Two-Temperature Thermoelastic Damping of a Gold Nano-Beam Resonator with Variable Young’s Modulus

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This paper deals with the thermoelastic damping (Q-factor) of a gold nano-beam resonator and is based on two-temperature thermoelasticity models. An explicit formula of the Q-factor has been derived when Young’s modulus is variable as a function of the room temperature. The length of the beam and Young’s modulus has been studied with comparison being made between the Biot model and the Lord-Shulman model (L-S). The numerical results show that the values of beam length, the relaxation time parameter, and the two-temperature parameter have a strong influence on the thermoelastic damping quality factor.

1. INTRODUCTION

The governing equations of the coupled thermoelasticity theory (Biot model) consist of the equation of motion, which is a hyperbolic partial differential equation, and of the energy conservation equation, which is parabolic based on the classical Fourier’s law of heat conduction. This model of heat conduction generates the infinite speed of the thermal wave propagation inside the thermeastic medium.1

Lord and Shulman (L-S) introduced the first generalization to thermoelasticity, which is called the “theory of generalized thermoelasticity with one relaxation time,” for an isotropic body.10 Among this theory, a modification law of heat conduction including both the heat flux and its time derivative (Cattaneo’s heat conduction), which is called non-Fourier’s law of heat conduction, replaces the conventional Fourier’s law. The heat equation of this theory is hyperbolic, which fixed the paradox of infinite speeds of propagation inherent in both.3

Youssef introduced a new theory of two-temperature generalized thermoelasticity and proved the general uniqueness theorem for the initially mixed boundary value problems.18 Sharma and Marin studied the effect of distinct conductive and thermodynamic temperatures on the reflection of plane waves in micropolar elastic half-space. Sharma and Marin showed that the effect of the two temperature parameters have a significant impact on amplitude ratios.14

Many applications, such as mechanical signal processing, scanning probe microscopes, and ultrasonic mass detection became based on micro- and nanoelectromechanical resonators. The most critical parameter of a microresonator is its Q-factor, or quality factor, and it’s closely related to the accuracy of measurement in many applications. A higher the Q-factor leads to both less energy being dissipated during vibrations and an increase in the resonator’s sensitivity. Thus, the study of the energy dissipation mechanism is of great significance for the development and improvement of the design for micro/Nano-electromechanical resonators.5,6

The inherent loss consists of thermoelastic effects and internal friction. The external loss includes support loss, gas damping, and surface loss. Intrinsic material damping due to thermoelastic coupling effect is the mechanism that imposes an upper limit on the achievable Q-factor for micro-electromechanical system (MEMS) resonators. In general, thermoelastic damping, support loss, and surface loss are the three primary sources of energy dissipation in MEMS.5–9

Zener was the first researcher on the thermoelastic damping problem that analyzed the problem by treating the viscoelastic material and obtaining the formula for the Q-factor of thermoelastic damping.9 Lifshitz and Roukes gave a formula of Q-factor for thermoelastic damping based on classical Fourier thermal conduction theory. Their model predicts that there is a peak of thermoelastic damping, which occurs at micrometer scale.5

When the beam height goes up to more than 100 µm or down to nanometer scales, the thermoelastic damping will decrease accordingly. However, the experimental results show that the Q-factor tends to decrease monotonously when the dimensions of microresonators go down to nanometer scale.5

Thermoelastic damping has been studied in many works, some of which use the classical theory of thermoelasticity based on Fourier’s law of heat conduction to calculate the thermoelastic damping.6,8,12–17 In the classical theory of thermoelasticity, the Fourier law of heat conduction is of a parabolic type that offers the infinite speed of propagation for the thermal wave. To eliminate this paradox, several non-classical theories have been formulated to admit the finite speed of thermal wave propagation, which is called the second sound theory for adding a first time derivative of the heat flux into the Fourier’s law of heat conduction.9 Sun et al. analyzed the thermoelastic damping of a beam resonator based on the generalized thermoelastic theory with one relaxation time.13 Sharma and Sharma studied damping in micro-scale circular plate resonators by the L-S theory of generalized thermoelasticity theory.13 Marin represented that the amplitude of the vibrations decays exponentially with the distance to the base. This decay estimate is similar to that of the Saint-Venant type.13
2. VARIABLE YOUNG’S MODULUS

Variable Young’s modulus of some materials was measured between 293 K and 973 K using the impulse excitation method, and by comparing with literature, data was reported. The data was measured with

\[ E(T) = E_0 - bT e^{(-T_0/T)}. \]  

(1)

The values of the fitting parameters “\( E_0 \)” and “\( T_0 \)” were related to the Debye temperature and the parameter “\( b \)” represented the harmonic character of the bond.2,4,5

Farraro and Rex4 found that no departure from linearity was detected when they studied the dependence of Young’s modulus on the temperature, and determined the linear relation

\[ E(T) = E_0 - E_1 T; \]  

(2)

where \( E_0 \) was Young’s modulus in the ordinary case and \( E_1 \) was a constant value of the order \( 10^4 \) and they measured it for pure Nickel, Platinum, and Molybdenum.2,4,5

Following this, we considered the fact that Young’s modulus depends on the temperature by using the following function

\[ E(T) \approx E(T_0) \approx E_0 (1 - \gamma T_0); \]  

(3)

where \( \gamma = E_1/E_0 \) was a constant value.2,4

3. FORMULATION OF NANO-BEAM EQUATIONS BASED ON TWO-TEMPERATURE MODEL

The formula of the two-temperature heat conduction model, which was derived by Youssef, took the form [4]:

\[ q_i(r,t) + \tau_0 \frac{\partial q_i(r,t)}{\partial t} = -K T_{C,i}^C (r,t); \]  

(4)

and

\[ T^C(r,t) - T^D(r,t) = \beta T_{C,i}^C (r,t); \]  

(5)

where \( q_i \) was the heat flux vector, \( T^C \) was the absolute conductive temperature, \( r = r(x_i) \) the position vector, \( K \) the thermal conductivity of the material, \( \tau_0 \) was the relaxation time, \( T^D \) was the absolute dynamical temperature, and \( \beta \) was the two-temperature parameter.

The heat flux, temperature and volumetric strain for a thermoelastic isotropic body had the following relation:5

\[ -q_{i,j}(r,t) = \rho C_v \frac{\partial T^D(r,t)}{\partial t} + \frac{T_0 \alpha E(T_0) \partial e(r,t)}{1 - 2\nu}; \]  

(6)

where \( e \) was the volumetric strain, \( \alpha \) was the thermal expansion coefficient, \( C_v \) was the specific heat at constant volume, \( E(T_0) \) was Young’s modulus as a function of the room temperature \( T_0 \), \( \nu \) was Poisson’s ratio, and \( \rho \) represented mass density.

Equation (4) gave the following equation

\[ q_{i,j}(r,t) + \tau_0 \frac{\partial q_{i,j}(r,t)}{\partial t} = -K T_{C,i}^C (r,t); \]  

(7)

When substituting from Eq. (7) into Eq. (6), we obtained

\[ \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left[ \rho C_v \frac{\partial T^D(r,t)}{\partial t} + \frac{T_0 \alpha E(T_0) \partial e(r,t)}{1 - 2\nu} \right] = \frac{K T_{C,i}^C (r,t)}{\partial t}. \]  

(8)

We then re-wrote Eq. (5) in the form

\[ \theta(r, t) = \varphi(r, t) - \beta \varphi_{i,i}(r, t); \]  

(9)

where \( \varphi(r, t) = (T^C(r, t) - T_0) \) and \( \theta(r, t) = (T^D(r, t) - T_0) \).

Hence, Eq. (8) took the form

\[ \frac{K T_{C,i}^C (r,t)}{\partial t} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left[ \rho C_v \frac{\partial \theta(r,t)}{\partial t} + \frac{T_0 \alpha E(T_0) \partial e(r,t)}{1 - 2\nu} \right]. \]  

(10)

When substituting from Eq. (9) into Eq. (10), we got

\[ \varphi_{i,i} + \frac{\beta \rho C_v}{K} \left( \frac{\partial \varphi_{i,i}}{\partial t} + \tau_0 \frac{\partial^2 \varphi_{i,i}}{\partial t^2} \right) = \frac{\rho C_v}{K} \left( \frac{\partial \varphi}{\partial t} + \tau_0 \frac{\partial^2 \varphi}{\partial t^2} \right). \]  

(11)

4. THE RECTANGULAR NANO-BEAM RESONATOR

We considered the small flexural vibrations of a thin elastic beam of length \( L \) and a rectangular cross-section of dimensions \( h \). We took the x-axis along the axis of the beam, the y-axis along the thickness and the z-axis along the width direction. Thus, the strain tensor and volumetric strain took the forms:7–9

\[ e_{xx} = -\frac{\partial^2 w}{\partial x^2}; \]  

(12a)

\[ e_{yy} = e_{zz} = \nu \frac{\partial^2 w}{\partial x^2} + (1+\nu)\alpha \theta; \]  

(12b)

and

\[ e = e_{xx} + e_{yy} + e_{zz} = 2(1+\nu)\alpha \theta - (1-2\nu)\frac{\partial^2 w}{\partial x^2}; \]  

(13)

where \( w \) was the deflection of the beam.

The beam was free of stress, deformation and was kept at a uniform temperature \( T_0 \) in a state of equilibrium. The beam underwent flexural vibrations of small amplitudes about the x-axis in the x-y plane such that the deflection was consistent with the linear Euler-Bernoulli beam theory.

For this case, the equation of motion with thermoelastic coupling for the beam was given by7–9,17

\[ \frac{\partial^2 w}{\partial t^2} + \frac{E(T_0)I}{\rho A} \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial^2 w}{\partial x^2} \right) + \frac{E(T_0)\alpha}{\rho A} \frac{\partial^2 I_T}{\partial x^2} = 0; \]  

(14)

\[ A = h \times b \] is the area of the cross-section \( I \) and \( I_T \) are the moment of inertia and thermal moment of the beam, respectively, which are given by7–9

\[ I = \int_A y^2 dy dz; \]  

(15)

and

\[ I_T = \int_A y \theta dy dz. \]  

(16)
An Euler-Bernoulli beam Eq. (11), by using Eq. (13), took the form
\[ \frac{\partial^2 \varphi_{ii}}{\partial t^2} + \frac{\beta}{\chi} \left( \frac{\partial \varphi_{ii}}{\partial t} + \tau_0 \frac{\partial^2 \varphi_{ii}}{\partial t^2} \right) + \frac{\Delta_E E^*}{\alpha} \left( \frac{\partial^3 w}{\partial x^2 \partial t} + \tau_0 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) = \frac{1}{\chi} \left( \frac{\partial^2 \varphi}{\partial t} + \tau_0 \frac{\partial^2 \varphi}{\partial t^2} \right)^2 + \frac{2 \Delta_E E^*}{\chi \alpha} (1 + v) \left( \frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} \right); \]

where \( \chi = K/\rho C_v \) was thermal diffusivity of the material and \( \Delta_E = T_0 E_0 a^2 / \rho C_v \) was the relaxation strength of Young’s modulus.

Because the temperature gradients in the plane of the cross-section along the y-direction were much larger than those along the x-direction and that no gradients existed in the z-direction, we replaced \( \varphi_{ii} \) with \( \frac{\partial^2 \varphi}{\partial y^2} \). Also, \( \Delta_E \) was thermal diffusivity of the material and \( \Delta_E = T_0 E_0 a^2 / \rho C_v \) and the thermal moment took the form
\[ \phi = \frac{\Delta_E E^* \partial^2 W}{\alpha} \left( y - \frac{\sin(ky)}{k \cos(kh/2)} \right). \]

By substituting the expressions in Eqs. (19) into Eq. (18), we obtained
\[ \frac{\partial^2 \varphi}{\partial y^2} + \frac{\beta}{\chi} \frac{\partial^3 \varphi}{\partial x \partial y \partial t} + \frac{\tau_0}{\chi} \frac{\partial^4 \varphi}{\partial x^2 \partial t^2} = \frac{1}{\chi} \left( \frac{\partial^2 \varphi}{\partial t} + \tau_0 \frac{\partial^2 \varphi}{\partial t^2} \right)^2 + \frac{2 \Delta_E E^*}{\chi \alpha \beta} (1 + v) \left( \frac{\partial \theta}{\partial t} + \tau_0 \frac{\partial^2 \theta}{\partial t^2} \right). \]

To know the effect of thermoelastic coupling on the harmonic vibrations of the beam resonator, we assumed that:

\[ w(x, t) = W(x) e^{i \omega t}; \]
\[ \varphi(x, y, t) = \varphi(x, y) e^{i \omega t}; \]
\[ \theta(x, y, t) = \theta(x, y) e^{i \omega t}. \]

By substituting the expressions in Eqs. (19) into Eq. (18), we obtained
\[ \left( 1 + \frac{\beta \omega}{\chi} - \frac{\beta \tau_0 \omega^2}{\chi^2} \right) \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\chi} \left( i \omega - \tau_0 \omega^2 \right) \left( \frac{\partial^2 \varphi}{\partial t} + \frac{\partial^2 W}{\partial x^2} \right). \]

The last Eq. (20) was written as a differential equation in the form
\[ \frac{\partial^2 \phi}{\partial y^2} - \left( \frac{i \omega - \tau_0 \omega^2}{\chi + \beta \omega - \beta \tau_0 \omega^2} \right) \phi = \left( \frac{i \omega - \tau_0 \omega^2}{\alpha + \beta \omega - \beta \tau_0 \omega^2} \right) \frac{\partial^2 W}{\partial x^2}. \]

Hence, the solution of Eq. (21) took the form
\[ \phi = A_1 \cos(ky) + A_2 \sin(ky) + \frac{\Delta_E E^* (1 + f(\omega))}{\alpha} \frac{\partial^2 W}{\partial x^2}; \]

where
\[ k = \sqrt{\frac{\omega}{\chi} - \frac{\beta \omega - \beta \tau_0 \omega^2}{\alpha}}; \]
\[ A_1 = \frac{\xi \eta}{h \eta}; \]
\[ A_2 = \frac{\xi}{h \eta} a_2; \]
\[ \xi = h \sqrt{\frac{\omega}{2 \chi}}; \]
\[ \eta = \sqrt{a_1 + a_2^2} + a_2^2; \]
\[ \omega = \omega_0 \sqrt{1 + \Delta_E E^* (1 + f(\omega))}; \]

where \( \omega_0 \) was the isothermal value of frequency given by
\[ \omega_0 = \frac{g_0}{h} \sqrt{\frac{E_0 E^*}{2 \rho}}; \]

\[ f(\omega) = \frac{24 (1 + \beta k^2)}{h^4 k^4} \left( \frac{h k}{2} - \frac{\tan \left( \frac{h k}{2} \right)}{2} \right) \frac{\partial^2 W}{\partial x^2}. \]

Alternatively, we simplified the last equation to be in the form
\[ \omega^2 W = \frac{E_0 E^* I}{\rho A} \left[ 1 + \Delta_E E^* (1 + f(\omega)) \right] \frac{\partial^2 W}{\partial x^2}; \]

where the complex function \( f(\omega) \) took the form
\[ f(\omega) = \left( \frac{h k}{2} - \frac{\tan \left( \frac{h k}{2} \right)}{2} \right); \]

\[ \omega = \omega_0 \sqrt{1 + \Delta_E E^* (1 + f(\omega))}; \]

\[ \omega_0 = \frac{g_0}{h} \sqrt{\frac{E_0 E^*}{12 \rho}}; \]
ends. resonator, which is made of Gold (Au) and clamped at two
thermal relaxation time $\tau$. Using the Taylor series for only the first order and obtained
$\Delta$, the beam height $h$.

NUMERICAL RESULTS AND DISCUSSIONS

The relationships between the variations of the Q-factor of the beam height $h$, dimensionless variable for $\xi$ different values of the two-temperature parameter $\beta = \{0.0, 10^{-10}\}$ and the thermal relaxation time $\tau_0 = \{0.0, 10^{-10}\}$ of a microbeam resonator, which is made of Gold (Au) and clamped at two ends.

Material properties of Gold (Au) are listed as follows [14]:

$$K = 3188\text{W/(mK)};$$  \hspace{1cm} (40a)

$$\alpha = 14.2(10)^{-6}\text{K}^{-1};$$  \hspace{1cm} (40b)

$$T_0 = 300\text{K};$$  \hspace{1cm} (40c)

and

$$q_\alpha L = \{4.73, 7.853, 10.996, \ldots\}; \quad n = 1, 2, 3, \ldots \quad (35)$$

Since $\Delta E \ll 1$ we expanded the right-hand side of Eq. (33) by using the Taylor series for only the first order and obtained

$$\omega = \omega_0 \left[ 1 + \frac{\Delta E_0}{2} \left( 1 + f(\omega_0) \right) \right]. \quad (36)$$

Because thermoelastic damping is feeble, that is, we replaced $f(\omega)$ with $f(\omega_0)$. The frequency relation Eq. (36) took the form

$$\omega = \omega_0 \left[ 1 + \frac{\Delta E_0}{2} \left( 1 + f(\omega_0) \right) \right]. \quad (37)$$

The amount of thermoelastic damping was expressed regarding the inverse of the quality factor:7.9

$$Q^{-1} = 2 \frac{|Im(\omega)|}{Re(\omega)}. \quad (38)$$

After separating the real and imaginary parts of $\omega$, the thermoelastic damping of a beam resonator in the context of two-temperature parameters was then given by Eq. (39) (see on the top of the next page).

5. NUMERICAL RESULTS AND DISCUSSIONS

Figures 1–4 represent the Q-factor concerning the full range of beam height $h$ ($10^{-9} \leq h \leq 10^{-4}$) based on two different models of thermoelasticity; coupled thermoelasticity (Biot model) and generalized thermoelasticity with one relaxation time (L-S model) in the context of the two-temperature thermoelasticity model (Youssef model). Figure 1 represents the results based on Biot’s model ($\tau_0 = 0.0$) with different values for Young’s modulus parameter $\gamma$ where the solid line represents the case of constant Young’s modulus, $\gamma = 0$, and the dashed line represents the case of variable Young’s modulus, $\gamma \neq 0$, model with one temperature type $\beta = 0.0$ we find that $\gamma$ has a significant effect on the Q-factor of the nanobeam resonator. In Fig. 2, the same situation happened when the results based on L-S model ($\tau_0 = 10^{-10}$). In the two models, when Young’s modulus is variable, the Q-factor decreases up to the peak point, then the Q-factor increases. Figure 3 and 4 compare between Biot model and L-S model based on two-temperature type respectively. We found that, in the two cases where Young’s modulus is variable, the Q-factor increases. Figures 5–8 represent the Q-factor with respect to the wide range of the dimensionless variable $\xi (1 \leq \xi \leq 10)$ based on two different models of thermoelasticity; coupled thermoelasticity (Biot model) and generalized thermoelasticity with one relaxation time (L-S model) in the context of the two-temperature thermoelasticity model (Youssef’s model). Figures 5 and 6 represent Biot model ($\tau_0 = 0.0$) and L-S model ($\tau_0 = 10^{-10}$) with one temperature type ($\beta = 0.0$) respectively. We found that the variability of Young’s modulus has very small effects on the Q-factor distribution with respect to the dimensionless variable $\xi$. Figures 7 and 8 compare between Biot model ($\tau_0 = 0.0$) and L-S model ($\tau_0 = 10^{-10}$) based on two-temperature type ($\beta = 10^{-10}$) respectively. We found that the variability of Young’s modulus has significant effects on the Q-factor distribution where it increases its value.
\[ Q^{-1} = \frac{24E_0 E^* \alpha^2 T_0}{C_\nu} \left[ \frac{\alpha_2}{\xi^2 \left( \eta^2 + \frac{\alpha_1^2}{\eta^2} \right)^2} - \frac{3\eta \alpha_2 - \frac{3\eta^3}{\eta^2}}{\xi^3} \left[ \xi^3 \left( \eta^2 + \frac{\alpha_1^2}{\eta^2} \right)^2 \left( \cos(\eta \xi) + \cosh \left( \frac{\xi \alpha_2}{\eta} \right) \right) \right] \right] - \frac{24E_0 E^* \alpha^2 T_0 a}{C_\nu h^2} \left[ \frac{\alpha_2}{\xi^2 \left( \eta^2 + \frac{\alpha_1^2}{\eta^2} \right)} \left[ \cos(\eta \xi) + \cosh \left( \frac{\alpha_2 \eta}{\xi a} \right) \right] \right]. \] (39)

![Figure 3](image3.png)  
Figure 3. The Q-factor based on two-temperature model of Biot.

6. CONCLUSION

Young’s modulus parameter \( \gamma \) has significant effects on the Q-factor of the nanobeam resonator in the context of the Biot and L-S models of one temperature and also of two-temperatures. The variability of Young’s modulus has minimal effects on the Q-factor distribution concerning the dimensionless variable \( \xi \).

REFERENCES


