1. INTRODUCTION

Machine components like steam turbine blades, compressor vanes, helicopter blades, propeller of aircrafts, manipulator of different robots, horn antennas, etc. rotate about a certain axis. Some of these components function at an elevated temperature and hub radius is almost linear for different revolution speeds. Hamilton’s principle is used to develop the equation of motion and accompanying end conditions. Then, the non-dimensional form of the equation of motion and the end conditions are found. Galerkin’s process is used to find a number of Hill’s equations from the non-dimensional equations. The parametric instability regions are acquired by means of the Saito-Otomi conditions. The consequences of the variation parameter, revolution speed, temperature grade, and hub radius on the instability regions are examined for both static and dynamic load case and represented by a number of plots. The legitimacy of the results is tested by plotting different graphs between displacement and time using the Runge-Kutta fourth-order method. The results divulge that the stability is increased by increasing the revolution speed; however, an increase in the variation parameter leads to destabilization in the system and for same parameters, the stability is less in the case of a variable temperature grade than that of a constant temperature grade condition.
duced on this type of beam because it will be economical, as less quantity of material will be required due to the exponentially varying profile and will be more efficient because there will be less drag force due to the curved profile.

2. SYSTEM MODELLING

2.1. Kinematics and Equation of Motion

As shown diagrammatically in Fig. 1, an exponentially converging beam having a circular cross-section is rotating about Z-axis and capable of vibrate in X-Z plane. The beam had a length \( l \), base diameter \( D \), was attached to a hub of radius \( B_0 \), which was subjected to an axial live load of \( W(t) = W_0 + W_1 \cos(ft) \) at \( x' = B_0 + l \) along the centroid of the cross-section.

The succeeding assumptions were made for the analysis of the system:

1. The beam material was homogeneous and isotropic;
2. The transverse displacement \( \Delta(x, t) \) was small and equal for all points of a cross-section;
3. The beam followed the Euler-Bernoulli beam theory;
4. Longitudinal displacement of the beam was abandoned;
5. Stable-one dimensional variable temperature grade was assumed to be present along the centroidal axis of the beam and the variation of temperature grade along vertical direction was neglected;
6. Extension and rotary inertia influences were insignificant.

The terminologies for total kinetic energy, total potential energy and work done were as follows:

\[
T = \frac{1}{2} \int_0^l \rho(A)(\frac{\partial \Delta}{\partial t})^2 \partial x + \frac{1}{2} \int_0^l \rho(A)N^2 \Delta^2 \partial x; \tag{1}
\]

\[
U = \frac{1}{2} \int_0^l T(x)I(x)(\frac{\partial \Delta}{\partial x})^2 \partial x + \frac{1}{2} \int_0^l \rho(A)N^2(B_0 + x)(\int_0^x (\frac{\partial \Delta}{\partial x})^2 \partial x); \tag{2}
\]

\[
W = \int_0^l W(t)(\frac{\partial \Delta}{\partial x})^2 \partial x. \tag{3}
\]

Hamilton’s principle was used to derive the boundary conditions and equation of motion as follows:

\[
\delta \int_{t_1}^{t_2} (T - U + W) = 0. \tag{4}
\]

Using the Eqs. (1), (2), and (3) in the Eq. (4), the equation of motion was obtained as:

\[
\begin{align*}
[E(x)I(x)\Delta_{x,x}]_{x,x} + \rho(A)\Delta_{x,t} + N^2I(x)\Delta_{x,x} - & V(x_1)\Delta_{x,x} + W(t)\Delta_{x,x} = 0; \tag{5}
\end{align*}
\]

In that case,

\[
V(x_1) = \frac{1}{2} \rho(A)N^2[(B_0 + l)^2 - (B_0 + x)^2]. \tag{6}
\]

The edge conditions at \( x' = B_0 \) and \( x' = B_0 + l \) were:

\[
\begin{align*}
\left[ E(x)I(x)\Delta_{x,x} \right]_{x|x=1} + W(t)\Delta_{x,x} = 0; \\
\text{or} \quad [E(x)I(x)\Delta_{x,x}]_{x=1} = 0; \\
\text{and} \quad \Delta_{x} = 0 \text{ or } \Delta = 0; \tag{7}
\end{align*}
\]

where \( \Delta_{x,x} = \frac{\partial \Delta}{\partial x} \), \( \Delta_{x,x} = \frac{\partial^2 \Delta}{\partial x^2} \), \( \Delta_{x,t} = \frac{\partial \Delta}{\partial t} \text{ and } \Delta_{tt} = \frac{\partial^2 \Delta}{\partial t^2} \).

Using the various dimensionless parameters, the dimensionless equation of motion was expressed as:

\[
\begin{align*}
[T(\eta)S(\eta)\zeta''(\eta) + m(\eta)\zeta''(\eta)] & + \{r_gN_0^2 + w(\tau)\zeta''(\eta) + N_2^2[g(\eta)\zeta''(\eta)] = 0; \tag{8}
\end{align*}
\]

where:

\[
\begin{align*}
r_g = \frac{I(\eta)}{A l^2}, \quad N_0^2 & = \frac{\rho(A)N^2 l^2}{E l^2}; \\
N_2^2 & = \frac{V(s)l^2}{E l^2}, \quad I(\eta) = I(\eta); \\
E(\eta) & = E_0S(\eta) \text{ and } A(\eta) = A_0m(\eta). \\
\end{align*}
\]

The boundary conditions were:

\[
\begin{align*}
[T(\eta)S(\eta)\zeta''(\eta)] + w(\tau)\zeta''(\eta) & = 0; \\
\text{or} \quad [T(\eta)S(\eta)\zeta''(\eta)]_{\eta=1} = 0; \\
\text{and} \quad \zeta(0, \tau) = 0 \text{ or } \zeta'(0, \tau) = 0. \tag{9}
\end{align*}
\]

In order to obtain the Eqs. (8) & (9), the non-dimensional parameters used were:

\[
\eta = \frac{x}{l}, \quad \zeta = \frac{\Delta}{l}, \quad \tau = \frac{ct}{l};
\]

where \( c^2 = \frac{E(x)I(x)}{\rho(A)x^4} \), \( \frac{\partial \Delta}{\partial x} = \frac{\partial \zeta}{\partial \eta} \), \( \frac{\partial^2 \Delta}{\partial x^2} = \frac{\partial^2 \zeta}{\partial \eta^2} \); \( \frac{\partial^2 \zeta}{\partial \eta^2} = \frac{1}{c^2} \left( \frac{\partial \zeta}{\partial \eta} \right)^2 \); \( \frac{\partial^2 \zeta}{\partial \eta \partial t} = c \left( \frac{\partial \zeta}{\partial \eta} \right) \left( \frac{\partial \zeta}{\partial \eta} \right) \); \( w(x) = \frac{W(t)}{E l^2} \); \( w(t) = w_0 + w_1 \cos(\tau t) \) and \( b_0 = \frac{B_0}{l} \) etc.

2.2. Series Solution to the Equation of Motion

The inexact solution could be presumed as:

\[
\zeta(\eta, \tau) = \sum_{i=1}^{\infty} \zeta_i(\eta)s_i(\tau). \tag{10}
\]

In this case, we had to choose the function of time \( s_i(\tau) \) and the coordinate function \( \zeta_i(\eta) \) in such a way that most of the edge conditions in Eq. (9) and the equation of motion would be satisfied. It was further anticipated that \( \zeta_i(\eta) \) could be characterized by using a number of functions (Eq. (10)), which must fulfill the conditions acquired from Eq. (8) by cancelling the expressions comprising of \( w_0 \) and \( w(\tau) \). The coordinate function for the different edge conditions was estimated by the expressions specified in the Table 1.26

The subsequent matrix equation of motion in Eq. (11) was obtained by applying the Galerkin’s method and replacing a series of solutions in the non-dimensional equation:

\[
[M] \{\ddot{s}\} + [K]\{s\} - \left\{ \begin{array}{c} w_0[H] - w_1 \cos(\tau t)[H] \end{array} \right\} \{s\} = \{0\}. \tag{11}
\]

In the Eq. (11), \( \ddot{s} = \frac{\partial^2 \zeta}{\partial \tau^2} \), \( \{s\} = \{s_1 \ldots s_n\}^T \) and the different matrix coefficients could be written as:

\[
\begin{align*}
M_{ij} & = \int_0^l \delta(\eta)\zeta_i(\eta)\zeta_j(\eta) \partial \eta; \\
K_{ij} & = \int_0^l \left[ \frac{T(\eta)S(\eta)\zeta_i''(\eta)\zeta_j''(\eta)}{N_2(\eta)} \right] \partial \eta; \\
H_{ij} & = \int_0^l \{\zeta_i'(\eta)\zeta_j'(\eta)\} \partial \eta \text{ for } i, j = 1, 2, 3, \ldots, N.
\end{align*}
\]
2.3. Formulation for Static Buckling Load

By substituting \( \{\bar{s}\} = \{0\} \) and \( w_1 = 0 \) in Eq. (11), we had an eigenvalue problem of:

\[
[K]^{-1} [H] \{s\} = \frac{1}{w_0} \{s\}. \tag{12}
\]

The static buckling loads \( w_0 \) for the first few modes were found as the real parts of the reciprocal of the eigenvalues of \([K]^{-1} [H]\). Then a number of graphs were plotted between \( w_0 \) and the rotational speed, taper parameter, and temperature grade.

2.4. Formulation for Dynamic Instability Regions

By considering \([L]\) to be the modal matrix of \([M]^{-1}[K]\), and by introducing the linear coordinate conversion \( \{s\} = [L]\{u\} \), we had:

\[
\{\bar{u}\} + [f_n^2]\{u\} + w_1 \cos(\bar{T}\tau)[B]\{u\} = \{0\}. \tag{13}
\]

In this case, \( \{u\} \) was a number of new generalized coordinates, \( [f_n^2] \) was a special matrix analogous to \([M]^{-1}[K]\) and \( [B] = -[L]^{-1}[M]^{-1} [H][L] \). The Eq. (13) could be expressed as:

\[
\bar{u}_n + f_n^2 + w_1 \cos(\bar{T}\tau) \sum_{m=1}^N b_{mn} u_m = 0; \tag{14}
\]

where \( n = 1, 2, 3, ..., N \).

The Eq. (14) characterized a system of \( N \) that combined Hill’s equations with complex factors. The complex terms are:

\[
f_n = f_{n,R} + j f_{n,I}; \quad b_{n,m} = b_{n,m,R} + j b_{n,m,I}.\]

Using the Saito-Otomi conditions, the constraints of the instability regions for simple and combination resonances were obtained for an undamped case.  

Table 1.

<table>
<thead>
<tr>
<th>SN</th>
<th>End condition</th>
<th>Coordinate functions for ( j = 1, 2, 3, ..., r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Pinned-Pinned</td>
<td>( \zeta(\eta) = \sin(\pi \eta) )</td>
</tr>
<tr>
<td>2</td>
<td>Clamped-Pinned</td>
<td>( \zeta(\eta) = 2(j + 2)(j+1) \eta^{j+1} - (4j + 6) \eta^j + 2(j+1) \eta^j )</td>
</tr>
<tr>
<td>3</td>
<td>Clamped-Clamped</td>
<td>( \zeta(\eta) = \eta^{j+1} + 2 \eta^j \eta + \eta^j )</td>
</tr>
</tbody>
</table>

1. Simple resonance

\[
\left| \frac{T}{2} - \bar{T}_{\mu,R} \right| < \frac{1}{4} \frac{w_1 b_{\mu,R}}{\bar{T}_{\mu,R}}; \tag{15}
\]

2. Combination resonance of sum type \((\nu \neq \mu)\)

\[
\left| \frac{T}{2} - \frac{1}{2} (\bar{T}_{\mu,R} + \bar{T}_{\nu,R}) \right| < \frac{w_1}{4} \sqrt{\frac{b_{\mu,R} b_{\nu,R}}{\bar{T}_{\mu,R} \bar{T}_{\nu,R}}}; \tag{16}
\]

3. Combination resonance of subtraction type

\[
\left| \frac{T}{2} - \frac{1}{2} (\bar{T}_{\nu,R} - \bar{T}_{\mu,R}) \right| < \frac{w_1}{4} \sqrt{\frac{b_{\mu,R} b_{\nu,R}}{\bar{T}_{\mu,R} \bar{T}_{\nu,R}}}. \tag{17}
\]

The system stability was investigated for the various boundary conditions with the help of Eqs. (15), (16), and (17). Then the regions of instability were analysed.

3. RESULTS AND OBSERVATION

The diameter of the exponentially tapered beam having circular cross-section is supposed to change, agreeing to the expression \( d(\eta) = D(e^{-\beta \eta}) \). In this case, \( D \) is base diameter and \( \beta \) is variation parameter. Subsequently, the mass distribution is \( m(\eta) = e^{-2\beta \eta} \) and \( A(\eta) = A * m(\eta) \). The temperature variation is supposed to follow the relation \( \delta(\eta) = \delta(1 - \eta) \). Considering \( \delta \) as the reference temperature, which is the temperature at the base, variation of the modulus of elasticity will follow the relation, \( E(\eta) = E[1 - \psi(\eta)(1 - \eta)] \), \( E(\eta) = E * S(\eta), 0 \leq \alpha \delta \leq 1 \), where \( \psi = \alpha \delta \) is the thermal grade parameter and \( \psi(\eta) = \psi/e^{-2\beta \eta} \). Numerical results are acquired for different values of the parameters like revolution speed parameter, temperature grade, variation parameter and hub radius.
If a certain change in the values of the parameter leads to narrowing the instability regions shifts the regions towards the higher values of excitation frequencies or reduces the number of instability regions, then it can be concluded that the stability of system has improved. Otherwise, the stability is reduced.

3.1. Dynamic Stability Plot

In order to avoid the clumsiness of the figures, only two values of parameters are shown, but the trend remains same for other values of the parameters.

The increase in the variation parameter moves the instability areas of a beam with clamped-pinned (C-P), clamped-clamped (C-C), and pinned-pinned (P-P) conditions toward a lower frequency region, as shown in Figs. 2, 3, and 4.

For all the three end arrangements, a surge in the revolving speed parameter reduces the width of the instability areas significantly and moves those to a higher excitation frequency (\( \bar{f} \)) region, as shown in Figs. 5, 6, and 7.

For all three end conditions, it is found that the instability of the system is more in the case of a variable temperature grade in comparison to a constant temperature grade, as shown in Figs. 8, 9, and 10.

Further in the dynamic analysis of the system, it is found that the stability of the system is more in a C-C case than that of a C-P and P-P case for all the parameters considered.

Change in the hub radius has no effect on the instability regions for all three edge conditions because the change in the hub radius does not affect the centre of gravity of the hub, which always lies on the central axis of revolution and hence, the stabilizing centrifugal force is mainly due to the rotating beam.

3.2. Static Stability Plot

It is observed that a surge in the revolving speed parameter increases the static load factor, as shown in Fig. 11.

The rise in the variation parameter as well as the thermal grade parameter decreases the static load factor for all three situations, as shown in Figs. 12 and 13.

A number of graphs were plotted for different edge arrangements and different modes of frequency by solving Eq. (13)
using the Runge-Kutta fourth-order method, with the help of the MATLAB program.

Figure 14 shows the variation of amplitude with respect to time. When the frequency is chosen from the instability zone (from first mode \((2f_1)\) of Fig. 3), the amplitude continues increasing.

When the frequency is chosen from outside the instability area (from first mode \((2f_1)\) of Fig. 3), the amplitude continues decreasing, as shown in Figure 15.

4. CONCLUSIONS

The static and dynamic stability of an exponentially tapered revolving beam having a circular cross-section is exposed to an axial live excitation and constant as well as a variable temperature grade under several edge arrangements has been investigated computationally by developing a MATLAB code. The instability areas are analysed using the Runge-Kutta fourth-order method, which certified the results to be true. The investigation leads to the following conclusions.
For all end arrangements, an increase in the revolving speed parameter makes the beam more stable due to an increase in the centrifugal force which decreases the effect of the external load but the stability of clamped-clamped beam increases more rapidly than clamped-pinned and pinned-clamped beams. The increase in the variation parameter leads to a decrease in the stability of the lower excitation frequency regions less significantly than the higher excitation regions, but the reverse occurs in the case of diverging beams due to an increase of the variation parameter; the rate of decrease of flexural rigidity is less for lower excitation frequency regions and the rate of decrease is more for higher frequency regions. For a certain temperature grade parameter, stability decreases if it is taken as a constant, and again it is further decreases if taken as a variable for a converging taper; this is due to a decrease of stiffness for the variable temperature grade in comparison to a constant temperature grade. It is also concluded that the hub radius has no effect on the dynamic instability. The static load factor increases with an increase in the rotational speed. The increase in the temperature grade along the positive X-axis decreases the stiffness of the system and an increase in the variation parameter along a positive X-axis decreases the flexural rigidity of the system. Due to these two reasons, the static load factors decrease with an increase in the temperature grade as well as in the variation parameter. This research can be useful for the vibration isolation of a rotating non-uniform beams with a high surrounding temperature and moderate rotational speeds as well as the design of rotor blades with a high strength to weight ratio, by choosing the suitable parameters obtained from the computational analysis.

REFERENCES

4. Horway, G. Chord wise and beam wise bending frequencies


