are not modeled in it.

Also, with the definition of an objective function of transferability for head to seat vibrations and to use a genetic algorithm, seat parameters have been optimized. The presented results show that seat to head vibrations transferability in both horizontal and vertical direction has been improved by adding the headrest. According to the results in the previous section, these achievements can be concluded:

- The horizontal vibration transferability has been reduced up to 50% in comparison with the base model (without headrest), in different objective functions (RMS or maximum vibration transferability).
- The vertical vibration transferability has been reduced up to 50% in comparison with the base model (without headrest), in different objective functions (RMS or maximum vibration transferability).

In general, concerning the transferability reduction in both objective functions and horizontal and vertical directions, it can
be concluded that the proposed model is desirable for minimizing head injuries caused by vibrations and to optimize the design of headrest parameters. However, the headrest has better performance in the horizontal direction in comparison with the vertical direction.

REFERENCES

Table 11. The comparison of the headrest adding effect.

<table>
<thead>
<tr>
<th>Objective function</th>
<th>Vibration direction</th>
<th>Harsha model (WO headrest)</th>
<th>Proposed model (W headrest)</th>
<th>Improvement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 0.5 \cdot \text{RMS}(STH_x) + 0.5 \cdot \text{RMS}(STH)$</td>
<td>Mixed vertical and horizontal</td>
<td>0.51</td>
<td>0.49</td>
<td>3.9</td>
</tr>
<tr>
<td>$y = 0.3 \cdot \text{RMS}(STH_x) + 0.7 \cdot \text{RMS}(STH)$</td>
<td>Mixed vertical and horizontal</td>
<td>0.61</td>
<td>0.55</td>
<td>9.8</td>
</tr>
<tr>
<td>$y = \max(STH_x)$</td>
<td>Vertical</td>
<td>1.00</td>
<td>0.50</td>
<td>50</td>
</tr>
<tr>
<td>$y = \text{RMS}(STH_x)$</td>
<td>Horizontal</td>
<td>0.48</td>
<td>0.24</td>
<td>50</td>
</tr>
</tbody>
</table>


Equilibrium and Forced Vibration of an Axially Moving Belt with Belt-Pulley Contact Boundaries

Hu Ding

Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai, 200072.

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Axially moving materials have usually dealt with classic boundary conditions, i.e. zero boundaries, such as the simply supported and the fixed ends. In this paper, the dynamics responses of the axially moving belt with belt-pulley contact boundary conditions are studied for the first time. Therefore, due to the fact that non-homogeneous terms are included in the boundary conditions, the traditional generalized eigenvalue method is no longer applicable. In this work, the belt is numerically discretized by using the differential quadrature method (DQM). Iterative schemes are proposed for determining the equilibrium configuration. Harmonic inertia excitation is considered to be the vertical motion of the whole system. The steady-state responses of the forced vibration are also numerically solved by applying the DQM. The parametric effects on the equilibrium configuration and the steady-state response are investigated. The numerical investigations reveal that the radius of the support pulley has significant effects on both the equilibrium configuration and the transition phase of the transverse vibration of the axially moving belt under inertia excitation.

1. INTRODUCTION

The objective of this work was to investigate the vibration of an axially moving belt under belt-pulley contact boundary conditions. Axially moving belts are common constituent elements in many engineering systems such as magnetic tapes, textile fibers, and power transmission belts.1–5 One of the most important problems in these moving belts is the occurrence of large unwanted bending vibrations due to vary excitations, such as the effect of pulley eccentricity, multi-pulse excitation, a time varying velocity and a harmonic axial tension, and external force excitation.6–10 To ensure that these belts are operating under stable working conditions, it is imperative to understand the vibration characteristics of the moving belts.

Many studies on the free vibration and forced vibration response of axially moving materials can be found in the literature.11–13 Traditionally, these investigations were focused on the moving materials with classic boundary conditions and either simply supported or fixed ends.14–16 Wickert obtained the pattern of equilibrium for an axially moving beam with a simply supported end at supercritical transport speed.17 Sedighi and Eipakchi determined the natural frequency and critical speed of an axially moving beam.18 Ghayesh, Amabili and Farokhi conducted post-buckling analysis for an axially moving beam undergoing a transverse harmonic excitation.19 Yao and Zhang carried out the reliability and sensitivity analysis of an axially moving beam with simply supported boundary conditions.20 Based on a ’cantilever’ type boundary conditions, Kelleche, Tatar and Khemmoudj studied the relationship between the dissipation produced by the viscoelastic material and the transverse vibrations caused by the axial motion of the beam.21 All the research above on the vibration of axially moving systems assumes that the beam was simply supported. As a result, axially moving materials are considered to be free to bend at the boundary.

By considering the structures with the fixed ends, which were assumed to be unbendable at the border, Oz studied the vibrations of an axially moving beam with variable velocity.22 Yang and Chen investigated the free non-linear vibration and parametric excitation vibration of axially moving beams.23,24 Zhang et al. calculated the non-trivial static equilibrium and the steady-state response for moving beams in the supercritical transport speed range under clamped boundary conditions.25 In order to describe a more general boundary, Chen and Yang proposed hybrid supports.26 The boundary condition can describe the constraints between the two classical boundary conditions, and can degrade to the simply-supported boundary and the fixed boundary. Based on the hybrid supports, Ding and Chen studied the stability of axially accelerating viscoelastic beams.27 Yang and Yang presented an exact solution for the supercritical configurations of axially moving beams with fixed boundary conditions.28 More generally, Park and Chung presented a study on the dynamic analysis of an axially moving beam with intermediate spring supports.29 Bagdadi and Uslu proposed that axially moving beams have simple and clamped support conditions as a combination of ideal and non-ideal boundaries with a weighting factor.30 However, all of the above-mentioned studies on axially moving materials were for homogeneous boundary conditions. The study of homogeneous boundary conditions, to a certain extent, can explain the vibration of axially moving material. However, the non-homogeneous boundary conditions that are closer to the pulley-contact actual situation will have an effect on the vibration characteristics of the axially moving belt and are always unknown.

In light of the lack of research for the axially moving belts
under pulley-contact non-homogeneous boundary conditions, the present paper studied the effects of the non-homogeneous boundaries on the free and forced vibrations of the axially moving belt. The differential quadrature method was adopted to numerically solve the equilibrium configuration and the forced vibration responses. The effects of the radius of the support pulley on the static and dynamics of the bending vibrations of the axially moving belts were mainly investigated.

2. DYNAMIC MODEL

The mechanical dynamics model of an axially moving belt with length $l$ and belt-pulley contact boundary conditions is shown in Fig. 1, where the symbols $x$ and $t$ were, respectively, the axial and time coordinates. $r_1$ and $r_2$ were the radius of the left and right support pulleys, and $P_0$ were the moving speed and the initial static axial tension of the belt and were considered to be constant and uniform, respectively. The whole system was subjected to a vertical harmonic displacement excitation $B \cos(\Omega t)$, where $B$ and $\Omega$ were, respectively, the amplitude and frequency of the excitation.

By only considering the bending vibration of the belt described by the transverse displacement $w(x, t)$, the following governing equation of the transverse vibration of the moving belt was derived by using Newton’s second law of motion

$$\rho A\left(w_{tt} + 2cw_{xt} + c^2w_{xx}\right) + M_{xx}(x, t) - P_0 w_{xx} = B\rho A\Omega^2 \cos(\Omega t); \quad (1)$$

where $\rho$ and $M$, respectively, represented the density and the bending moment of the belt. $A$ was the cross-sectional area. Moreover, a comma preceding $x$ or $t$ denoted partial differentiation with respect to $x$ or $t$. The material of the viscoelastic belt was assumed to obey the Kelvin constitution relation. The linear moment-curvature relationship was adopted based on the Euler-Bernoulli theory

$$M(x, t) = \left(E + \eta \frac{\partial}{\partial t}\right) I w_{xx}; \quad (2)$$

where $E$ and $I$ were Young’s modulus and the area moment of inertial of the moving belt, $EI$ was used to account for the bending stiffness, respectively. Therefore, the governing equation of the bending vibration of the axially moving belt is derived as

$$\rho A\left(w_{tt} + 2cw_{xt} + c^2w_{xx}\right) - P_0 w_{xx} + \left(E + \eta \frac{\partial}{\partial t}\right) I w_{xxxx} = B\rho A\Omega^2 \cos(\Omega t); \quad (3)$$

with the boundary conditions as following

$$w(0, t) = 0, w(l, t) = 0, EI w_{xx}(0, t) = -EI/r_1, EI w_{xx}(l, t) = -EI/r_2. \quad (4)$$

One can find that there were non-homogeneous boundary conditions. The major goal of this work was to disclose the influence of these non-homogeneous boundary conditions on the vibration characteristics of the axially moving belt. By defining the following dimensionless variables and parameters

$$x \leftrightarrow \frac{x}{l}; \quad (5a)$$

$$w \leftrightarrow \frac{w}{l}; \quad (5b)$$

$$t \leftrightarrow \frac{t}{\sqrt{\frac{P_0}{\rho A l^2}}}; \quad (5c)$$

$$c \leftrightarrow \frac{c}{\sqrt{\frac{\rho A}{P_0}}}; \quad (5d)$$

$$\alpha = \frac{l_\eta}{l^3 \sqrt{\rho A P_0}}; \quad (5e)$$

$$k_f = \sqrt{\frac{E l}{P_0 l^2}}; \quad (5f)$$

$$b = \frac{B}{l}; \quad (5g)$$

$$\omega = \Omega \sqrt{\frac{\rho A l^2}{P_0}}; \quad (5h)$$

the dimensionless equation was derived as

$$w_{tt} + 2cw_{xt} + c^2w_{xx} + k_f^2 w_{xxxx} + \alpha w_{xxxxx} = b\omega^2 \cos(\omega t); \quad (6)$$

with the dimensionless boundary conditions as following

$$w(0, t) = 0; \quad (7a)$$

$$w(l, t) = 0; \quad (7b)$$

$$w_{xx}(0, t) = -1/r_1; \quad (7c)$$

$$w_{xx}(l, t) = -1/r_2; \quad (7d)$$

where the dimensionless parameter, $k_f^2$ denoted the bending stiffness of the moving belt, $\alpha$ accounted for the dynamic viscosity.
3. THE EQUILIBRIUM CONFIGURATION

In order to study the effect of the non-homogeneous boundary conditions, the non-trivial equilibrium configuration was determined first. By disregarding all time-related items, the equilibrium configuration \( \ddot{w}(x) \) of equation (6) satisfied

\[
(c^2 - 1) \dddot{w} + k_f^2 \ddot{w}'' = 0. 
\]

(8)

Correspondingly, the boundary conditions of the equilibrium equation were described as

\[
\begin{align*}
\dot{w}(0) &= 0; \\
\dot{w}(1) &= 0; \\
\ddot{w}''(0) &= -l/r_1; \\
\ddot{w}''(1) &= -l/r_2.
\end{align*}
\]

(9a) - (9d)

The equilibrium solution was solved by using the differential quadrature method (DQM). The following algebraic equations were obtained by numerical discretization

\[
\begin{align*}
\ddot{w}_1 &= 0; \\
\sum_{k=1}^{N} A_{2k}^{(2)} \ddot{w}_k &= -\frac{l}{r_1}; \\
(c^2 - 1) \sum_{k=1}^{N} A_{jk}^{(2)} \ddot{w}_k + k_f^2 \sum_{k=1}^{N} A_{jk}^{(4)} \ddot{w}_k &= 0, \\
(j &= 3,4,\ldots,N-2); \\
\sum_{k=1}^{N} A_{(N-1)k}^{(2)} \ddot{w}_k &= -\frac{l}{r_2}; \\
\ddot{w}_N &= 0;
\end{align*}
\]

(10a) - (10e)

where the first-order differential quadrature weighting coefficients were calculated as

\[
A_{jk}^{(1)} = \begin{cases} \\
\prod_{m=1 \atop m \neq j}^{N} \frac{(x_j - x_m)}{x_j - x_m} & \text{for } k \neq j \\
\prod_{m=1 \atop m \neq j}^{N} \frac{1}{x_j - x_m} & \text{for } k = j \\
\sum_{k=1 \atop k \neq j}^{N} \frac{1}{x_j - x_k} & \text{for } k = j \\
\end{cases}
\]

(11)

where \( x_1 - x_N \) were the discrete sampling points. The second and higher order derivatives weighting coefficients were calculated respectively by recurrence relationship

\[
A_{jk}^{(2)} = \sum_{m=1}^{N} A_{jm}^{(1)} A_{mk}^{(1)} \text{ for } j,k = 1,2,\ldots,N;
\]

(12)

and

\[
A_{jk}^{(r)} = \sum_{m=1}^{N} A_{jm}^{(1)} A_{mk}^{(r-1)} = \sum_{m=1}^{N} A_{jm}^{(r-1)} A_{mk}^{(1)} \text{ for } r = 3,4,\ldots,N \text{ and } j,k = 1,2,\ldots,N.
\]

(13)

For a given set of initial iterative values, the algebraic equations was solved by using the following iterative schemes

\[
\ddot{w}_1 = 0; \\
\ddot{w}_2 = -\frac{l}{r_1} - \sum_{k=1 \atop k \neq 2}^{N} A_{2k}^{(2)} \ddot{w}_k; \\
\ddot{w}_j = \frac{(c^2 - 1) \sum_{k=1 \atop k \neq j}^{N} A_{jk}^{(2)} \ddot{w}_k + k_f^2 \sum_{k=1 \atop k \neq j}^{N} A_{jk}^{(4)} \ddot{w}_k}{(c^2 - 1) A_{jj}^{(2)} + k_f^2 A_{jj}^{(4)}} \ddot{w}_k, \\
(j &= 3,4,\ldots,N-2); \\
\ddot{w}_{N-1} = -\frac{l}{r_2} - \sum_{k=1 \atop k \neq N-1}^{N} A_{N-1,k}^{(2)} \ddot{w}_k; \\
\ddot{w}_N &= 0.
\]

(14a) - (14e)

The physical and geometrical properties of the example moving belt are listed in Table 1. The initial iterative values were set as

\[
\ddot{w}_j = 0.0001 \sin(\pi x_j), \quad (j = 1,2,\ldots,N).
\]

(15)

Table 1. Properties of the axially moving belt with contact boundaries.

<table>
<thead>
<tr>
<th>Item</th>
<th>Notation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of belt</td>
<td>l</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>E</td>
<td>2 \times 10^5 \text{ N/m}^2</td>
</tr>
<tr>
<td>Width</td>
<td>b</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Height</td>
<td>h</td>
<td>0.01 m</td>
</tr>
<tr>
<td>Density</td>
<td>\rho</td>
<td>1200 \text{ kg/m}^3</td>
</tr>
<tr>
<td>Static tension</td>
<td>P_y</td>
<td>400 N</td>
</tr>
<tr>
<td>Viscous damping</td>
<td>\alpha</td>
<td>5 \times 10^{-5} \text{ N s/m}</td>
</tr>
<tr>
<td>Radius of left pulley</td>
<td>r_l</td>
<td>0.03 m</td>
</tr>
<tr>
<td>Radius of right pulley</td>
<td>r_r</td>
<td>0.05 m</td>
</tr>
<tr>
<td>Axial speed</td>
<td>c</td>
<td>10 \text{ m/s}</td>
</tr>
<tr>
<td>Amplitude of excitation</td>
<td>B</td>
<td>0.0002 m</td>
</tr>
</tbody>
</table>

In this paper, the number of iterations was always set as 100000. Figure 2 presents the comparison of the equilibrium configurations between different discrete sampling points. Figures 2(a) and 2(b), respectively, show the comparison with the different scale and the same scale. The numerical results in Fig. 2 illustrate that the equilibria calculated with different sampling numbers are completely coincidental. Although Fig. 2(a) shows that the support pulley significantly affects the equilibrium, Fig. 2(b) demonstrates that the displacement of the equilibrium is rather small. However, the degree of bending of the belt could be great in the vicinity of the boundaries.

Figures 3 and 4 respectively present the effects of the radius of the support pulley and axially moving speed on the equilibrium configuration. Figure 3 clearly shows that the equilibrium displacement increases with the decreasing radius of...
moving belt on the equilibrium configuration are described in Figs. 5 and 6, respectively. Interestingly, Fig. 5 demonstrates that the displacement of the equilibrium increases with the increasing belt stiffness. The numerical results in Fig. 6 shows that a strong initial tension can make the static deformation smaller.

4. FORCED VIBRATION

The governing partial differential equation of the bending vibration of the axially moving belt with non-homogeneous boundary conditions can be numerical solved. A series of ordinary differential equations can be obtained by using the differential quadrature method (see Eq. (16), on top of the next site).

By discretizing the temporal variables and setting the fixed temporal step as $5 \times 10^{-5}$, $u(x_j, t)$ ($j = 2, 3, \ldots, N - 1$) was numerically solved using the four-order Runge-Kutta method for given parameter values. Furthermore, the initial values for all numerical examples are set as

$$u(x, 0) = 0;$$

$$u_{,t} (x, 0) = 0.0001.$$
For an odd $N$, $u(x_{(N+1)/2}, t)$ was the transverse displacement of the midpoint of the axially moving belt. In the following numerical examples, the number of sampling points $N$ is set as 15.

The time histories of the bending vibration of the midpoint of the axially moving belt were presented in Fig. 7 with the excitation frequency $\Omega = 400$ Hz. In Fig. 7(a) and Fig. 7(b), the numerical results describe that a response depending on the initial conditions (19) occurs at the beginning phase, then the transition phase appears, and finally a steady-state response phase forms. Since the axial speed affects the equilibrium configuration and the natural frequencies, Fig. 7(a) shows that the equilibrium position and the steady-state response amplitude of the bending vibration both vary with the axial speed. As shown in Fig. 7(b), the radius of the pulley only changes the equilibrium position. However, the amplitude of the steady-state response does not change with the radius of the support pulley. Nonetheless, the support pulley significantly affects the transition phase of the transverse vibration of the axially moving belt.

In order to determine the stable steady-state response amplitude, the time histories of the axially moving belt were simulated in the time interval of $[0 \text{ s}, 7 \text{ s}]$. For ensuring that transient stage has died away, the time series of the moving belt in the time interval of $[0 \text{ s}, 6.5 \text{ s}]$ were discarded. The amplitudes of the steady-state response of the belt were determined by recording the local maximums of $w_{max} = w[(N + 1)/2, t]$.
The effects of the axial speed and the height of the cross section of the belt on the amplitude-frequency relationship are presented in Figs. 10 and 11, respectively. Figures 10 and 11 illustrate that the increase in the axial speed and the height causes the resonant area to move to the low frequency region. However, the resonance intensity is weakened at the same time. In light of this, the larger cross-sectional height corresponds to a smaller resonant frequency. This is different from the static belt. Figures 12 and 13 respectively show the influences of Young’s modulus and the initial tension of the axially moving belt on the amplitude-frequency relationship of the transverse bending vibration. As shown in Figs. 12 and 13, the frequency of resonance and the intensity of resonance both increase as the stiffness and the initial tension of the moving belt increase.

5. CONCLUSIONS

For a wide range of belt-pulley coupled dynamic systems, the dynamics of axially moving belts with belt-pulley contact boundary conditions are rarely involved because it is difficult to deal with the non-homogeneous terms. The goal of this work is to study the bending vibration of the axially moving belt with pulley support boundary conditions. The differential quadrature method is applied to discretize the moving belt. Then, the equilibrium configuration caused by the non-homogeneous boundary conditions is numerically solved by proposing an iterative scheme. The forced vibration response is investigated by the DQM. The numerical results show that the equilibrium position and the transition phase of the bending vibration of the moving belt are significantly affected by the support pulleys. Moreover, this work interestingly finds that the larger axial speed or cross-sectional height corresponds to a smaller resonant frequency and a weaker resonant intensity. Furthermore, the larger Young’s modulus, or initial tension, draws a stronger resonance intensity.
ACKNOWLEDGEMENTS

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H. Ding, et al.: EQUILIBRIUM AND FORCED VIBRATION OF AN AXIALLY MOVING BELT WITH BELT-PULLEY CONTACT BOUNDARIES


Application of a Bandpass Filter for the Active Vibration Control of High-Speed Rotors

Miroslav Pawlenka, Miroslav Mahdal and Jiří Tůma
Department of Control Systems and Instrumentation at VŠB-TU Ostrava, Czech Republic.

Adam Bureček
Department of Hydromechanics and Hydraulic Equipment at VŠB-TU Ostrava, Czech Republic.

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This study concerns the active vibration control of journal bearings, which are also known as sliding bearings. The control system contains a non-rotating loose bushing, the position of which is controlled by piezoelectric actuators. For governing the respective orthogonal direction of the journal motion, the control algorithm realizes a proportional controller in parallel with a bandpass filter of the IIR type. The bandpass filter is of the second order and its centre frequency is self-tuned to be the same as the whirl frequency that results from the instability of the bearing journal due to the oil film. The objective of active vibration control is to achieve the highest operational speed of the journal bearing at which the motion of the rotor is stable. The control algorithm for the active vibration control is implemented in Simulink and realized in a dSPACE control system.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_n$</td>
<td>Denominator determinants</td>
</tr>
<tr>
<td>CPB</td>
<td>Constant Percentage Band</td>
</tr>
<tr>
<td>$f_{\text{rot}}$</td>
<td>Rotor rotational frequency</td>
</tr>
<tr>
<td>$f_{\text{whirl}}$</td>
<td>Frequency of whirl instability</td>
</tr>
<tr>
<td>$G(s)$</td>
<td>Transfer function of the filter</td>
</tr>
<tr>
<td>$G(j\omega)$</td>
<td>Frequency transfer function of the filter</td>
</tr>
<tr>
<td>IIR</td>
<td>Infinite Impulse Response</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$n$-order integral for calculating the variance</td>
</tr>
<tr>
<td>$j$</td>
<td>Imaginary unit</td>
</tr>
<tr>
<td>$K$</td>
<td>Gain factor</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
</tr>
<tr>
<td>$N_n$</td>
<td>Numerator determinants</td>
</tr>
<tr>
<td>rpm</td>
<td>Revolutions per minute</td>
</tr>
<tr>
<td>$s$</td>
<td>Complex $s$-plane</td>
</tr>
<tr>
<td>$S_{xx}(\omega)$</td>
<td>Power spectral density</td>
</tr>
<tr>
<td>$T_0$</td>
<td>Time constant</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Damping ratio</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular frequency</td>
</tr>
<tr>
<td>$\sigma_y^2$</td>
<td>Variance at the output</td>
</tr>
<tr>
<td>$\sigma_z^2$</td>
<td>Variance at the input</td>
</tr>
</tbody>
</table>

1. INTRODUCTION

Currently, hydrodynamic sliding bearings are highly advantageous owing to their high radial load and operating loads at high shaft rotation speeds. At high speeds, however, there is an adverse phenomenon. Namely, the excitation of shaft vibrations called an oil whirl. This vibration occurs when the rotational speed exceeds a threshold, which is related to the radial clearance and lubricating oil viscosity. At present, the massive deployment of high-speed hydrodynamic bearings is partially limited by the aforementioned unfavourable vibration phenomenon that can vibrate the entire machine and lead to damage. The vibrations produced when the threshold speed is exceeded due to the oil film properties, as evidenced by theoretical analysis.\(^1\)\(^2\) There is a passive approach to suppress the resulting vibrations, such as a structural change of the bearing bushing, e.g., to an elliptical or a lemon form, by creating segments or inserting grooves.

In this study, the focus is on realizing vibration suppression with the use of an active vibration control system, the principle of which is a non-rotating loose bushing, whose position is controlled by piezoelectric actuators also called piezo actuators shortly. For the tests, a test stand was designed to achieve maximum shaft speeds of up to 24 000 rpm. Throughout this range, it was possible to monitor the occurrence of shaft instability by measuring the movements of the bearing journal. The objective of the entire research was to suppress the shaft vibrations and increase the boundary of instability to a higher shaft rotation speed by using a suitable control algorithm and piezoelectric actuators. The entire control algorithm being programmed in the dSPACE control system as a real-time system made it necessary to resolve several other sub-tasks. One of these tasks was to apply an automatically tuneable bandpass second-order filter of the IIR type, which should improve the efficiency of the active vibration control system. The proportional feedback controller provides attenuation of the disturbances over a wide frequency range while the bandpass filter, in parallel with the proportional feedback controller, allows a selective increase of the proportional gain of the controller in the narrow frequency range where the bearing journal tends to vibrate.
2. PROBLEM DEFINITION

At present, efforts are being made for extending the life of machines and improving their operational efficiency. This requirement is also closely related to active vibration control. Active vibration suppression can lead to longer service life, as well as higher machine rotational speeds. From the industrial perspective, active vibration control is desirable because vibrations are particularly problematic for machine tools, etc. There are requirements for high-precision machining and high-speed cutting.

We now highlight some articles reported worldwide that address this issue. This study described a test device that tested vibration damping using magnetic actuators, see Fig. 1. The magnetic bearing was a part of the Bently Nevada Rotorkit of the RK 4 type. The test device for testing the sliding bearings with active vibration control using piezo actuators is shown in Fig. 2. The literature describes its design. The bearing journal rotated at approximately 5000 rpm. The active vibration control system used a feedback signal, which was the velocity or acceleration of the bearing housing. The bearing was intended to be used in a flywheel for energy storage to achieve long-term service life. The preferred means for rotor control were magnetorheological liquids. However, the delayed response of the liquid viscosity to the magnetic field change didn’t allow this method to be used for the closed-loop control of high-speed rotors. The first paper dealing with actively controlled hydrodynamic slide bearings was published in 2002. The vibration damping of hydrodynamic bearings using magnetostrictive actuators, but the usability of the test device was only for low revolution speeds. The article solved the theoretical problem of the stability of the control loop but does not describe a suitable actuator. By the 1990s, NASA had employed piezoelectric pushers (actuators) for active vibration damping. The shaft, in this case, is supported by ball bearings. Development of piezo actuators continues. Literature that focuses on the active vibration control of rotors is reported in publications.

The research of the active vibration control of journal bearings that use piezo actuators began at the VSB Technical University of Ostrava and Prague’s Techlab Company in 2007. We created an original test facility that uses bearings very similar to their industrial design. The test rig allowed for the testing and verification of control algorithms to efficiently suppress the rotor vibrations due to the rotor motion instability. Piezo actuators were used for the positioning of non-rotating loose bearing bushing.

3. DESCRIPTION OF TEST STAND

The test set-up was composed of a shaft which was supported by two sliding radial hydrodynamic bearings, and it enabled the monitoring of rotor motion and its control as is shown in Fig. 3. The span of bearing pedestals was 200 mm, the bearing diameter was 30 mm, and the length to diameter ratio of the journal bearings was equal to 0.77. For the results presented here, the radial clearance was 55 μm. In operation, the lubricating oil entered under pressure into the gap between the bearing bushing and the journal. The oil inlet was located...
Measurement of the shaft position relative to the bearing body was carried out in two perpendicular directions rotated by 90 degrees from each other which are also perpendicular to the axis of the shaft. Proximity probes are located near each of two bearings.

The bearing bushing was shifted in the radial direction by the linear piezo actuators of the P-844.60 type. The P-884.60 was a product of the Physik Instrumente Company. The piezo actuator required a low voltage amplifier with the 120 V peak value at the output. The piezo actuator travel range was 90 µm, the pushing force was 3000 N, and the pulling force was 700 N. Piezo actuator holders were designed to eliminate their bending and torsional load. The design of the holder is available in a paper.\(^\text{15}\)

The three-phase induction motor drove the rotor. The frequency inverter of the Commander SKA1200075 (Control Techniques) type powered the driving motor. The maximum frequency of the inverter was 400 Hz. Thus, the maximum motor rotational speed was 24 000 rpm. The power of the induction motor of the ATAS FT4C52G type was 500 W. The power of the frequency inverter was 750 W. The laser speed sensor (Tacho) measured up to 250 000 rpm, which suited the specified speed range. The Kalman filter reduced possible small errors of the rotational speed measurement.\(^\text{16–18}\)

As it was mentioned, the position of the journal was measured using proximity sensors. The principle of the proximity sensors was based on the capacitive sensors of the capaNCDT CS05 type originated from the MICRO-EPSILON company, see Fig. 4.\(^\text{19}\) These sensors had a measuring range of 0.5 mm. The capaNCDT system was based on the principle of the parallel plate capacitor. Changing the distance of the two plates determined the capacitance change of the capacitor. For conductive materials, the tip of the sensor and the shaft surface formed the two electrodes. This theoretical principle was realized almost ideally in practice by designing the sensors as guard ring capacitors. In this case, it dealt with two capacitors. The conductive shaft surface connected these two capacitors in series. The second electrode of both capacitors was connected to a control unit that contained a demodulator. The advantage of this distance measurement solution was that no target grounding was required. If we measured by two sensors simultaneously, both the two control units must be synchronized. The measurement error of the capacitance sensors is less than 1 µm.

The entire layout of the individual elements of the bearing housing body, such as the sensors, the actuators, the bearing bushing and the journal, can be seen in Fig. 5. Active vibration control of the journal bearing enabled the movable bearing bushing. As mentioned above, the bushing did not rotate, and the piezo actuators provided the bushing movement in two radial directions.

The desired journal position in the horizontal and vertical directions in a plane perpendicular to the axis of the shaft was compared with the actual position. The calculated control error was transformed by the controller into magnitudes of two manipulated variables, which were the voltages for the piezo actuator amplifiers. All calculations were performed in the dSPACE signal processor. Amplified voltages govern the piezo actuator positions.
The gain at the centre frequency of the dependent on the damping ratio was zero. The width of the permeable frequency band was passed through the filter in continuous time, see Fig. 7. Now, we introduced white noise as an input signal and to calculate the variance of the filter’s output signal, i.e., after the signal has passed through the filter in continuous time, see Fig. 7. Now, we will present the derivation and calculation of the integral for calculating the variance at the output \( \sigma_y^2 \), which was based on the literature:

\[
\sigma_y^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(j\omega)G(-j\omega)S_{xx}(\omega)d\omega; \tag{2}
\]

where \( G(j\omega) \) was the cross-spectral density and \( S_{xx}(\omega) \) was the power spectral density for white noise and was equal to \( \sigma_x^2 \). Therefore, it was necessary to determine the ratio of the variation in the output to that in the input.

The filter coefficients in the numerator were denoted as \( B(s) \) and in the denominator as \( A(s) \). Then, we formulated

\[
A(s) = T_0^2 s^2 + 2\xi T_0 s + 1 = a_0 s^2 + a_1 s + a_2; \\
B(s) = KT_0 s = b_0 s; \\
a_0 = T_0^2, \quad a_1 = 2T_0, \quad a_2 = 1, \quad b_0 = KT_0. \tag{3}
\]

The substitution of \( s = j\omega \) was introduced; the transfer function of the filter had the form

\[
G(j\omega) = \frac{jKT_0 \omega}{-T_0^2 \omega^2 + j2\xi T_0 \omega + 1} \\
\omega = \frac{1}{T_0}, \quad G\left(j\frac{1}{T_0}\right) = \frac{K}{2\xi}. \tag{4}
\]

The integral for the variance calculation changed after the substitution to

\[
\sigma_y^2 = \sigma_x^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(s)G(-s)ds. \tag{5}
\]

The general formula for calculating the integral in continuous time was

\[
I_n = \frac{1}{2\pi j} \int_{-\infty}^{+\infty} \frac{C(s)}{A(s)A(-s)} ds = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{C(j\omega)}{A(j\omega)A(-j\omega)} d\omega. \tag{6}
\]

The numerator of the integrated function was calculated as follows

\[
C(s) = B(s)B(-s) = \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} b_i b_k s^{i+k} (-1)^k. \tag{7}
\]

The filter being second order, \( n = 2 \), so that

\[
C(s) = \sum_{i=0}^{2-1} \sum_{k=0}^{2-1} b_i b_k s^{i+k} (-1)^k = KT_0 s (-KT_0 s); \\
C(s) = -K^2 T_0^2 s^2 = c_0 s^2, \quad c_0 = -K^2 T_0^2, \quad c_1 = 0. \tag{8}
\]

![Figure 6. Bode plot of the bandpass filters frequency function \( \xi = 0.05 \) (left column) and \( \xi = 0.5 \) (right column).](image)

**4. BANDPASS FILTER IN ACTIVE VIBRATION CONTROL**

The preceding chapter contains a detailed description of the test rig. Active vibration control has been tested in the past as well.\(^{14,15}\) Improving the control algorithm has become the focus of doctoral dissertations that discuss the improvement of the signal-to-noise ratio of the feedback signal. The main interest was to prove the theoretical assumptions experimentally.\(^{20}\)

A bandpass filter belongs to the class of digital filters. Its upper and lower cut-off frequencies limit the bandwidth of a bandpass filter. For the control of active vibration damping, a second-order bandpass filter was used to avoid the unnecessarily long phase delays in the feedback loop. A first order bandwidth filter cannot be designed, and a higher order implies a longer unwanted delay. The Bode plot of the second-order bandpass filter is shown in Fig. 6. When we adapted the filter parameters, the second-order bandpass filter was also tuneable. The purpose of the bandpass filter was to track a disturbance whose frequency changed proportionally to the rotational speed of the rotor.

The second-order bandpass filter has the transfer function

\[
G(s) = \frac{Y(s)}{X(s)} = \frac{T_0 s}{T_0 s^2 + 2\xi T_0 s + 1}; \\
s = j\omega, \quad \omega = \frac{1}{T_0}, \quad \omega = 2\pi f_{\text{whir}}. \tag{1}
\]

The frequency function of the bandpass filter to gain \( K = 1 \) and time constant \( T_0 = 1 \), and two different damping ratios \( \xi \) are plotted in Fig. 6. The gain at the centre frequency of the bandpass filter, at the angular frequency \( \omega = 1/T \) was equal to 1. For angular frequency \( \omega = 0 \) and \( \omega \to \infty \), the gain was zero. The width of the permeable frequency band was dependent on the damping ratio \( \xi \).

The next step in verifying the function of this filter was to introduce white noise as an input signal and to calculate the variance of the filter’s output signal, i.e., after the signal has passed through the filter in continuous time, see Fig. 7.
The result of the calculation of the integral Eq. (6) was the following formula

$$I_n = \frac{(-1)^{n+1} N_n}{2D_n};$$

(9)

where the denominator and numerator were determinants

$$D_n = \begin{vmatrix} d_{00} & d_{01} & \cdots & d_{0n} \\ d_{10} & d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & \cdots & d_{nn} \end{vmatrix}, \quad D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix};$$

$$N_n = \begin{vmatrix} c_0 & d_{12} & \cdots & d_{1n} \\ c_1 & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & d_{n2} & \cdots & d_{nn} \end{vmatrix}, \quad N_2 = \begin{vmatrix} c_0 & d_{12} \\ c_1 & d_{22} \end{vmatrix};$$

(10)

with individual elements

$$d_{mr} = \begin{cases} a_{2m-r}; & 0 \leq 2m - r \leq n \\ 0; & 2m - r < 0, \; 2m - r > n \end{cases};$$

$$d_{11} = \begin{cases} a_{2-1}; & 0 \leq 2 - 1 \leq 2 \\ 0; & 2 - 1 < 0, \; 2 - 1 > 2 \end{cases} = a_4;$$

$$d_{22} = \begin{cases} a_{4-2}; & 0 \leq 4 - 2 \leq 2 \\ 0; & 4 - 2 < 0, \; 4 - 2 > 2 \end{cases} = a_2;$$

$$d_{21} = \begin{cases} a_{4-1}; & 0 \leq 4 - 1 \leq 2 \\ 0; & 4 - 1 < 0, \; 4 - 1 > 2 \end{cases} = 0;$$

$$d_{12} = \begin{cases} a_{2-2}; & 0 \leq 2 - 2 \leq 2 \\ 0; & 2 - 2 < 0, \; 2 - 2 > 2 \end{cases} = a_0. \quad (11)$$

After being assigned to the determinants

$$D_2 = \begin{vmatrix} 2\xi T_0 & T_0^2 \\ 0 & 1 \end{vmatrix} = 2\xi T_0;$$

$$N_2 = \begin{vmatrix} -K^2 T_0^2 & T_0^2 \\ 0 & 1 \end{vmatrix} = -K^2 T_0^2. \quad (12)$$

Now, these expressions were used in a relationship for calculating the integral, which may be considered as a bandwidth ratio given by the quotient between the variances of the filter’s output and input signals, according to:

$$I_z = \frac{\sigma_y^2}{\sigma_z^2} = \frac{K^2}{4\xi T_0}. \quad (13)$$

The maximum bandpass filter gain at resonance was as follows:

$$G\left(\frac{1}{T_0}\right) = \frac{jK}{-1 + j2\xi + 1} = \frac{K}{2\xi}. \quad (14)$$

If a bandpass filter was not used, it would be necessary to set the gain for the proportional controller to compensate for the bandpass filter. The variance of the input noise at the output of the proportional feedback increased \((K/2\xi)^2\) times. With the bandpass filter, it increased only \(K^2/4\xi T_0\) times. The ratio of both the ratios is equal to \(\xi/T_0\). The noise power at the bandpass filter output relating to the noise power at the input was limited by \(\xi/T_0\) times. The bandpass filter improves the signal to noise ratio.

The filter effect was such that the other noise frequency of the filter tuning frequency did not reach the manipulated variable. Therefore, the operation is expected to be quieter.

The bandwidth depended on the damping ratio \(\xi\). Therefore, we plotted the dependence of the variance ratio on damping ratio \(\xi\).

To plot the ratio of the variances in Fig. 8, the values of \(K/T_0\) were set to 1. These values affected the amplitude of the filtered signal.

5. APPLICATION OF THE FILTER

Rotor instability of the whirl type referred to a phenomenon in which the journal axis circulates within the bearing bushing at a reduced speed relative to the rotor rotational speed \(f_{\text{rotor}}\). The frequency of the mentioned circulation was named \(f_{\text{whirl}}\). This frequency depends on the journal’s frequency according to this approximate formula

$$f_{\text{whirl}} = (0.42 \text{ to } 0.48) \times f_{\text{rotor}}. \quad (15)$$

The full cascade spectrum of the journal axis motion for a run-up from 0 to 12 000 rpm is shown in Fig. 9. The spectra demonstrated the presence of a frequency of whirling in the measured signals during increasing rpm after crossing the threshold of instability. The full spectrum was calculated using the FFT for the coordinates of the journal axis in a complex plane that was perpendicular to the journal axis. For example, the FFT time signal decomposed the signal to harmonic components; the full spectrum represented the decomposition of the journal
motion inside the bearing bushing bore on elementary orbits with different rotational frequencies, see a book. The positive frequencies corresponded with the phasor rotation in the positive direction, and the negative frequencies corresponded with the phasor rotating in the opposite direction. An example of the full spectrum of the journal motion without any control demonstrated the elementary orbits of the journal axis, which rotated at \( \pm 45 \)X fraction of the rotor rotational speed. The onset of the instability of the whirl type was excited by a disturbance at that frequency. It was a reason for closing the feedback only for the narrow frequency band around the whirl frequency, which also increased the signal-to-noise ratio for the feedback signal.

The whirl frequency was the centre frequency to which the bandpass filter was tuned. The rotors supported on the journal bearings were not operated at a constant rotational speed, so automatically adapting the centre frequency of the bandpass filter was required after rotor speed changes. The bandpass filter of the active vibration control system was realized in the time domain by Simulink, which was implemented in dSPACE. The Laplace transfer function, given by Eq. (1), had to be converted to the second-order differential equation with coefficients which depend on the centre frequency as well. After conversion, the mathematical model of the filter was given by the following differential equation:

\[
\frac{d^2 y}{dt^2} = \left(0.45 \times 2\pi f_{\text{rotor}}\right)^2 \left[\frac{K}{0.45 \times 2\pi f_{\text{rotor}}} \frac{dx}{dt} - \frac{2\zeta}{0.45 \times 2\pi f_{\text{rotor}}} \frac{dy}{dt} - y\right].
\]  

As can be seen in Fig. 10, the bandpass filter is directly programmed in the Simulink environment. The input for the filter reset is omitted to make block diagram simpler. The gain \( K \) and damping ratio \( \zeta \) (ksi) were considered as constants. The expression of \( 0.45 \times 2\pi f_{\text{rotor}} \) was replaced by the identical \( \omega_{\text{whirl}} \) angular frequency. By decreasing the value of the damping ratio \( \zeta \), the width of the bandpass of the filter also decreased. The implementation of the bandpass filter in the control algorithm is depicted in Fig. 10. The block diagram of active vibration control is shown in Fig. 11. Parallel connection of the bandpass filter and proportional controller allowed for experimentation with the weighting factors of these feedbacks.

### 6. EXPERIMENTS

The instability of the rotor motion resulted from the property of the oil film. The operating range of the journal bearing speed was extended by the use of active vibration control, i.e., by introducing feedback between the position of the journal and the bushing. There were three possible combinations for the arrangement of the controller feedback. The experiment enabled the determining of the type of feedback that maximizes the operating speed range. The time ramp at a constant rate of the rpm increase was selected as a set point of the control loop. To compare the effectiveness of active vibration control with operation without this system, it was necessary to specify the maximum possible rotor speed without vibration of the whirl type. Almost every time the instability of the journal motion
occurred before reaching a speed of 3000 rpm. During the unstable movement, the bearing journal moved in an almost circular path with a diameter that was given by clearance in the bearing. Many tests confirmed that the movement of the bearings in the bushing is unstable when the rotational speed exceeds the limit of 2500 rpm. Radial clearance and oil temperature were the same.

As mentioned above, different combinations of the connections of the bandpass filter and proportional controller were tested. Here are the measurements for the controller parameters that led to the best results. Figure 11 shows the control with bandpass filters without the parallel proportional controller (the gain of the proportional controller is set to 0). Instability of the journal motion occurs when 6553 rpm is exceeded. The cursor indicates the onset of instability in Figs. 12, 13 and 14. Variable $X$ of the cursor data was the time in seconds and $Y$ was the value in rpm for diagrams showing the stability margin. The scale for the vertical position of the bearing journal was in reverse order to show the visible upward stroke of the rotor axis after the run-up.

In the proportional control, the instability occurred after crossing approximately 7407 rpm. In Fig. 13, the control only with the proportional controller is presented. In Fig. 14, the control with the combination of the proportional controller and the bandpass filter is presented. By combining the proportional controller and the bandpass filter, the best results are achieved. Table 1 lists the results of the experiments. It becomes clear that active vibration control is adapted to improve the functional properties of bearings by applying electronics to replace passive measures against whirl vibration that cannot be easily and inexpensively manufactured in comparison to the cylindrical bushing. The prototype of an actively controlled bearing is the first fully functional device in the world.

7. CONCLUSIONS

This article describes the implementation of the proportional controller and the second-order bandpass filter to compensate disturbances in a limited frequency band. The bandpass filter increases the feedback gain in the narrow frequency band and, in this way, increases the threshold rotation speed at which the ‘whirl’ type instability occurs. The experiments proved that the filter has a positive effect on active vibration control. The highest rotational speed at stable operation (approximately 9300 rpm) was obtained by combining the proportional controller and the bandpass filter. The stable operational range is three or four times greater than for the sliding bearing without any active vibration control. With the use of the proportional feedback and without the filter, we could achieve a speed of approximately 7400 rpm. The Kalman filter smoothes the instantaneous rotational speed signal that inputs to the bandpass filter because the measurement of rpm is a lightly corrupted by a random error. The speed is evaluated from the tacho-signal, which is a string of pulses. The linear interpolation of the trigger level with the rising edge of the pulses enhances the calculation of the length of the time interval between the pulses of the tacho-signal.
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Seyed Hamed Seyed Hosseini received his BS degree from Islamic Azad of University in Mashhad, Iran, in 2007. He is currently an MS student at Imam Khomeini International University in Qazvin, Iran. His research interests are viscoelastic, vibration, smart materials, and nano- and micromechanic.

Yu-yang Chai is a PhD candidate in the School of Astronautics, Harbin Institute of Technology, China. His research interests include structural dynamics, nonlinear dynamics, and aeroelastic flutter analysis and control.

Feng-ming Li received the PhD degree from Harbin Institute of Technology, China, in 2003. He is a full professor in the College of Mechanical Engineering at Beijing University of Technology, Beijing, PR China. Dr. Li is currently an adjunct professor at the School of Astronautics, Harbin Institute of Technology. His research interests include structural dynamics, nonlinear dynamics, structural vibration control, elastic dynamics and aeroelastic analysis of structures. Dr. Li has published over 120 peer-reviewed international journals including JAM, JSV, JMPS, and IJSS. His scientific articles have obtained more than 700 citations in the past 5 years by other scholars in the Science Citation Index.

Zhi-guang Song received his PhD degree in general and fundamental mechanics from the School of Astronautics, Harbin Institute of Technology, China, in 2014. He was awarded the Alexander von Humboldt Research Fellowship in 2015, and he is working at the Dynamics and Vibrations Group in the Department of Mechanical Engineering, Technische Universität Darmstadt. Dr. Song’s current research interests include vibration and control, nonlinear dynamics, aerothermoelastic analysis, and active flutter control of structures. He has authored over 20 peer-reviewed scientific articles on the dynamic and control field.

Cristina Castejón is an associate professor at the Department of Mechanical Engineering of the University Carlos III, Madrid. She got a master’s degree as an industrial engineer in 1998 with laude mention and a PhD degree in industrial technologies at University of Carlos III in 2002. Her research interests cover aspects of mechanics of robots, mechanics of machinery, and monitoring and diagnosis of rotary elements. She is a member of MAQLAB research group, ‘Pedro Juan de Lastanosa’ research institute, both belonging to Carlos III University. Also, she belongs to AEIM (Spanish mechanical engineering association) and IFToMM.
María Jesús Gómez is an assistant professor at the Department of Mechanical Engineering of the University Carlos III (Madrid). She got a master’s degree as an industrial engineer in 2008 with laude mention, a master’s degree of machine and transports engineering in 2011 with laude mention, and a PhD in mechanical engineering at University of Carlos III in 2014. Her areas of expertise are diagnosis of rotating machines, vibration in rotating machinery, signal processing, biomechanics, and mechanics of robots. She is a member of MAQLAB research group that belongs to University Carlos III. She also belongs to AEIM (Spanish mechanical engineering association) and to the rotordynamics division of IFToMM.

Juan Carlos García Prada is a head professor at the Department of Mechanical Engineering in the University Carlos III, Madrid. He got a master’s degree as an industrial engineer in 1985 at the Polytechnic of Madrid University and a PhD in mechanical engineering at The National Distance Education University (UNED) in 1991. His professional activities have been related to mechanism and machines cinematic and dynamic, vibration in rotating machinery, signal processing, and mechanics of robots. He is a member of MAQLAB research group and ‘Pedro Juan de Lastanosa’ research institute, both belonging to Carlos III University. Also, he belongs to AEIM (Spanish mechanical engineering association) and IFToMM.

Eduardo Corral is an assistant professor at the Department of Mechanical Engineering of the University Carlos III (Madrid). He got a master’s degree as an industrial engineer in 2008 with laude mention, a master’s degree of machine and transports engineering at 2011 with laude mention, and a PhD in mechanical engineering at University of Carlos III in 2015. His areas of expertise are multibody dynamics, mechatronics, and diagnosis of rotating machines. He is a member of MAQLAB research group that belongs to University Carlos III. He also belongs to AEIM (Spanish mechanical engineering association) and to the multibody division of IFToMM.

Madhusmita Pradhan received a BTech in mechanical engineering from OSME, Keonjhar, India in 2009 and an MS in Machine Design and Analysis from Veer Surendra Sai University of Technology (VSSUT), Burla, India in 2013. Now she is continuing her PhD in mechanical engineering at VSSUT, Burla, India from 2014. In 2009, she joined the department of mechanical engineering, CEB Bhubaneswar, India as a lecturer. In 2013, she joined the department of mechanical engineering, CVRCE Bhubaneswar, India as an assistant professor. Since May 2015 she has been with department of mechanical engineering, VSSUT, Burla, India as an assistant professor. Her current research interest includes mechanical vibration analysis.

Mrunal Kanti Mishra received a BTech in mechanical engineering from Sikha ‘O’ Anusandhan (SOA) University, Bhubaneswar, India in 2013 and the MS (Gold Medallist) in machine design & analysis specialisation in mechanical engineering from VSSUT, Burla, India. He is currently pursuing his PhD in mechanical engineering from Indian Institute of Technology (IIT), Kharagpur, India. He has co-authored several publications on parametric vibration. His current research focuses on flexible bionic manipulator.
Pushparaj Dash completed HSC and ISc in 1982 and 1984 respectively. He completed his BE in mechanical engineering in 1988 from UCE (now VSSUT), Burla, India and MS engineering in machine design and analysis from REC (now NIT), Rourkela, India in 1990. He got his doctorate degree in mechanical engineering from IIT, Kharagpur, India in 2001. He worked as lecturer from 1990 to 2002 at IGIT, Sarang, India, as reader from 2002 to 2010 at VSSUT, and from 2010 onwards worked as professor in the department of mechanical engineering at VSSUT. He has published around thirty research articles in the area of structural vibration and rotor dynamics in different reputable international and national journals and conference proceedings.

K. Renji joined the Structures group of ISRO Satellite Centre (ISAC), Bangalore in 1986 after completing his M Tech in machine dynamics from the Indian Institute of Technology (IIT), Madras. He was responsible for the structural dynamic (vibration, acoustic and shock) tests of various spacecrafts. He obtained his PhD in mechanical engineering from IIT, Madras in 1999. Currently, he is an outstanding scientist. He is the group director of the Structures wing of ISAC/ISRO. He made significant contributions to the fields of design, development and realization of various spacecraft structures. His areas of interest include vibration, acoustics, nonlinear vibrations, dynamic testing, non-destructive testing, finite element method, statistical energy analysis, boundary element method, passive vibration control, spectral elements, structural design, structural materials and manufacturing. He has guided many for their MS and PhD degrees. He has several publications in international journals.

S. Josephine Kelvina Florence obtained her bachelor’s degree in civil engineering with honors from Bharathiar University in 2002. She obtained her master’s degree in structural engineering from Anna University in 2004. She is pursuing her PhD from Anna University. She started her professional career as a scientist/engineer at the Vikram Sarabhai Space Centre (VSSC), Trivandrum in 2005. She moved to the ISRO Satellite Centre (ISAC), Bangalore in 2010 and her current work focuses on the structural analysis of spacecraft structures. Her research interests include soft computing, finite element method, dynamics and statistical energy analysis.

Sameer Deshpande obtained his M Tech in applied mechanics from NIT Bhopal in 2005, bachelor’s degree from Pune University in 2000. He joined the ISRO Satellite Center as a scientist / engineer in 2006 and is presently serving in various capacities in the area of structural dynamics. He is working in the dynamic tests section and is mainly responsible for spacecraft dynamic tests. He has five conference / journal papers. His areas of interest include modal testing, experimental structural dynamics and micro vibration. He is a member of the Indian Society for Advance Material and Process Engineers (ISAMPE).

Mohamed El-Morsy is an associate professor in the Mechanical Design Department at Helwan University. He received his PhD degree in mechanical design engineering from Helwan University in 2010. He worked as a senior researcher at Czech Technical University in Prague. He conducts many projects with the Science and Technology Development Fund (STDF) in Egypt, the European Social Fund in the Czech Republic and Ministry of Education, and the Youth and Sports of the Czech Republic. His current research is focused on the development of machine noise and vibration from the point of view of maintenance, fault diagnosis, and prognosis.
Gabriela Achtenová is an associate professor of mechanical engineering at the Czech Technical University (CTU) in Prague. She received her MS degree from CTU in Prague in 1994 and PhD in 1998 at CTU in Prague (Czech Republic) and ESEM (Orléans, France). Her research areas are design and testing of automotive transmissions and analysis and synthesis of power split mechanisms for automotive applications.

Cheng Wang is an associate professor of school of mechanical engineering, University of Jinan. He graduated from Northwestern Polytechnical University in China in April 2010 and received PhD candidate in engineering. His current research interests include mechanical transmission, dynamic analysis of mechanical systems, etc. The projects he hosted include the National Natural Science Foundation of China, the Shandong Province Young and Middle-Aged Scientists Research Awards Fund, The China Postdoctoral Science Foundation funded project, Beijing Postdoctoral Science Foundation funded project and the Project of Shandong Province Higher Educational Science and Technology Program and so on. He has published 20 papers, 8 of which have been retrieved by SCI and 6 of which have been retrieved by EI. 3 invention patents he applied for have been authorized.

Shouren Wang is a professor of school of mechanical engineering, University of Jinan. He graduated from Shandong University in June 2006 and received PhD candidate in engineering. His current research interests include protection and control of friction, wear and lubrication of mechanical components and so on. The projects he hosted include the Taishan Scholar Project Special Funds, the National Natural Science Foundation of China, the Shandong key research and development plan, the Shandong Natural Science Foundation and so on. He has published 86 papers, 38 of which have been retrieved by SCI and 30 of which have been retrieved by EI. 27 invention patents he applied for have been authorized.

Gaoqi Wang is a lecturer of school of mechanical engineering, University of Jinan. He graduated from Shandong University in July 2015 and received PhD candidate in engineering. His current research interests include the research work of dental and department of orthopedics materials, biomechanics, biologic tribology, structural design and so on. The project he hosted includes the Shandong Province Young and Middle-Aged Scientists Research Awards Fund. Participating projects include the National Natural Science Foundation of China, the Taishan Scholar Project Special Funds, the Shandong key research and development plan and so on. He has published 8 papers, 5 of which have been retrieved by SCI and 3 of which have been retrieved by EI. 10 invention patents he applied for have been authorized.

Ashraf M. Zenkour graduated from Mansoura University, Egypt in mathematics in 1985 and was awarded the MS and PhD degrees from the same university in 1989 and 1995, respectively. He is a professor of Applied Mathematics at Kafrelsheikh University, Egypt and he is currently a Professor of Applied Mathematics at King Abdulaziz University, Saudi Arabia. His research interests are in the areas of structural stability, vibration, plated structures, and shells. He is the author or co-author of over 250 scientific publications, and has received more than 4000 citations with 31 h-factor (ISI Web of Science). He is a reviewer of many international journals in solid mechanics and applied mathematics, and an editorial member of many journals. In addition, he has delivered various lectures at national and international conferences. Professor Zenkour’s research papers have been cited in many articles and textbooks.
About the Authors

Eshagh Derakhshan

Mahboobeh Fakhrzarei

Shahram Derakhshan

Rakesh Ranjan Chand is a Ph.D. scholar of the Department of Mechanical Engineering at IIT(BHU), Varanasi, India. He received a B.Tech. in Mechanical Engineering from Government College of Engineering, Keonjhar, India in 2014 and M.Tech. in Mechanical Engineering with specialization Machine Design and Analysis from Veer Surendra Sai University of Technology, Burla in 2017. His research interests span parametric vibration analysis and control, smart structures, wave propagation, energy extraction etc. Much of his work has been on the design of rotating systems subjected to parametric vibration with several boundary conditions under a wide range of external operating conditions. Mr. Chand within this short period as a research scholar, has authored a research article which is in press in the Journal of Vibration Engineering and Technology (JVET) and co-authored a conference paper which is to be presented in the International Conference on Vibration Problems (ICOVP-2017) at IIT Guwahati.

Pravat Kumar Behera is a Ph.D. scholar of the Department of Aerospace Engineering at IIT Kharagpur, India. He received a B.Tech. in Mechanical Engineering from BPUT, Odisha, India in 2013 and M.Tech. in Mechanical Engineering with specialization Machine Design and Analysis from Veer Surendra Sai University of Technology, Burla in 2017. His research interests span both parametric vibration analysis of rotating structures and control. Mr. Behera has co-authored a research article which is in press in the Journal of Vibration Engineering and Technology (JVET) and also co-authored a conference paper which is to be presented in the International Conference on Vibration Problems (ICOVP-2017) at IIT Guwahati.
Nanfei Wang is a PhD candidate in the Department of Thermal Engineering from Tsinghua University, Beijing, China. Major in mechanical vibration and dynamic analysis of rotating machine; finite element analysis; condition monitoring and fault diagnosis; dynamic modeling and simulation.

Dongxiang Jiang is Professor in the Department of Thermal Engineering, Tsinghua University, Beijing, China. He received his BS in electronic engineering from the Shenyang Polytechnic University in 1983. He received his MS in electrical engineering from Harbin Institute of Technology in 1989. He received his PhD in astronautics and mechanics from Harbin Institute of Technology in 1994. He worked as an assistant engineer and an engineer at Harbin Research Institute of Electrical Instrumentation for six years. He was a post-doctoral fellow in the Department of Thermal Engineering, Tsinghua University from 1994 to 1996. His research interests include condition monitoring and diagnostics for machinery and wind power.

Yizhou Yang is a PhD student in Department of Thermal Engineering from Tsinghua University, Beijing, China. His current research interests include: condition monitoring, fault diagnosis, signal processing and rotor dynamics.

Kamran Behdinan is Professor in Mechanical Engineering from the University of Victoria in British Columbia in 1996 and has considerable experience in both academic and industrial settings. Dr. Behdinan was appointed to the academic staff of Ryerson University in 1998, tenured and promoted to the level of associate professor in 2002 and subsequently to the level of Professor in 2007 and has served as the director of the aerospace engineering program (2002–03), and the founding Chair of the newly established Department of Aerospace Engineering (2007–03 and 2007–11). He has joined the Department of Mechanical & Industrial Engineering, University of Toronto, in the rank of Full Professor in September 2011. He is the NSERC Chair in multidisciplinary engineering design, sponsored by NSERC, University of Toronto, and thirteen companies including Bombardier, Pratt & Whitney Canada, Goodrich, Magna, Ford, and DRDC Toronto. He is the founding director of the Institute for Multidisciplinary Design & Innovation (UT-IMDI) an industry centered project-based learning institute in partnership with major aerospace and automotive companies.


Malesela K. Moutlana is a lecturer in Mechanical Engineering at the Durban University of Technology. He graduated B.S. from Massachusetts Institute of Technology and M.Sc. from University of Kwa-Zulu Natal. He started working in the semi-conductor manufacturing sector with ASTeX(Pty) and served as a research engineer at Eskom(Pty). Mr. Moutlana decided to change his profession and moved to an academic and research environment in Durban, South Africa; where he lectures and conducts research. His current research interests are piezo-electric actuators and sensors and vibrations of nano structures. He is also a member of the Golden Key International Honour Society.

Sarp Adali is Sugar Millers Professor of Mechanical Design in Mechanical Engineering at the University of KwaZulu-Natal (UKZN), Durban, South Africa. He received a BSc degree from Middle East Technical University, Ankara, Turkey and PhD from Cornell University, NY, USA. After working at Council for Scientific and Industrial Research between 1977-1983, he joined UKZN in 1984. Presently his research areas are composite materials, design optimization and nanomechanics. Prof Adali published more than 180 journal papers, is a member of the editorial boards of six international journals and a Fellow of ASME.

Xiuxian Jia is currently at Purdue University, West Lafayette, Indiana, USA, as a joint PhD in mechanical engineering, and she is also currently in a PhD program of automotive engineering at Dalian University of Technology (Dalian, China). She got the BS degree in automotive engineering at Yantai University. She then studied for a PhD degree at Dalian University of Technology. She has been researching the vibration and noise control of structures by applying acoustic black holes (ABH). Her research interests include the vibration and noise control of lightweight structures and ABH theory.

Yu Du studied sound and vibration in the vibration and acoustics lab (VAL) at Virginia Tech. He graduated with a PhD degree in mechanical engineering in 2003. He then worked as a research scientist and acoustic engineer at Adaptive Technologies, Inc. (Blacksburg, Virginia, USA) and Knowles Electronics (Itasca, Illinois, USA). Since 2010, Dr. Du became a professor in the School of Automotive Engineering at Dalian University of Technology (Dalian, China). His current research interests include Vehicle NVH, the vibration and noise control of lightweight structures, acoustic signal processing for ADAS and MEMS transducer design for sound and vibration applications.

Ye Yu is currently at Purdue University, West Lafayette, IN, USA, as a joint PhD in the School of Aeronautics and Astronautics. He got his BS degree in automotive engineering and double BS degree in English at Dalian University of Technology. He then graduated with an MS degree in automotive engineering Mr. Yu is currently in a PhD program for automotive engineering at Dalian University of Technology. He has been researching advanced composite structures, automotive structure design, optimization and structural vibration control and adhesive analysis. Mr. Yu’s research interests include fiber-reinforcement polymer composite, failure analysis of sandwich structures, vibration of sandwich structures, and automotive acoustic and vibration control.
Kunmin Zhao graduated with a PhD degree in mechanical engineering from Ohio State University in 1999. He then worked as a senior project engineer at General Motors (Detroit, Michigan, USA) until 2008. He then worked as a director of stamping and die operations engineering at Hybrid Kinetic Motors (Los Angeles, California, USA) and Chrysler (Detroit, Michigan, USA) until 2013. Since 2013, Dr. Zhao became a professor in the School of Automotive Engineering at Dalian University of Technology (Dalian, China). His research interests include advanced forming processes, stamping die design and manufacturing, automotive acoustic and vibration control, the application of additive manufacturing (3D printing) technology in automobile and aviation.

Hamdy M. Youssef is a Professor of Applied Mathematics. He has 22 years of experience of teaching mathematics for undergraduate and postgraduate students in Alexandria University (full-time), Umm Al-Qura University (full-time) and Arab Academy for Science and Technology (part-time). He is also a consultant of Umm Al-Qura University for the scientific chairs unit for 3 years. He published more than 100 published papers in international journals and conferences.

Alaa Abdel Wahed Hassan Abdel Bary is a Professor of Applied Mathematics. He has 30 years of experience of teaching mathematics for undergraduate and postgraduate students in Arab Academy for Science and Technology and Maritime Transport. He is also Vice president for post graduate studies and scientific research. He published more than 120 published papers in international journals and conferences.

Mangesh S. Kotambkar

Zhen Wang received his bachelor’s degree in electric information engineering from the Harbin Engineering University, Harbin, P. R. China in 2011. He received his master’s degree in design and construction of naval architecture and ocean structure from China Ship Research and Development Academy, P. R. China in 2014. He received his PhD in mechanic engineering from the department of Building Services Engineering of the Hong Kong Polytechnic University, Hong Kong, P. R. China in 2018. His research interests include vibration isolation, active control, periodic structures, and room acoustics.
Cheuk Ming Mak was awarded his bachelor’s degree in building services and environmental engineering and his Doctor of Philosophy in acoustics from the University of Liverpool. He received his Postgraduate Diploma in acoustics and noise control from the Institute of Acoustics of United Kingdom. He is currently a full professor in the Department of Building Services Engineering at the Hong Kong Polytechnic University. His research interests include building acoustics, noise and vibration, indoor environmental quality, and the application of computational fluid dynamics to building and urban environments. He has committed to research for over 25 years and has produced over 200 publications. He currently serves as a member of the editorial advisory board of Building and Environment and an associate editor of the International Journal of Acoustics and Vibration. He is a fellow of the Hong Kong Institution of Engineers and is the former chairman of the Hong Kong Institute of Acoustics.

Dayi Ou received his PhD from the Department of Building Services Engineering from the Hong Kong Polytechnic University, Hong Kong, in 2011. He currently works as an associate professor at the School of Architecture of Huqiao University. His current research interests include vibration isolation, sound insulation, and building acoustics.

Mohammad Javanbakht received his BS degree in mechanical engineering from Islamic Azad University, Iran in 2014 and his MS degree in mechanical engineering from Sirjan University of Technology, Sirjan, Iran in 2017. His research interests include suspension systems, optimal control, fuzzy logic and evolutionary algorithms.

Mohammad Javad Mahmoodabadi received his BS and MS degrees in mechanical engineering from Shahid Bahonar University of Kerman, Iran in 2005 and 2007, respectively. He received his PhD degree in mechanical engineering from the University of Guilan, Rasht, Iran in 2012. During his research, he was a visiting scholar in the Robotics and Mechatronics Group, University of Twente, Enchede, the Netherlands for six months. At present, he is an assistant professor of mechanical engineering at the Sirjan University of Technology, Sirjan, Iran. His research interests include optimization algorithms, non-linear and robust control, and computational methods.

Fan Yang has a BSME degree from Southwest Jiaotong University and is currently a ph.D. candidate in Mechanical Engineering at the SWJTU. His research focuses on modeling and simulation of noise control technologies for fluid power.
Bin Deng is a professor at the Southwest Jiaotong University. He began at SWJTU in 1984. He earned his PhD in 2004 from the SWJTU. His research interests include control and simulation of electro-hydraulic integration system and water hydraulics.

Rongxing Wu received his BS and MS degrees in engineering mechanics from the Department of Mechanics and Engineering Science, School of Mechanical Engineering and Mechanics, Ningbo University in 2005 and 2008, respectively. He received his PhD in engineering mechanics from Ningbo University in 2012. Currently he is an associate professor in the Department of Architectural Engineering, Ningbo Polytechnic. His research interests include piezoelectric resonators and acoustic wave devices.

Ji Wang received his BS in structural engineering in 1983 from Gansu University of Technology (currently Lanzhou University of Technology). From 1983 to 1988, he was a structural engineer at the 11th Institute of Project Planning and Research in Xi’an, Shaanxi, China. From 1988 to 1990, he was a visiting scientist at Argonne National Laboratory working on structural analysis. From 1990 to 1995, he was a graduate student at Princeton University studying high frequency vibrations of piezoelectric plates. He received his MS and PhD degrees in civil engineering from Princeton in 1993 and 1996, respectively. From 2002, he has been a Qian River Fellow Professor at Ningbo University, Ningbo, China and the founding director of the Piezoelectric Device Laboratory. His research interests include physical acoustic waves in piezoelectric resonators and computational methods.

Jianke Du received his BS in aircraft design from Beijing University of Aeronautics and Astronautics in 1992. From 1994 to 1997, he was a graduate student for his MS degree at Northwestern Polytechnic University and the Fourth Academy of CASC. He received his PhD in mechanics from Xi’an Jiaotong University in 2004. From 1992-1993 and 1997-1999, he was working on the structural analysis of composite material vessels at Xiangyang Dynamic Mechanical Corporation in Xi’an, Shaanxi, China. From 2007, he has been a professor at Ningbo University. His research interests include the surface acoustic waves in piezoelectric and piezomagnetic materials.

Jiashi Yang received his BE and ME in engineering mechanics in 1982 and 1985 from Tsinghua University, and his PhD in civil engineering in 1994 from Princeton University. He then worked as a postdoctoral fellow at the University of Missouri-Rolla and Rensselaer Polytechnic Institute. He was employed by Motorola, Inc. from 1995 through 1997 as an engineer. Since 1997 he has been an assistant, associate, and full professor in the Department of Mechanical and Materials Engineering at the University of Nebraska-Lincoln.
Hamid Gheibollahi attended the Isfahan University of Technology in 2008 and graduated with a degree in mechanical engineering in 2013. He has received his master's degree in automotive engineering from the Iran University of Science & Technology in 2016. His main research interests include the vibrations of the car and its damage to the driver’s body. His research on vibrations has been under the supervision of Dr. Masoud Masih-Tehrani. The title of his thesis is ‘Modeling and Analysis of Heavy Vehicle Driver Damages due to Vibrations’. In his thesis, he has presented a biodynamic model of a driver’s body, which can be used to determine the optimal stiffness and damping parameters of the seat.

Masoud Masih-Tehrani received his PhD degree in mechanical engineering with a major in hybrid energy storage systems (HESS) from the University of Tehran (UT), Tehran, Iran, in 2013. He is currently a faculty member and assistant professor at the School of Automotive Engineering, Iran University of Science and Technology (IUST), Tehran, Iran. His research interests include vehicle seat design, hybrid energy storage system, hybrid flywheel vehicles, heavy-duty vehicles, vehicle suspension systems, and vehicle control systems.

Mohammadmehdi Niroobakhsh obtained his master’s degree in mechanical engineering from the Iran University of Science & Technology. He is currently a PhD student in the Department of Civil & Mechanical Engineering at the University of Missouri-Kansas City and performing research on bone biomechanics.

Hu Ding is a Professor of Nanchang University of Technology, Nanchang, Jiangxi, China. Hi received his Bachelor Degree from HeFei University of Technology, in 2002, and his Master Degree from HeFei University of Technology in 2005. In 2008 he received his Doctor Degree from Shanghai University. He was a Visiting Professor at University of Toronto (2012–2013), Visiting Scholar at Duke University (2016–2017). His research activities and research interests include nonlinear vibration and control of continua, pipe conveying fluid, and axially moving continua. The main focus is on the geometric nonlinearity of the continua. He is also working on more effective suppression of the vibration of the elastic continua by introducing nonlinear factors, including nonlinear vibration absorption and nonlinear vibration isolation.
Miroslav Pawlenka completed his master’s degree at the Technical University of Ostrava (Czech Republic), Faculty of Mechanical Engineering in 2014. He then continued his PhD study, which was completed in 2018. Since 2015, he has worked with the Siemens Company as a PLC, and HMI programmer. He is an author and co-author of 9 publications (6 abroad).

Miroslav Mahdal completed his master’s degree at the Technical University of Ostrava, Faculty of Mechanical Engineering in 2006. He then completed his PhD degree in 2011 at the same university. After graduating, he became a researcher in the Department of Control Systems and Instrumentation. Later, he became an assistant professor in the Department of Control Systems and Instrumentation. He was involved in research in the field of automatic control, control of the mechatronic systems and wireless technologies (especially IEEE 802.15.4 standard). He has experience working with electronics and microprocessor technology, knowledge of the design and production of PCB. His research interests include active vibration control and energy harvesting problems. He is an author and co-author of 6 patents and more than 55 publications.

Jiří Tůma graduated from the Brno University of Technology (Czech Republic) in 1970. He then completed his PhD degree in 1977 at the same university. After graduating, he became a researcher in the field of control systems. Since 1988, he began working in an automotive company (Tatra Trucks), where he was involved in research regarding the noise and vibration of heavy-duty vehicles. In 1995, he joined the Technical University of Ostrava, Faculty of Mechanical Engineering where he became a full professor in 2001. He gives lectures on control systems, signal processing, active vibration control, and machine diagnostics. His research interests include the active vibration control of the journal bearings and flexible structures. He is the author of 16 patents, more than 200 publications, including a book on signal processing (in Czech), a chapter of Crocker’s Handbook on noise and vibration control, and a book on Vehicle gearbox noise and vibration (Wiley 2014).

Adam Bureček completed his master’s degree at the Technical University of Ostrava, Faculty of Mechanical Engineering in 2010. He then completed his PhD degree in 2013 at the same university. After graduating, he became a researcher in the Department of Hydromechanics and Hydraulic Equipment. Later, he became an assistant professor in the same department. His research interests include the field of the hydraulic systems, numerical simulation and measurement of dynamic properties of hydraulic systems, influence of air content on the area under investigation, verification of mathematical models based on experimental measurements and research of multiphase flow and fluid-structure interaction. He is an author and co-author of one patent, more than 20 research reports, 20 publications (8 abroad).