Exact Closed-Form Solution of Vibrations of a Generally Restrained Circular Plate with Crack and Weakened Along an Internal Concentric Circle

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A thin circular plate with generally restrained periphery and weakened along an internal concentric circle due to a crack, is considered in this paper for studying its vibration characteristics. The frequency for first six modes of plate vibration is computed for varying values of radius of crack, elastic rotational and translational restraints, and the extent of weakening duly simulated by considering the crack as a rotational restraint on the plate. From the results obtained, it is observed that for a plate with translation and rotational restraints, the internal weakening decreases the fundamental frequency by around 31 per cent.

NOMENCLATURE

K_{T1}	Translational spring stiffness at outer periphery;
K_{R1}	Rotational spring stiffness at outer periphery;
K_{R2}	Rotational spring stiffness at the cracked region;
T_{11}	Non-dimensional translational Flexibility parameter at the outer edge;
R_{11}	Non-dimensional rotational flexibility parameter at the outer edge;
R_{22}	Non-dimensional rotational flexibility parameter at the cracked region;
k	Non-dimensional frequency parameter;

1. INTRODUCTION

Many structural elements are composed of circular plates in aeronautical, civil, mechanical, and marine applications. The problem of determination of vibration characteristics of circular plates is basic to engineering design.[1–4] Several researchers reviewed the literature on vibrations of circular plates with basic edge conditions and internal strengthening.[5–9] Detection of structural damage through analytical and experimental investigations of vibration characteristics of cracked plates has become essential for solving design analysis problems of various types of mechanical systems. The natural frequencies and mode shapes of cracked elastic circular plates considerably differ from their healthy counterparts. In this respect, Dimarogonas has conducted a comprehensive literature search regarding the effects of cracks on the vibrations of various types and shapes of structures.[10] Papadopoulos has briefly described the history of the strain energy release rate (SERR) theory as well as different methods of crack identification.[11] Broda et al. have discussed various models of classical and non-classical crack- induced elastic, thermo-elastic, and dissipative nonlinearities.[12]

Lynn and Kumbasar investigated the problem of vibrations of cracked rectangular plates by presenting the solution for the Fredholm integral equation of the first kind and calculating numerically the drop in the natural frequency of vibration of plates due to the presence of cracks.[13] Petyt investigated experimentally the variation of fundamental frequency of vibration as the crack length changes and, the results were verified against analytical ones using finite element method.[14] Stahl and Keer, Hirano and Okazaki, Solecki and Yuan, and Dickinson studied further on vibrations of cracked plates using different methods of analysis.[15–18] Huang and Ma studied the problem of vibrations of circular plate with a radial crack using an optical measurement system known as the AF-ESPI method.[19]

Liew et al.[20] studied the problem of out-of-plane vibrations of cracked plates utilizing the domain decomposition method, confirming the results presented by Stahl and Keer, Hirano and Okazaki, and presented results for a wide range of crack length ratios.[15,16] Further, they examined the vibrations of a plate having a centrally located internal crack and reported results of frequency crossings. Ma and Huang recently studied the problem of vibrations of a square plate with an edge crack utilizing both experimental and finite element analysis.[21] They found that the variations in crack length to be having considerable influence on the natural frequencies and mode shapes of the plate. Si at el. recently studied the free vibrations of circular plates with radial side cracks considering the presence of water on one side utilizing Rayleigh-Ritz method.[22]

By the integration of stress intensity factors, the stiffness matrix was derived for the cracked plate by Qian et al. for carrying out the finite element analysis.[23] Utilizing Rayleigh's method and including effects of shear deformation and rotary inertia, Lee and Lim presented results for the natural frequencies and mode shapes of thick rectangular plate with a centrally located crack.[24] Krawczuk studied the influence of the crack location and crack length on the natural frequencies of both simply supported and cantilever rectangular plates.[25] The dynamic behaviour of cracked rectangular plates was investigated by Liew et al. analysing the free vibrations of rectangular plates either with a crack emanating from an edge or that which is centrally located.[26] Khadem and Rezaee carried out vibration analysis for crack detection in a rectangular plate subjected to uniform external loads.[27] Krawczuk et al. studied the fracture mechanics of a plate with an elasto-plastic through crack by using the finite element method.[28]

Based on Rayleigh's principle, Lee proposed a simple numerical method for computing the first natural frequencies of an annular plate with an internal concentric crack and applied the same for an annular plate with two opposite edges simply supported and the other two edges clamped.[29] The effect of the number and length of periodic radial cracks on the natural frequencies of an annular plate was investigated experimentally by Ramesh et al.[30] By modelling a surface peripheral crack as a local rotational flexibility, Anifantis et al. investigated the problem identifying free vibration characteristics of cracked annular plates.[31] Utilizing the Ritz method, Yuan et al. studied the influence of radial or circumferential cracks or slits through the full thickness on the natural frequencies of free vibration of circular and annular plates.[32] By using the optimum number of sector plate elements and joining them together with artificial spring elements, they obtained the flexibility matrix of sector-type element with radial crack and proved the applicability of the derived element in the dynamic analysis of annular plates with cracks eventually comparing the results with the experimental ones available in the literature.

Very few studies[33–35] exist in the literature on the vibrations of circular plate weakened along a concentric circle due to crack where Wang[33] and Yu[34] considered the basic boundary conditions and Bhaskara Rao and Kameswara Rao[35] considered an elastically restrained edge against translation. Due to internal notching, partial crack, or fatigue crack along a concentric circle, the plate may become weak in its bending strength. A hinge with an elastic rotational restraint can be considered to model the weakened position. In realistic engineering circumstances, a perfect boundary condition is hardly present. Therefore, when the boundary departs from such realistic situation, an elastically restrained edge must be considered.[36–39] Main intention of this work is therefore to study the effect of weakening of thin plate along concentric circle due to crack and the plate being elastically restrained along the outer edge against translation and rotation using exact method of solution approach. The natural frequencies of circular plate for varying values of translation restraint and rotational restraint along the plate periphery, along with the variations in the radius of weakened circle, Poisson's ratio and rotational restraint with hinge of cracked region are obtained for a wide range of non-dimensional parameters. The results are expressed in graphical and tabular formats for ease of use in understanding the design of such cracked and weakened circular plates in engineering industry.

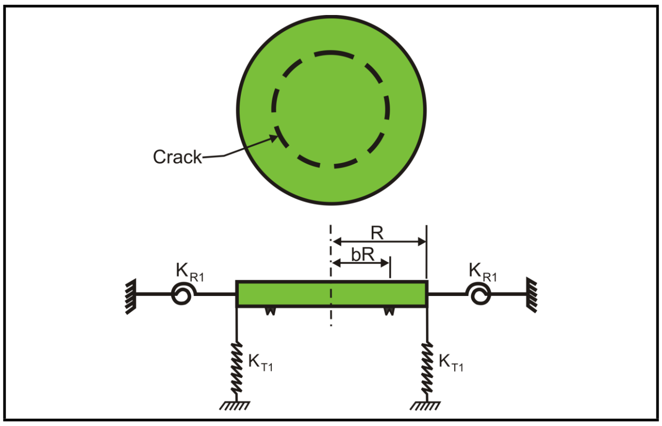

Figure 1. Generally restrained circular plate with crack.

2. MATHEMATICAL FORMULATION

The circular plate under consideration is of radius R, Poisson's ratio ν, density ρ, thickness h, and elastic constant E. Figure 1 is shows a circular plate that has an outer periphery, a generally restrained edge (at radius R), and an edge weakened along an internal concentric circle (at radius bR). Here, b is a fraction and is less than 1.

Here, all lengths are normalized with respect to R i.e., the radius of outer region is 1 and radius of inner cracked region is b. Here, r designates the distance measured from the centre of plate whose maximum value is R. Here subscript I represents outer region $b \le r \le 1$ and subscript II represents inner region $0 \le r \le b$. General form of lateral displacement of vibration of classical thin plate can be expressed as[2]

$$w = u(r)\cos(n\theta)e^{i\omega t}; \tag{1}$$

where n is the number of modal diameters, ω is the frequency, w is the transverse displacement, and t is time.

The function $u(r)$ is a linear combination of Bessel functions $J_n(kr)$, $Y_n(kr)$, $I_n(kr)$, $K_n(kr)$ and $k = R(\rho\omega^2/D)^{1/4}$; here D is flexural rigidity and k is the square root of the non-dimensional frequency.[3] General solutions for regions I & II are

$$u_I(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr); \tag{2}$$

$$u_{II}(r) = C_5 J_n(kr) + C_6 I_n(kr). \tag{3}$$

Considering the generally restrained edge at the outer periphery, the boundary conditions can be formulated as

$$M_r(r,\theta) = K_{R1}\frac{\partial w_I(r,\theta)}{\partial r}; \tag{4}$$

$$V_r(r,\theta) = -K_{T1} w_I(r,\theta); \tag{5}$$

where K_{R1} and K_{T1} are, respectively, the rotational and linear spring stiffness of the elastically restrained circular plate boundary. Here, bending moment $(M_r(r,\theta))$ & Kelvin-Kirchhoff's $(V_r(r,\theta))$ shear force can be represented as

$$M_r(r,\theta) = -\frac{D}{R}\left[\frac{\partial^2 w_I(r,\theta)}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial w_I(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_I(r,\theta)}{\partial \theta^2}\right)\right]; \quad (6)$$

$$V_r(r,\theta) = -\frac{D}{R^3}\left[\frac{\partial}{\partial r}\nabla^2 w_I(r,\theta) + (1-\nu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 w_I(r,\theta)}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_I(r,\theta)}{\partial \theta}\right)\right]; \quad (7)$$

From Eqs. (4), (6) and (5), Eq. (7) yields the following expressions

$$\left[\frac{\partial^2 w_I(r,\theta)}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial w_I(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_I(r,\theta)}{\partial \theta^2}\right)\right] = -R_{11}\frac{\partial w_I(r,\theta)}{\partial r}; \quad (8)$$

$$\left[\frac{\partial}{\partial r}\nabla^2 w_I(r,\theta) + (1-\nu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 w_I(r,\theta)}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w_I(r,\theta)}{\partial \theta}\right)\right] = T_{11} w_I(r,\theta); \quad (9)$$

Equations (8) and (9) can be presented as

$$\ddot{u_I}(r) + \nu\left[\dot{u_I}(r) - n^2 u_I(r)\right] = -R_{11}\dot{u_I}(r); \quad (10)$$

$$\dddot{u_I} + \ddot{u_I}(r) - \left[1 + n^2(2-\nu)\right]\dot{u_I}(r) + n^2(3-\nu)u_I(r) = -T_{11}u_I(r); \quad (11)$$

where $R_{11} = \frac{K_{R1}R}{D}$ & $T_{11} = \frac{K_{T1}R}{D}$ are, respectively, the non-dimensional rotational and translational spring parameters involving the springs constants K_{R1} & K_{T1}, which are the elastic spring stiffnesses simulating the elastic restraints at the circular plate outer periphery.

Apart from the generally restrained edge at the outer periphery, the continuity requirements[33] at $r = b$ are as follows

$$u_I(b) = u_{II}(b); \quad (12)$$

$$b\ddot{u_I}(b) + \nu\dot{u_I}(b) = b\ddot{u_{II}}(b) + \nu\dot{u_{II}}(b); \quad (13)$$

$$b^2\dddot{u_I}(b) - \left[1 + n^2(2-\nu) + \nu\right]\dot{u_I}(b) = b^2\dddot{u_{II}}(b) - \left[1 + n^2(2-\nu) + \nu\right]\dot{u_{II}}(b); \quad (14)$$

$$b^2\ddot{u_{II}}(b) + \nu\left[b\dot{u_{II}}(b) - n^2 u_{II}(b)\right] = b^2 R_{22}\left[\dot{u_I}(b) - \dot{u_{II}}(b)\right]; \quad (15)$$

where $R_{22} = \frac{K_{R2}R}{D}$ is the normalized spring constant with K_{R2} being the rotational spring stiffness, which is utilized for modelling the rotational restraint, created by the presence of circular crack at $r = b$. The non-trivial solutions to Eqs. (10)–(15) are sought. From Eqs. (2), (3), and (10)–(15) we derive the following equations.

$$\left[\frac{k^2}{4}P_2 + \frac{k}{2}(\nu + R_{11})P_1 - \left(\frac{k^2}{2} + \nu n^2\right)J_n(k)\right]C_1 + \left[\frac{k^2}{4}Q_2 + \frac{k}{2}(\nu + R_{11})Q_1 - \left(\frac{k^2}{2} + \nu n^2\right)Y_n(k)\right]C_2 + \left[\frac{k^2}{4}R_2 + \frac{k}{2}(\nu + R_{11})R_1 + \left(\frac{k^2}{2} - \nu n^2\right)I_n(k)\right]C_3 - \left[\frac{k^2}{4}S_2 - \frac{k}{2}(\nu + R_{11})S_1 + \left(\frac{k^2}{2} - \nu n^2\right)K_n(k)\right]C_4 = 0; \quad (16)$$

$$\left[\frac{k^3}{8}P_3 + \frac{k^2}{4}P_2 - \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)P_1 + \left(n^2(3-\nu) - \frac{k^2}{2} - T_{11}\right)J_n(k)\right]C_1 + \left[\frac{k^3}{8}Q_3 + \frac{k^2}{4}Q_2 - \frac{k}{2}\left(\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)Q_1 + \left(n^2(3-\nu) - \frac{k^2}{2} - T_{11}\right)Y_n(k)\right]C_2 + \left[\frac{k^3}{8}R_3 + \frac{k^2}{4}R_2 + \frac{k}{2}\left(\frac{3}{4}k^2 - n^2(2-\nu) + 1\right)R_1 + \left(n^2(3-\nu) + \frac{k^2}{2} - T_{11}\right)I_n(k)\right]C_3 + \left[-\frac{k^3}{8}S_3 + \frac{k^2}{4}S_2 + \frac{k}{2}\left(-\frac{3}{4}k^2 + n^2(2-\nu) + 1\right)S_1 + \left(n^2(3-\nu) + \frac{k^2}{2} - T_{11}\right)K_n(k)\right]C_4 = 0; \quad (17)$$

$$J_n(kb)C_1 + Y_n(kb)C_2 + I_n(kb)C_3 + K_n(kb)C_4 - J_n(kb)C_5 - I_n(kb)C_6 = 0; \quad (18)$$

$$\left[\frac{bk^2}{4}\dot{P}_2 + \frac{\nu k}{2}\dot{P}_1 - \frac{bk^2}{2}J_n(kb)\right]C_1 + \left[\frac{bk^2}{4}\dot{Q}_2 + \frac{\nu k}{2}\dot{Q}_1 - \frac{bk^2}{2}Y_n(kb)\right]C_2 + \left[\frac{bk^2}{4}\dot{R}_2 + \frac{\nu k}{2}\dot{R}_1 + \frac{bk^2}{2}I_n(kb)\right]C_3 + \left[\frac{bk^2}{4}\dot{S}_2 - \frac{\nu k}{2}\dot{S}_1 + \frac{bk^2}{2}K_n(kb)\right]C_4 - \left[\frac{bk^2}{4}\dot{P}_2 + \frac{\nu k}{2}\dot{P}_1 - \frac{bk^2}{2}J_n(kb)\right]C_5 - \left[\frac{bk^2}{4}\dot{R}_2 + \frac{\nu k}{2}\dot{R}_1 + \frac{bk^2}{2}I_n(kb)\right]C_6 = 0; \quad (19)$$

(See Eqs. (20) and (21) on the top of the next page.)

3. SOLUTION

Given the set of values of n, ν, T_{11}, R_{11}, R_{22}, and b, equations listed above are utilized in obtaining exact characteristics

$$\left[\frac{b^2k^3}{8}\dot{P}_3 - \frac{k}{2}\left(\frac{3b^2k^2}{4} + (1+n^2(2-\nu)+\nu)\right)\dot{P}_1\right]C_1 + \left[\frac{b^2k^3}{8}\dot{Q}_3 - \frac{k}{2}\left(\frac{3b^2k^2}{4}+(1+n^2(2-\nu)+\nu)\right)\dot{Q}_1\right]C_2 + \left[\frac{b^2k^3}{8}\dot{R}_3 + \frac{k}{2}\left(\frac{3b^2k^2}{4} - (1+n^2(2-\nu)+\nu)\right)\dot{R}_1\right]C_3 + \left[-\frac{b^2k^3}{8}\dot{S}_3 + \frac{k}{2}\left(-\frac{3b^2k^2}{4}+(1+n^2(2-\nu)+\nu)\right)\dot{S}_1\right]C_4 + \left[\frac{b^2k^3}{8}\dot{P}_3 + \frac{k}{2}\left(\frac{3b^2k^2}{4}+(1+n^2(2-\nu)+\nu)\right)\dot{P}_1\right]C_5 + \left[-\frac{b^2k^3}{8}\dot{R}_3+\frac{k}{2}\left(-\frac{3b^2k^2}{4}+(1+n^2(2-\nu)+\nu)\right)\dot{R}_1\right]C_6 = 0; \qquad (20)$$

$$\left[\frac{b^2kR_{22}}{2}\dot{P}_1\right]C_1 + \left[\frac{b^2kR_{22}}{2}\dot{Q}_1\right]C_2 + \left[\frac{b^2kR_{22}}{2}\dot{R}_1\right]C_3 - \left[\frac{b^2kR_{22}}{2}\dot{S}_1\right]C_4 - \left[\frac{b^2k^2}{4}\dot{P}_2 + \frac{kb}{2}(\nu+bR_{22})\dot{P}_1 - \left(\frac{b^2k^2}{2}+n^2\right)J_n(kb)\right]C_5 - \left[\frac{b^2k^2}{4}\dot{R}_2 + \frac{kb}{2}(\nu+bR_{22})\dot{R}_1 + \left(\frac{b^2k^2}{2}-n^2\right)I_n(kb)\right]C_6 = 0; \qquad (21)$$

where $P_1 = J_{n-1}(k) - J_{n+1}(k)$; $P_2 = J_{n-2}(k) + J_{n+2}(k)$; $P_3 = J_{n-3}(k) - J_{n+3}(k)$;
$Q_1 = Y_{n-1}(k) - Y_{n+1}(k)$; $Q_2 = Y_{n-2}(k) + Y_{n+2}(k)$; $Q_3 = Y_{n-3}(k) - Y_{n+3}(k)$;
$R_1 = I_{n-1}(k) + I_{n+1}(k)$; $R_2 = I_{n-2}(k) + I_{n+2}(k)$; $R_3 = I_{n-3}(k) + I_{n+3}(k)$;
$S_1 = K_{n-1}(k) + K_{n+1}(k)$; $S_2 = K_{n-2}(k) + K_{n+2}(k)$; $K_3 = K_{n-3}(k) + K_{n+3}(k)$;
$\dot{P}_1 = J_{n-1}(kb) - J_{n+1}(kb)$; $\dot{P}_2 = J_{n-2}(kb) + J_{n+2}(kb)$; $\dot{P}_3 = J_{n-3}(kb) - J_{n+3}(kb)$;
$\dot{Q}_1 = Y_{n-1}(kb) - Y_{n+1}(kb)$; $\dot{Q}_2 = Y_{n-2}(kb) + Y_{n+2}(kb)$; $\dot{Q}_3 = Y_{n-3}(kb) - Y_{n+3}(kb)$;
$\dot{R}_1 = I_{n-1}(kb) + I_{n+1}(kb)$; $\dot{R}_2 = I_{n-2}(kb) + I_{n+2}(kb)$; $\dot{R}_3 = I_{n-3}(kb) + I_{n+3}(kb)$;
$\dot{S}_1 = K_{n-1}(kb) + K_{n+1}(kb)$; $\dot{S}_2 = K_{n-2}(kb) + K_{n+2}(kb)$; $\dot{S}_3 = K_{n-3}(kb) + K_{n+3}(kb)$;

equations by suitably eliminating coefficients of C_1, C_2, C_3, C_4, C_5, and C_6. The values of non-dimensional frequency parameters k are obtained by solving the exact characteristic equation by utilizing a root search method based on bisection method and coding the same appropriately in MATHEMATICA.

4. RESULTS AND DISCUSSIONS

Poisson's ratio employed here is 0.3. The values (R_{11} and T_{11}) are chosen to cover both classical and nonclassical boundary conditions. Also, the values of rotational spring R_{22} are chosen to simulate the intensity of the crack appropriately. A smaller value of R_{22} represents that the crack is very small and higher value of R_{22} represents a concentric rigid ring support. Frequency values for various values of R_{22} keeping R_{11} and T_{11} constant $[R_{11} = T_{11} = 0.0001]$ are computed. The first frequencies $[k]$ of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 0.0001$ are computed. For $b = 1$ and $R_{22} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, the first frequency of $n = 0$, the modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 2.31479 $[n = 2]$, 3.00049 $[n = 0]$, 3.52684 $[n = 3]$, 4.52488 $[n = 1]$, 4.67279 $[n = 4]$, and 5.7874 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and generally restrained edge against translation and rotation occurs at $n = 2$ mode. The variation of fundamental frequency of plate for varying values of radius of weakened circle and rotational restraint parameter of hinge are given in Fig. 2, for $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 3.00049. For the remaining values of R_{22}, the frequency parameter decreases except when $b = 0$ or 1.

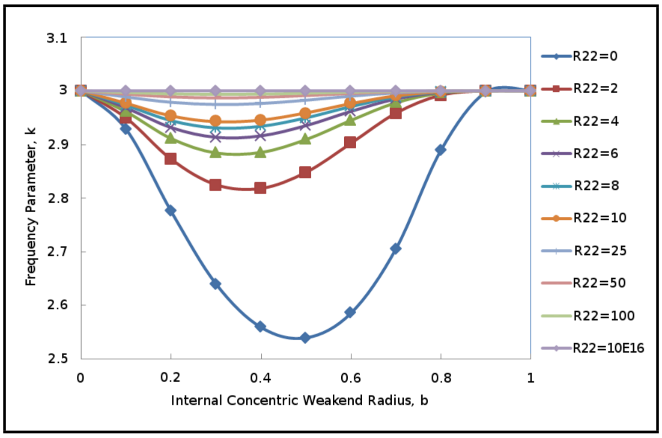

Figure 2. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 0.0001$, $\nu = 0.3$, and $n = 0$.

The frequency values for various values of R_{22} keeping R_{11} and T_{11} constant $[R_{11} = T_{11} = 2]$ are computed. First frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 2$ are computed. For $b = 1$ and $R_{22} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, the first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 1.39366 $[n = 0]$, 1.82795 $[n = 1]$, 2.73255 $[n = 2]$, 3.78917 $[n = 3]$, 4.86708 $[n = 4]$, and 5.94356 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at

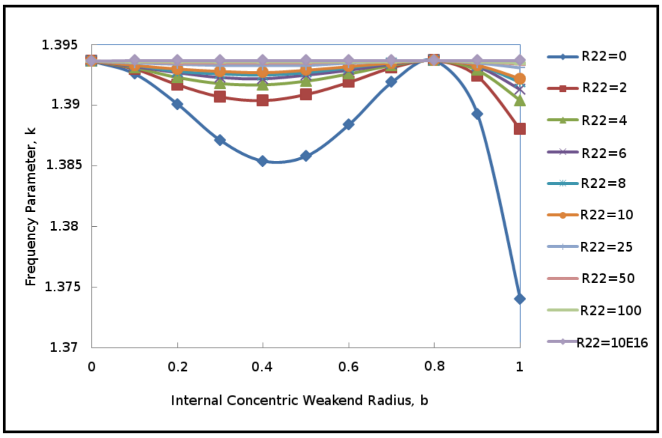

Figure 3. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 2$, $\nu = 0.3$, and $n = 0$.

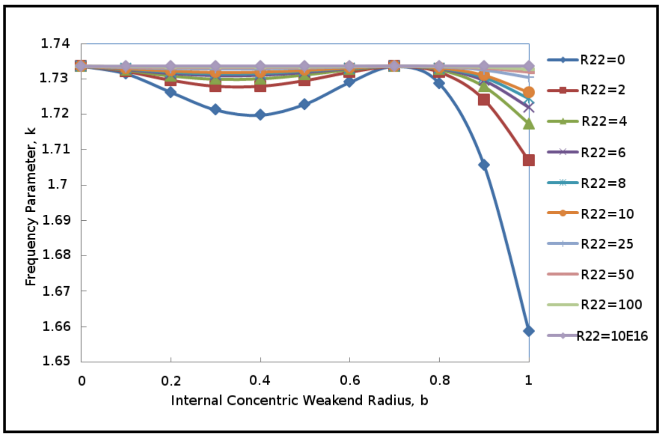

Figure 4. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 5$, $\nu = 0.3$, and $n = 0$.

$n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 1.39366. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 3. There is an optimum location within the plate.

Internal weakening decreases fundamental frequency 1.39366, which is the fundamental frequency of plate without the weakening by less than 1% (0.5948%). For a given value of R_{22}, the frequency k decreases from 1.39366 to 1.38537, increases to 1.39366, and finally decreases to 1.374 as the radius of the weakened circle varies from 0 to 1. The local maximum frequency 1.39366 occurs at $b = 0.8$. Thus $b = 0.8$ is the optimum radius if the plate needs to be notched, such as a closed hatch. The internal weakening has minute effect [decreases fundamental frequency by less than 1% (0.5948%)] on the fundamental frequency when $0 \leq b \leq 0.4$. Where as it has little effect on fundamental frequency [decreases the fundamental frequency by less than 2% (1.41%)] when $b > 0.8$.

The frequency values for various values of R_{22} keeping R_{11} and T_{11} constant $[R_{11} = T_{11} = 5]$ are computed. The first frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 10$ are computed. For $b = 1$ and R_{22} 0, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, the first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 1.73363 $[n = 0]$, 2.16554 $[n = 1]$, 2.97556 $[n = 2]$, 3.97353 $[n = 3]$, 5.01943 $[n = 4]$, and 6.07607 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 1.73363. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 4. There is an optimum location within the plate.

Internal weakening decreases the fundamental frequency 1.73363, which is fundamental frequency of the plate without the weakening by less than 1% (0.8%). For a given value of R_{22}, the frequency k decreases from 1.73363 to 1.71975, increases to 1.73363, and finally decreases to 1.65872 as radius b of weakened circle varies from 0 to 1. Here, local maximum frequency 1.73363 occurs at $b = 0.7$. Thus $b = 0.7$ is optimum radius if the plate needs to be notched, such as a closed hatch. Here, internal weakening has minute effect [decreases fundamental frequency by less than 1% (0.8%)] on fundamental frequency when $0 \leq b \leq 0.4$. Where as it has more effect on fundamental frequency [decreases the fundamental frequency by less than 5% (4.32%)] when $b > 0.7$.

The frequency values for various values of R_{22} keeping R_{11} and T_{11} constant $[R_{11} = T_{11} = 10]$ are tabulated. First frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 10$ are presented in Tables 1 to 6. For $b = 1$ and $R_{22} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 2.03159 $[n = 0]$, 2.46241 $[n = 1]$, 3.19071 $[n = 2]$, 4.13574 $[n = 3]$, 5.15511 $[n = 4]$ and 6.19719 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 2.03159. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 5. There is an optimum location within the plate.

Internal weakening decreases fundamental frequency 2.03159, which is fundamental frequency of plate without weakening by less than 9% (8.5647%). For a given value of R_{22} , the frequency k decreases from 2.03159 to 2.00913, increases to 2.03063 and finally decreases to 1.85759 as radius b of weakened circle varies from 0 to 1. Here, local maximum frequency 2.03063 occurs at $b = 0.7$. Thus $b = 0.7$ is the optimum radius if plate needs to be notched, such as a closed hatch. The internal weakening has little effect [decreases fundamental frequency by less than 1% (1.11%)] on fundamental frequency when $0 \leq b \leq 0.4$. Where as it has more effect on fundamental frequency [decreases the fundamental frequency by less than 9% (8.5647%)] when $b > 0.7$.

The frequency values for various values of R_{22} keeping R_{11} & T_{11} constant $[R_{11} = T_{11} = 50]$ are computed. First frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 10$ are computed. For $b = 1$

Table 1. First frequency parameter, ($n = 0$) for different R_{22} and internal concentric weakened radius parameter, b , $R_{11} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{22} \to 0$	2	4	6	8	10	25	50	100	10^6
0	2.03159	2.03159	2.03159	2.03159	2.03159	2.03159	2.03159	2.03159	2.03159	2.03159
0.1	2.02739	2.02869	2.02929	2.02979	2.02999	2.03029	2.03089	2.03119	2.03139	2.03159
0.2	2.0178	2.024	2.02639	2.0276	2.02839	2.02889	2.03039	2.03089	2.0312	2.03159
0.3	2.00962	2.0214	2.025	2.0267	2.0277	2.0284	2.03019	2.0308	2.03119	2.03159
0.4	2.00913	2.02221	2.0257	2.0273	2.0282	2.0288	2.03039	2.0309	2.03129	2.03159
0.5	2.01691	2.0259	2.028	2.029	2.0296	2.0299	2.03089	2.03119	2.03139	2.03159
0.6	2.02778	2.03019	2.03069	2.03099	2.03109	2.03119	2.03139	2.03149	2.03149	2.03159
0.7	2.03063	2.03127	2.03138	2.03139	2.03149	2.03149	2.03149	2.03159	2.03159	2.03159
0.8	2.00906	2.02368	2.02678	2.02809	2.02889	2.02939	2.03059	2.03109	2.03129	2.03159
0.9	1.94995	2.00144	2.01312	2.01831	2.02121	2.02301	2.0279	2.0297	2.0306	2.03159
1	1.85759	1.96079	1.98721	1.99928	2.00616	2.01065	2.02252	2.02691	2.0292	2.03159

Table 2. First frequency parameter, ($n = 1$) for different R_{22} and internal concentric weakened radius parameter b, $R_{11} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{22} \to 0$	2	4	6	8	10	25	50	100	10^6
0	2.46239	2.46239	2.46239	2.46239	2.46239	2.46239	2.46239	2.46239	2.46239	2.46241
0.1	2.44903	2.45442	2.45672	2.45802	2.45882	2.45932	2.46101	2.46161	2.46201	2.46241
0.2	2.4067	2.43805	2.44684	2.45093	2.45333	2.45492	2.45912	2.46071	2.46151	2.46241
0.3	2.33785	2.41649	2.43436	2.44215	2.44654	2.44943	2.45682	2.45952	2.46092	2.46241
0.4	2.26443	2.39335	2.42069	2.43247	2.43906	2.44335	2.45433	2.45822	2.46032	2.46241
0.5	2.21468	2.37421	2.40883	2.4239	2.43238	2.43776	2.45193	2.45702	2.45972	2.46241
0.6	2.20233	2.36444	2.40195	2.41871	2.42819	2.43428	2.45034	2.45623	2.45932	2.46241
0.7	2.22629	2.36744	2.40286	2.41902	2.4283	2.43428	2.45024	2.45613	2.45922	2.46241
0.8	2.27925	2.384	2.41253	2.4258	2.43348	2.43857	2.45204	2.45703	2.45972	2.46241
0.9	2.35036	2.41172	2.42968	2.43816	2.44325	2.44654	2.45543	2.45882	2.46052	2.46241
1	2.42166	2.44274	2.44943	2.45272	2.45472	2.45602	2.45952	2.46091	2.46161	2.46241

Table 3. First frequency parameter, ($n = 2$) for different R_{22} and internal concentric weakened radius parameter b, $R_{11} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{22} \to 0$	2	4	6	8	10	25	50	100	10^6
0	3.1908	3.1908	3.1908	3.1908	3.1908	3.1908	3.1908	3.1908	3.1908	3.1908
0.1	3.1948	3.1939	3.19331	3.19291	3.19261	3.19241	3.19161	3.19121	3.19101	3.19071
0.2	3.20557	3.20009	3.19759	3.1962	3.1952	3.1945	3.19251	3.19171	3.19121	3.19071
0.3	3.21492	3.20346	3.19938	3.19729	3.19599	3.1952	3.1927	3.19181	3.19131	3.19071
0.4	3.19851	3.19426	3.19298	3.19239	3.19209	3.1919	3.1912	3.19101	3.19091	3.19071
0.5	3.12095	3.16202	3.17262	3.17752	3.18041	3.18221	3.18701	3.18881	3.18981	3.19071
0.6	2.98278	3.10605	3.1376	3.15208	3.16036	3.16565	3.17993	3.18522	3.18792	3.19071
0.7	2.83663	3.04051	3.0953	3.12085	3.13562	3.1452	3.17105	3.18063	3.18562	3.19071
0.8	2.73415	2.98644	3.05879	3.09331	3.11347	3.12674	3.16277	3.17634	3.18343	3.19071
0.9	2.7072	2.96328	3.04172	3.07984	3.10249	3.11736	3.15838	3.17404	3.18223	3.19071
1	2.77645	2.98702	3.05558	3.08961	3.10997	3.12345	3.16087	3.17524	3.18283	3.19071

Table 4. First frequency parameter, ($n = 3$) for different R_{22} and internal concentric weakened radius parameter b, $R_{11} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{22} \to 0$	2	4	6	8	10	25	50	100	10^6
0	4.13572	4.13572	4.13571	4.13572	4.13571	4.13572	4.13571	4.13572	4.13571	4.13572
0.1	4.13584	4.13584	4.13574	4.13574	4.13574	4.13574	4.13574	4.13574	4.13574	4.13574
0.2	4.13774	4.13724	4.13684	4.13664	4.13654	4.13644	4.13604	4.13594	4.13584	4.13574
0.3	4.14541	4.14182	4.14013	4.13923	4.13863	4.13813	4.13684	4.13634	4.13604	4.13574
0.4	4.15824	4.14818	4.1444	4.14231	4.14111	4.14022	4.13773	4.13674	4.13624	4.13574
0.5	4.15298	4.14426	4.14139	4.1399	4.13911	4.13851	4.13693	4.13634	4.13604	4.13574
0.6	4.07023	4.10553	4.11613	4.12123	4.12424	4.12614	4.13154	4.13354	4.13464	4.13574
0.7	3.88275	4.01772	4.05886	4.07873	4.09041	4.0982	4.11926	4.12715	4.13135	4.13574
0.8	3.65521	3.90077	3.98002	4.01925	4.04271	4.05828	4.1014	4.11787	4.12666	4.13574
0.9	3.46495	3.79616	3.90776	3.96405	3.99799	4.02075	4.08433	4.10899	4.12207	4.13574
1	3.38479	3.75423	3.8792	3.94249	3.98072	4.00627	4.07784	4.10559	4.12037	4.13574

Table 5. First frequency parameter, ($n = 4$) for different R_{22} and internal concentric weakened radius parameter b, $R_{11} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{22} \to 0$	2	4	6	8	10	25	50	100	10^6
0	5.1551	5.1551	5.1551	5.1551	5.1551	5.1551	5.1551	5.1551	5.1551	5.1551
0.1	5.15511	5.15511	5.15511	5.15511	5.15511	5.15511	5.15511	5.15511	5.15511	5.15511
0.2	5.15531	5.15531	5.15521	5.15521	5.15521	5.15521	5.15512	5.15511	5.15511	5.15511
0.3	5.1573	5.1567	5.15631	5.15611	5.15591	5.15581	5.15541	5.15531	5.15521	5.15511
0.4	5.16545	5.16158	5.15979	5.1588	5.1582	5.1577	5.15631	5.15581	5.15541	5.15511
0.5	5.17786	5.16793	5.16406	5.16197	5.16068	5.15978	5.1572	5.15621	5.15571	5.15511
0.6	5.15333	5.15422	5.15445	5.15467	5.15477	5.15478	5.155	5.15501	5.15511	5.15511
0.7	5.00234	5.07817	5.10375	5.11664	5.12434	5.12943	5.14372	5.14922	5.15212	5.15511
0.8	4.7276	4.92987	5.00226	5.0394	5.06206	5.07734	5.12017	5.13694	5.14583	5.15511
0.9	4.4342	4.76097	4.88337	4.94766	4.98739	5.01435	5.09122	5.12167	5.13794	5.15511
1	4.22661	4.65672	4.81367	4.89544	4.94555	4.9794	5.07574	5.11368	5.13385	5.15511

Table 6. First frequency parameter, ($n = 5$) for different R_{22} and internal concentric weakened radius parameter b, $R_{11} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{22} \to 0$	2	4	6	8	10	25	50	100	10^6
0	6.19718	6.19718	6.19718	6.19718	6.19718	6.19718	6.19718	6.19718	6.19718	6.19718
0.1	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719
0.2	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719	6.19719
0.3	6.19768	6.19748	6.19748	6.19739	6.19739	6.19739	6.19729	6.19729	6.19719	6.19719
0.4	6.20095	6.19976	6.19917	6.19877	6.19848	6.19838	6.19778	6.19749	6.19739	6.19719
0.5	6.21167	6.20622	6.20374	6.20235	6.20146	6.20076	6.19888	6.19808	6.19769	6.19719
0.6	6.21357	6.20646	6.2037	6.20212	6.20124	6.20055	6.19877	6.19798	6.19759	6.19719
0.7	6.11202	6.15145	6.16596	6.17347	6.17807	6.18117	6.18998	6.19349	6.19529	6.19719
0.8	5.8271	5.99107	6.05448	6.08814	6.10891	6.12309	6.16354	6.17961	6.1882	6.19719
0.9	5.4652	5.77154	5.89685	5.96505	6.00808	6.03754	6.1237	6.15845	6.17732	6.19719
1	5.17181	5.61856	5.79369	5.88735	5.94577	5.98571	6.10064	6.14657	6.17123	6.19719

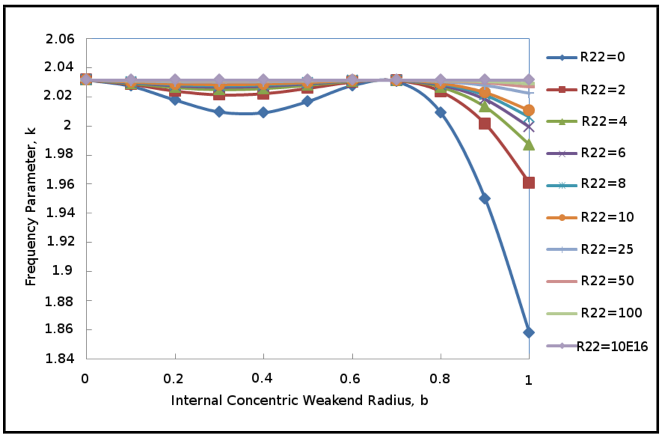

Figure 5. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 10$, $\nu = 0.3$, and $n = 0$.

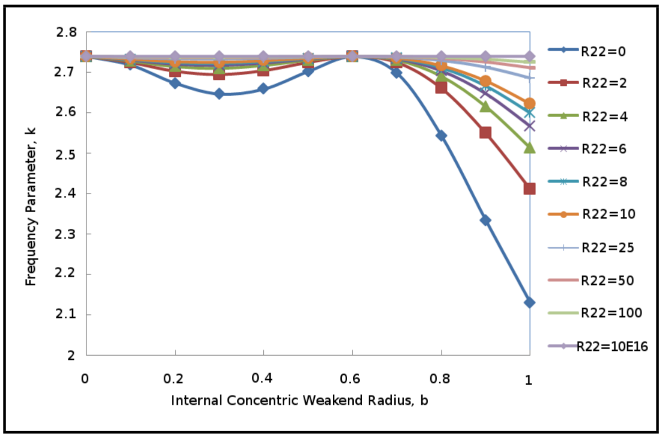

Figure 6. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 50$, $\nu = 0.3$, and $n = 0$.

and $R_{22} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 2.73971 [$n = 0$], 3.36961 [$n = 1$], 3.93897 [$n = 2$], 4.67971 [$n = 3$], 5.56452 [$n = 4$], and 6.52888 [$n = 5$]. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 2.73971. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 6. There is an optimum location within the plate.

Internal weakening decreases fundamental frequency 2.73971, which is fundamental frequency of the plate without weakening by less than 4% (3.4%). For a given value of R_{22}, the frequency k decreases from 2.73971 to 2.64629, increases to 2.73971 and finally decreases to 2.12973 as radius b of weakened circle varies from 0 to 1. Here, local maximum frequency 2.73971 occurs at $b = 0.6$. Thus, $b = 0.6$ is optimum radius if plate needs to be notched, such as a closed hatch. Here, the internal weakening has considerable effect [decreases the fundamental frequency by less than 4% (3.4%)] on fundamental frequency when $0 \leq b \leq 0.3$. Where as it has more effect on fundamental frequency [decreases the fundamental frequency by less than 23% (22.264%)] when $b > 0.6$.

The frequency values for various values of R_{22} keeping R_{11} & T_{11} constant [$R_{11} = T_{11} = 100$] are computed. First frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 100$ are computed. For $b = 1$ and $R_{22} = 0$, the frequency of plate is as that of plate without having weakening crack. For a given value of b & ν, the first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0.

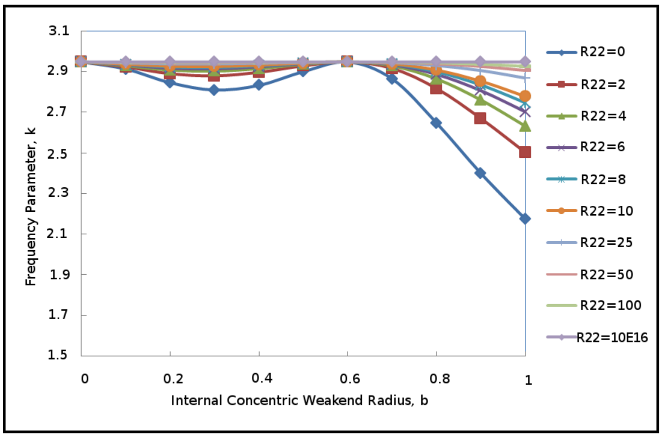

Figure 7. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 100$, $\nu = 0.3$, and $n = 0$.

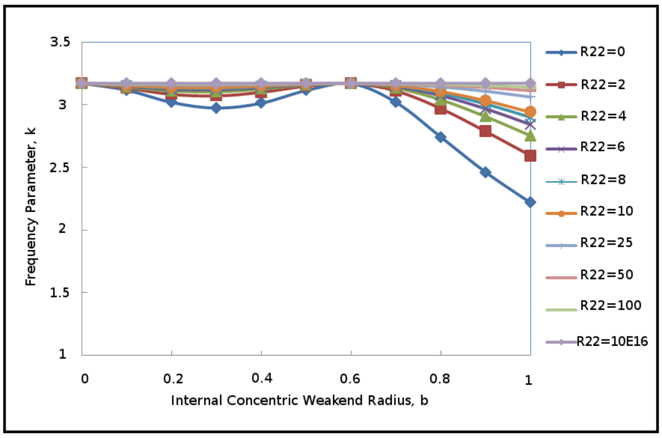

Figure 8. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 1000$, $\nu = 0.3$, and $n = 0$.

When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 2.94749 $[n = 0]$, 3.80332 $[n = 1]$, 4.41124 $[n = 2]$, 5.0759 $[n = 3]$, 5.86747 $[n = 4]$, and 6.75891 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 2.94749. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 7. There is an optimum location within the plate.

Here, the internal weakening decreases fundamental frequency 2.94749 that is fundamental frequency of plate without weakening by less than 5% (4.69%). For a given value of R_{22}, the frequency k decreases from 2.94749 to 2.80925, increases to 2.94749 and finally decreases to 2.17453 as radius b of weakened circle varies from 0 to 1. Here, local maximum frequency 2.94749 occurs at $b = 0.6$. Thus, $b = 0.6$ is optimum radius if the plate needs to be notched, such as a closed hatch. Here, internal weakening has considerable effect [decreases fundamental frequency by less than 5% (4.69%)] on fundamental frequency when $0 \leq b \leq 0.3$. Where as it has more effect on fundamental frequency [decreases the fundamental frequency by less than 27% (26.22%)] when $b > 0.6$.

The frequency values for various values of R_{22} keeping R_{11} & T_{11} constant $[R_{11} = T_{11} = 1000]$ are computed. First frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 1000$ are computed. For $b = 1$ and $R_{22} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are 3.17078 $[n = 0]$, 4.52162 $[n = 1]$, 5.68836 $[n = 2]$, 6.7132 $[n = 3]$, 7.61683 $[n = 4]$, and 8.43981 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 3.17078. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 8. There is an optimum location within the plate.

Internal weakening decreases fundamental frequency 3.17078, which is the fundamental frequency of plate without the weakening by less than 7% (6.22%). For a given value of R_{22}, the frequency k decreases from 3.17078 to 2.97341, increases to 3.17078 and finally decreases to 2.21675 as radius b of weakened circle varies from 0 to 1. Here, local maximum frequency 3.17078 occurs at $b = 0.6$. Thus $b = 0.6$ is optimum radius if a plate needs to be notched, such as a closed hatch. The internal weakening has significant effect [decreases the fundamental frequency by less than 7% (6.22%)] on fundamental frequency when $0 \leq b \leq 0.3$. Where as it has enormous effect on fundamental frequency [decreases the fundamental frequency by less than 31% (30.0856%)] when $b > 0.6$.

The frequency values for various values of R_{22} keeping R_{11} & T_{11} constant $[R_{11} = T_{11} = 100000]$ are computed. First frequencies of $n \leq 5$ modes with $R_{22} = 0$, 2, 4, 6, 8, 10, 25, 50, and 10^{16} and $R_{11} = T_{11} = 100000$ are computed. For $b = 1$ and $R_{22} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{22} is increased from value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are 3.19596 $[n = 0]$, 4.61 $[n = 1]$, 5.90356 $[n = 2]$, 7.13955 $[n = 3]$, 8.33993 $[n = 4]$, and 9.51523 $[n = 5]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on generally restrained edge against translation and rotation occurs at $n = 0$ mode. As $R_{22} \to \infty$ (as the spring becomes rigid), the frequency parameter stays at 3.19596. For the remaining values of R_{22}, the variation of frequency parameter is shown in Fig. 9. There is an optimum location within the plate.

Here, internal weakening decreases fundamental frequency 3.19596, which is the fundamental frequency of plate without weakening by less than 7% (6.39%). For a given value of R_{22}, the frequency k decreases from 3.19596 to 2.9916, increases to 3.19596 and finally decreases to 2.22145 as radius b of weakened circle varies from 0 to 1. Here, local maximum frequency 3.19596 occurs at $b = 0.6$. Thus $b = 0.6$ is optimum radius if the plate needs to be notched, such as a closed hatch. Here, internal weakening has significant effect [decreases fundamental frequency by less than 7% (6.39%)] on fundamental frequency

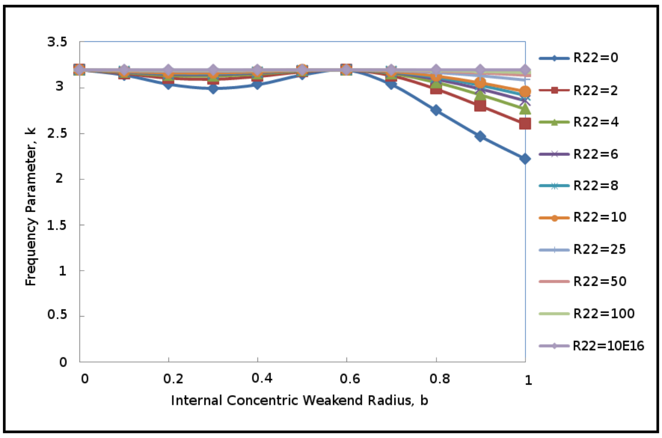

Figure 9. Fundamental frequency and concentric weakened radius parameter for different R_{22} and $R_{11} = T_{11} = 100000$, $\nu = 0.3$, and $n = 0$.

when $0 \leq b \leq 0.3$. Where as it has enormous effect on fundamental frequency [decreases the fundamental frequency by less than 31% (30.49%)] when $b > 0.6$.

A reduction in fundamental frequency increases from 1% (at lower values of R_{22} , i.e. $R_{22} = 2$) to 8% (at higher values of R_{22}, i.e. $R_{22} = 10^5$) through 4% (for $R_{22} = 50$), 5% (for $R_{22} = 100$), 7%(for $R_{22} = 1000$) at lower values of i nternal concentric weakened radius parameter, b. A reduction in fundamental frequency increases as rotational restraint parameter R_{22} increases.

A reduction in fundamental frequency increases from 2% (at lower values of R_{22}, i.e. $R_{22} = 2$) to 31% (at higher values of R_{22}, i.e. $R_{22} = 10^5$) through 23% (for $R_{22} = 50$), 27% (for $R_{22} = 100$), 31%(for $R_{22} = 1000$) at higher values of i nternal concentric weakened radius parameter, b. A reduction in fundamental frequency increases with increase in rotational spring stiffness parameter R_{22}. The rotational spring stiffness parameter R_{22} has immense influence on percentage decrease in fundamental frequency parameter, at higher values of internal concentric weakened radius parameter b that greatly decreases the fundamental frequency.

In all the cases discussed above, when $b = 1$, the structure is corresponding to circular plate with elastic edge. Fundamental frequencies for $R_{22} = 0$ case, which impersonates a frictionless hinge/trough circular crack, are shown in the Table 7.

The frequency values for various values of R_{11} keeping R_{22} & T_{11} constant ($R_{22} = T_{11} = 10$) are tabulated. First frequencies $[k]$ of $n \leq 5$ modes with $R_{11} = 0, 2, 4, 6, 8, 10, 25,$ 50, and 10^{16} and $R_{22} = T_{11} = 10$ are shown in Tables 8 to 13. For $b = 1$ and $R_{11} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, first frequency of $n = 0$ modal frequency converges to that of plate without weakening as R_{11} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 1.85759 $[n = 0\,]$, 2.43992 $[n = 1\,]$, 2.98961 $[n = 2\,]$, 3.84106 $[n = 3\,]$, 4.84171 $[n = 4\,]$, and 5.88973 $[n = 5\,]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on elastically restrained edge occurs at $n = 0$ mode.

The frequency values for various values of T_{11} keeping R_{11} & R_{22} constant ($R_{11} = R_{22} = 10$) are tabulated. First frequencies $[k]$ of $n \leq 5$ modes with $T_{11} = 0, 2, 4, 6, 8, 10, 25,$ 50, and 10^{16} and $R_{11} = R_{22} = 10$ are shown in Tables 14 to 19. For $b = 1$ and $T_{11} = 0$, the frequency of plate is same as that of plate without having weakening crack. For a given value of b & ν, the first frequency of $n = 0$ modal frequency converges to that of plate without weakening as T_{11} is increased from a value of 0. When $\nu = 0.3$, first six frequencies of plate without weakening are obtained as 1.56837 $[n = 1\,]$, 2.69693 $[n = 2\,]$, 3.5696 $[n = 0\,]$, 3.78149 $[n = 3\,]$, 4.8612 $[n = 4\,]$ and 5.89884 $[n = 5\,]$. Notice that the fundamental frequency of plate weakened along an internal concentric circle and resting on elastically restrained edge occurs at $n = 1$ mode.

According to the author's acquaintance, the results for circular plate with generally restrained edge conditions presented here, are quite new and are not available in literature. Hence, results could be compared only with those available in the literature as follows. (i) For the basic boundary such as simply supported and clamped plate[33] by setting the translational and rotational restraints with $T_{11} \to \infty$ & $R_{11} \to 0$ and $T_{11} \to \infty$ & $R_{11} \to \infty$, respectively. Here, internal weakening decreases fundamental frequency by less than 1% when b is 0 or 1 for simply supported plate and less than 1% for clamped plate. (ii) For the basic boundary such as movable edge and free[34] by setting the translational and rotational restraints with $T_{11} \to 0$ & $R_{11} \to \infty$ and $T_{11} \to 0$ & $R_{11} \to 0$, respectively. Here, internal weakening decreases fundamental frequency by less than 1% when $0 \leq b \leq 0.26$ for the plate with movable edge and less than 1% for the free plate. (iii) For non-classical boundary such as translational restrained edge[35] by setting rotational restraint $R_{11} \to 0$. Here, an internal weakening decreases fundamental frequency by less than 12% for a circular plate with translational restrained edge.

5. CONCLUSIONS

The exact vibration solutions of plates that are weakened along internal circle and that have generally restrained edges are presented. It is observed that the fundamental frequency of plate weakened along internal circle due to a crack and resting on generally restrained edges against translation and rotation occurs at $n = 0$ mode corresponding to $R_{22} = 0, 2, 4,$ 6, 8, 10, 25, 50, and 10^{16}, $R_{11} = T_{11} = 2, 5, 50, 10, 50,$ 100, 1000, and 10^5, and $R_{11} = 0, 2, 4, 6, 8, 10, 25, 50,$ and 10^{16}, $R_{22} = T_{11} = 10$ and at $n = 1$ mode corresponding to $T_{11} = 0, 2, 4, 6, 8, 10, 25, 50,$ and 10^{16} & $R_{11} = R_{22} = 10$. Here, an internal weakening greatly decreases the fundamental frequency by less than 31% (30.49%). In addition, the frequencies are given for variable elastic restraints (T_{11} & R_{11}) at boundary which simulate the translational and rotational restraints where $T_{11} \to \infty$ & $R_{11} \to \infty$ represents a clamped support and $T_{11} \to \infty$ & $R_{11} \to 0$ represents a simply supported boundary. These exact solutions serve as benchmark solutions for verifying approximate results by other methods. Here, results presented are useful in the design of hatches and doors used in various industrial applications such as aerospace and automobile.

In summary, in this paper, the effect on fundamental frequency due to the influence of the presence of a crack is in-

Table 7. Fundamental frequency parameters, k for a circular hinge ($R_{22} = 0$, and for different values of R_{11} and $\nu = 0.3$).

b	$R_{22} \to 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$R_{11} = T_{11} = 10^{-3}$	2.31479	2.31968	2.33405	2.3561	2.38104	2.40017	2.40176	2.37951	2.33862	2.29143	2.17876
$R_{11} = T_{11} = 0.5$	2.03159	2.02739	2.0178	2.00962	2.00913	2.01691	2.02778	2.03063	2.00906	1.94995	1.85759
$R_{11} = T_{11} = 2$	1.39361	1.39256	1.39006	1.38707	1.38537	1.38577	1.38837	1.39186	1.39365	1.38926	1.374
$R_{11} = T_{11} = 5$	1.73362	1.73143	1.72623	1.72124	1.71975	1.72285	1.72903	1.73351	1.7287	1.70556	1.65872
$R_{11} = T_{11} = 10$	2.03159	2.02739	2.0178	2.00962	2.00913	2.01691	2.02778	2.03063	2.00906	1.94995	1.94995
$R_{11} = T_{11} = 50$	2.73979	2.71805	2.67328	2.64629	2.65856	2.70238	2.73905	2.69805	2.5432	2.33372	2.12973
$R_{11} = T_{11} = 100$	2.94748	2.91268	2.84521	2.80925	2.83227	2.89985	2.94745	2.86418	2.6474	2.3993	2.17453
$R_{11} = T_{11} = 1000$	3.17077	3.11603	3.01805	2.97341	3.01324	3.11488	3.16778	3.01734	2.73996	2.45934	2.45934
$R_{11} = T_{11} = 10^5$	3.19596	3.13862	3.03715	2.9916	3.03373	3.13965	3.19206	3.03315	2.74966	2.46594	2.22145
$R_{11} = T_{11} = 10^{16}$	3.19616	3.13892	3.03735	2.9918	3.03393	3.13995	3.19235	3.03325	2.74976	2.46603	2.22145

Table 8. First frequency parameter, ($n = 0$) for different R_{11} and internal concentric weakened radius parameter b, $R_{22} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	1.85759	1.96996	2.00196	2.01702	2.0258	2.03159	2.04705	2.05282	2.05592	2.05902
0.1	1.8532	1.96776	2.00027	2.01553	2.02441	2.03029	2.04595	2.05184	2.05483	2.05802
0.2	1.84802	1.96527	1.99837	2.01393	2.02301	2.02889	2.04485	2.05074	2.05383	2.05702
0.3	1.84413	1.96387	1.99748	2.01324	2.02241	2.0284	2.04445	2.05044	2.05353	2.05682
0.4	1.84224	1.96387	1.99768	2.01354	2.02281	2.0288	2.04486	2.05094	2.05403	2.05722
0.5	1.84254	1.96517	1.99898	2.01484	2.02391	2.0299	2.04596	2.05194	2.05503	2.05822
0.6	1.84483	1.96727	2.00067	2.01633	2.02531	2.03119	2.04695	2.05274	2.05583	2.05892
0.7	1.84852	1.96926	2.00187	2.01703	2.0258	2.03149	2.04675	2.05233	2.05532	2.05832
0.8	1.85261	1.96995	2.00117	2.01563	2.0239	2.02939	2.04385	2.04924	2.05203	2.05482
0.9	1.8561	1.96766	1.99688	2.01034	2.01802	2.02301	2.03637	2.04136	2.04395	2.04654
1	1.85759	1.96079	1.98721	1.99928	2.00616	2.01065	2.02252	2.02691	2.0292	2.03159

vestigated. Also, the variation of fundamental frequency with respect to the location of the crack is investigated for different values of R_{11} and T_{11}. In addition, the influence of the intensity of the crack on fundamental frequency is investigated in detail. From the various results obtained and presented, it is found that the fundamental frequency decreases as the crack moves away from the centre of the plate and it reduces with the reduction in the intensity of the crack.

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Table 9. First frequency parameter, ($n = 1$) for different R_{11} and internal concentric weakened radius parameter b, $R_{22} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	2.44	2.45249	2.45729	2.45978	2.46129	2.46239	2.4654	2.46658	2.46719	2.46788
0.1	2.43772	2.44982	2.45442	2.45682	2.45832	2.45932	2.46222	2.46342	2.46402	2.46462
0.2	2.43442	2.44592	2.45023	2.45252	2.45392	2.45492	2.45762	2.45872	2.45932	2.45992
0.3	2.43063	2.44113	2.44513	2.44723	2.44853	2.44943	2.45193	2.45284	2.45343	2.45394
0.4	2.42654	2.43604	2.43955	2.44145	2.44255	2.44335	2.44555	2.44635	2.44685	2.44735
0.5	2.42315	2.43146	2.43456	2.43616	2.43706	2.43776	2.43967	2.44037	2.44077	2.44117
0.6	2.42155	2.42887	2.43157	2.43287	2.43378	2.43428	2.43588	2.43658	2.43688	2.43719
0.7	2.42286	2.42947	2.43187	2.43308	2.43378	2.43428	2.43568	2.43629	2.43659	2.43689
0.8	2.42745	2.43386	2.43616	2.43737	2.43807	2.43857	2.43987	2.44037	2.44067	2.44097
0.9	2.43423	2.44144	2.44394	2.44524	2.44604	2.44654	2.44795	2.44845	2.44875	2.44905
1	2.43992	2.44962	2.45282	2.45442	2.45532	2.45602	2.45772	2.45842	2.45882	2.45922

Table 10. First frequency parameter, ($n = 2$) for different R_{11} and internal concentric weakened radius parameter b, $R_{22} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	2.98959	3.09309	3.1382	3.16344	3.17951	3.1908	3.22393	3.23752	3.24499	3.25278
0.1	2.9909	3.0945	3.13971	3.16506	3.18123	3.19241	3.22565	3.23932	3.24671	3.25459
0.2	2.9927	3.09649	3.1418	3.16705	3.18332	3.1945	3.22784	3.24151	3.2489	3.25688
0.3	2.99399	3.09748	3.1426	3.16785	3.18392	3.1952	3.22833	3.24191	3.24929	3.25718
0.4	2.99359	3.09578	3.1402	3.16505	3.18082	3.1919	3.22433	3.23761	3.2449	3.25258
0.5	2.99089	3.08999	3.13271	3.15656	3.17173	3.18221	3.21316	3.22583	3.23272	3.24001
0.6	2.9858	3.07971	3.11974	3.1419	3.15597	3.16565	3.1942	3.20588	3.21217	3.21886
0.7	2.98001	3.06715	3.10368	3.12374	3.13642	3.1452	3.17076	3.18114	3.18673	3.19262
0.8	2.97673	3.05697	3.08991	3.10778	3.11896	3.12674	3.14911	3.15819	3.16308	3.16827
0.9	2.97953	3.05468	3.08452	3.10059	3.11057	3.11736	3.13712	3.14501	3.1493	3.1537
1	2.98961	3.06436	3.0929	3.10788	3.11716	3.12345	3.14151	3.1486	3.15239	3.15638

Table 11. First frequency parameter, ($n = 3$) for different R_{11} and internal concentric weakened radius parameter b, $R_{22} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	3.84103	3.97959	4.04817	4.08909	4.11635	4.13572	4.19552	4.22127	4.23555	4.25104
0.1	3.84106	3.97961	4.04819	4.08922	4.11637	4.13574	4.19564	4.2214	4.23567	4.25105
0.2	3.84156	3.98011	4.04879	4.08982	4.11697	4.13644	4.19634	4.22209	4.23637	4.25184
0.3	3.84295	3.98171	4.05048	4.09151	4.11876	4.13813	4.19813	4.22399	4.23826	4.25374
0.4	3.84514	3.98389	4.05257	4.0936	4.12085	4.14022	4.20012	4.22587	4.24015	4.25562
0.5	3.84683	3.98429	4.05216	4.09259	4.11945	4.13851	4.19741	4.22267	4.23675	4.25182
0.6	3.84533	3.97839	4.04368	4.08241	4.10797	4.12614	4.18204	4.2059	4.21918	4.23346
0.7	3.83864	3.96303	4.02312	4.05846	4.08172	4.0982	4.14851	4.16988	4.18166	4.19434
0.8	3.82927	3.94078	3.99349	4.02414	4.0442	4.05828	4.10111	4.11918	4.12917	4.13975
0.9	3.82608	3.92312	3.96744	3.9928	4.00917	4.02075	4.05529	4.06967	4.07756	4.08604
1	3.84106	3.92661	3.96355	3.98411	3.99719	4.00627	4.03303	4.04401	4.05	4.05629

Table 12. First frequency parameter, ($n = 4$) for different R_{11} and internal concentric weakened radius parameter b, $R_{22} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	4.84169	4.97687	5.05146	5.09868	5.13132	5.1551	5.23208	5.26682	5.2866	5.30826
0.1	4.84171	4.9769	5.05148	5.0987	5.13125	5.15511	5.2321	5.26684	5.28661	5.30828
0.2	4.84181	4.97699	5.05158	5.0988	5.13135	5.15521	5.23219	5.26694	5.28671	5.30838
0.3	4.84221	4.97749	5.05207	5.0994	5.13195	5.15581	5.23289	5.26764	5.28741	5.30908
0.4	4.8437	4.97918	5.05386	5.10119	5.13384	5.1577	5.23488	5.26973	5.2895	5.31126
0.5	4.84639	4.98167	5.05625	5.10347	5.13602	5.15978	5.23666	5.27131	5.29098	5.31265
0.6	4.84787	4.98086	5.05384	5.09997	5.13162	5.15478	5.22937	5.26292	5.28199	5.30286
0.7	4.84258	4.96808	5.03617	5.07891	5.10816	5.12943	5.19763	5.22809	5.24536	5.26413
0.8	4.82842	4.93914	4.99805	5.0345	5.05936	5.07734	5.13435	5.15952	5.1738	5.18928
0.9	4.81795	4.90771	4.95384	4.9819	5.00077	5.01435	5.05669	5.07526	5.08565	5.09683
1	4.84171	4.90831	4.94006	4.95863	4.97081	4.9794	5.00566	5.01675	5.02293	5.02952

Table 13. First frequency parameter, ($n = 5$) for different R_{11} and internal concentric weakened radius parameter b, $R_{22} = T_{11} = 10$, and $\nu = 0.3$.

b	$R_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	5.88972	6.01304	6.08684	6.13595	6.171	6.19718	6.28555	6.32739	6.35177	6.37902
0.1	5.88974	6.01305	6.08685	6.13598	6.17103	6.19719	6.28557	6.32741	6.35178	6.37894
0.2	5.88973	6.01305	6.08685	6.13598	6.17103	6.19719	6.28557	6.32741	6.35178	6.37904
0.3	5.88983	6.01315	6.08704	6.13617	6.17113	6.19739	6.28576	6.32761	6.35197	6.37924
0.4	5.89053	6.01395	6.08784	6.13707	6.17212	6.19838	6.28685	6.3287	6.35316	6.38033
0.5	5.89281	6.01623	6.09022	6.13945	6.1745	6.20076	6.28934	6.33118	6.35564	6.3829
0.6	5.89549	6.01811	6.0913	6.14003	6.17468	6.20055	6.28782	6.32906	6.35303	6.37979
0.7	5.89199	6.00912	6.07852	6.12445	6.15691	6.18117	6.26246	6.30061	6.32278	6.34735
0.8	5.87353	5.97629	6.0361	6.07515	6.10262	6.12309	6.1908	6.22216	6.24034	6.26041
0.9	5.85348	5.93148	5.97542	6.00358	6.02316	6.03754	6.08458	6.10595	6.11824	6.13182
1	5.88973	5.93358	5.95615	5.96983	5.97902	5.98571	6.00638	6.01547	6.02056	6.02606

Table 14. First frequency parameter, ($n = 0$) for different T_{11} and internal concentric weakened radius parameter b, $R_{11} = R_{22} = 10$, and $\nu = 0.3$.

b	$T_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	3.67597	1.40294	1.65497	1.81687	1.9368	2.03159	2.41131	2.64156	2.79104	2.95826
0.1	3.64793	1.40274	1.65458	1.81618	1.9358	2.03029	2.40753	2.63439	2.78059	2.94293
0.2	3.62648	1.40254	1.65418	1.81548	1.9347	2.02889	2.40364	2.62751	2.77063	2.92897
0.3	3.62838	1.40244	1.65398	1.81518	1.9343	2.0284	2.40225	2.62533	2.76804	2.92579
0.4	3.64843	1.40254	1.65408	1.81538	1.9346	2.0288	2.40374	2.62842	2.77283	2.93356
0.5	3.67026	1.40264	1.65448	1.81598	1.9355	2.0299	2.40713	2.63479	2.78219	2.9471
0.6	3.67445	1.40284	1.65488	1.81668	1.9365	2.03119	2.41052	2.64057	2.79005	2.95746
0.7	3.65221	1.40294	1.65498	1.81687	1.9367	2.03149	2.41091	2.64036	2.78875	2.95387
0.8	3.61401	1.40264	1.65438	1.81577	1.9351	2.02939	2.40453	2.6286	2.77152	2.92887
0.9	3.58008	1.40174	1.65218	1.81228	1.93021	2.02301	2.38799	2.60111	2.73496	2.88173
1	3.5696	1.39985	1.6479	1.80531	1.92054	2.01065	2.3588	2.55688	2.68026	2.81682

Table 15. First frequency parameter, ($n = 1$) for different T_{11} and internal concentric weakened radius parameter b, $R_{11} = R_{22} = 10$, and $\nu = 0.3$.

b	$T_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	1.66531	1.9257	2.10646	2.24722	2.36323	2.46239	2.94164	3.34731	3.72084	4.30624
0.1	1.66095	1.92223	2.10329	2.24415	2.36026	2.45932	2.93797	3.34204	3.71257	4.28781
0.2	1.65486	1.91734	2.0988	2.23976	2.35587	2.45492	2.93218	3.33315	3.69779	4.25278
0.3	1.64738	1.91145	2.09332	2.23447	2.35048	2.44943	2.92469	3.32157	3.67844	4.20901
0.4	1.6382	1.90437	2.08703	2.22838	2.3445	2.44335	2.91701	3.3103	3.6612	4.17802
0.5	1.62743	1.89669	2.08035	2.2223	2.33871	2.43776	2.91153	3.30434	3.65524	4.17933
0.6	1.61535	1.88891	2.07446	2.21752	2.33473	2.43428	2.91084	3.30754	3.66592	4.2176
0.7	1.60259	1.88222	2.07047	2.21522	2.33363	2.43428	2.91633	3.32031	3.69034	4.27481
0.8	1.58992	1.87754	2.06938	2.21642	2.33653	2.43857	2.927	3.33765	3.71545	4.30623
0.9	1.57824	1.87584	2.07207	2.2215	2.3433	2.44654	2.93826	3.34751	3.71704	4.26818
1	1.56837	1.87793	2.07845	2.22998	2.35258	2.45602	2.94125	3.33206	3.67021	4.15757

Table 16. First frequency parameter, ($n = 2$) for different T_{11} and internal concentric weakened radius parameter b, $R_{11} = R_{22} = 10$, and $\nu = 0.3$.

b	$T_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	2.83713	2.92015	2.99601	3.06574	3.13043	3.1908	3.54753	3.93833	4.39911	5.55402
0.1	2.83915	2.92207	2.99782	3.06747	3.13214	3.19241	3.54904	3.93984	4.40082	5.55823
0.2	2.84184	2.92467	3.00021	3.06986	3.13433	3.1945	3.55083	3.94143	4.4023	5.558
0.3	2.84323	2.92586	3.00121	3.07066	3.13512	3.1952	3.55072	3.94022	4.39889	5.53072
0.4	2.84083	2.92326	2.99851	3.06776	3.13192	3.1919	3.54602	3.93282	4.3856	5.4648
0.5	2.83185	2.91418	2.98932	3.05848	3.12254	3.18221	3.53454	3.91765	4.36235	5.39402
0.6	2.81409	2.89692	2.97236	3.04171	3.10588	3.16565	3.51769	3.89891	4.33973	5.37288
0.7	2.78795	2.87227	2.94901	3.01947	3.08463	3.1452	3.50103	3.88606	4.33318	5.4288
0.8	2.75621	2.84413	2.92387	2.99682	3.06418	3.12674	3.49286	3.88886	4.35193	5.52572
0.9	2.72378	2.81818	2.90311	2.98045	3.0516	3.11736	3.49983	3.9098	4.38672	5.54383
1	2.69693	2.80112	2.89373	2.97725	3.05349	3.12345	3.52218	3.93583	4.3971	5.40798

Table 17. First frequency parameter, ($n = 3$) for different T_{11} and internal concentric weakened radius parameter b, $R_{11} = R_{22} = 10$, and $\nu = 0.3$.

b	$T_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	3.95635	3.99457	4.03153	4.06736	4.10208	4.13572	4.35999	4.65502	5.07354	6.75599
0.1	3.95638	3.99461	4.03164	4.06747	4.1021	4.13574	4.36011	4.65503	5.07356	6.75632
0.2	3.95707	3.9953	4.03233	4.06807	4.1028	4.13644	4.36071	4.65573	5.07435	6.7591
0.3	3.95887	3.9971	4.03413	4.06986	4.10459	4.13813	4.3624	4.65732	5.07594	6.75865
0.4	3.96126	3.99948	4.03632	4.07205	4.10668	4.14022	4.36399	4.6583	5.07562	6.72428
0.5	3.96055	3.99848	4.03521	4.07074	4.10518	4.13851	4.36098	4.6531	5.06582	6.63323
0.6	3.94947	3.9872	4.02363	4.05896	4.0931	4.12614	4.34661	4.63523	5.04067	6.541
0.7	3.92153	3.95926	3.99569	4.03092	4.06506	4.0982	4.31817	4.60551	5.00837	6.53889
0.8	3.87662	3.91545	3.95298	3.98921	4.02434	4.05828	4.28395	4.57838	4.99242	6.65873
0.9	3.82431	3.86644	3.90706	3.94629	3.98412	4.02075	4.26259	4.57598	5.01536	6.75635
1	3.78149	3.8304	3.87711	3.92183	3.96485	4.00627	4.27486	4.61219	5.06654	6.61541

Table 18. First frequency parameter, ($n = 4$) for different T_{11} and internal concentric weakened radius parameter b, $R_{11} = R_{22} = 10$, and $\nu = 0.3$.

b	$T_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	5.05056	5.07222	5.09341	5.11427	5.13483	5.1551	5.29775	5.50518	5.83866	7.93035
0.1	5.05059	5.07216	5.09342	5.11428	5.13485	5.15511	5.29777	5.5052	5.83869	7.93038
0.2	5.05069	5.07226	5.09352	5.11438	5.13495	5.15521	5.29777	5.5053	5.83878	7.93097
0.3	5.05129	5.07285	5.09411	5.11498	5.13554	5.15581	5.29846	5.5059	5.83948	7.93393
0.4	5.05328	5.07484	5.09601	5.11687	5.13744	5.1577	5.30035	5.50778	5.84126	7.92598
0.5	5.05557	5.07713	5.09829	5.11915	5.13962	5.15978	5.30194	5.50866	5.84064	7.8587
0.6	5.05156	5.07292	5.09389	5.11445	5.13482	5.15478	5.29553	5.49986	5.82684	7.72757
0.7	5.02751	5.04857	5.06933	5.0897	5.10976	5.12943	5.26819	5.46933	5.79003	7.64986
0.8	4.97431	4.99558	5.01654	5.03711	5.05737	5.07734	5.2174	5.42034	5.74435	7.74586
0.9	4.90284	4.9259	4.94856	4.97082	4.99279	5.01435	5.16549	5.38401	5.73237	7.92328
1	4.84612	4.87388	4.90113	4.92779	4.95384	4.9794	5.1568	5.40656	5.79155	7.79604

Table 19. First frequency parameter, ($n = 5$) for different T_{11} and internal concentric weakened radius parameter b, $R_{11} = R_{22} = 10$, and $\nu = 0.3$.

b	$T_{11} \to 0$	2	4	6	8	10	25	50	100	10^6
0	6.12986	6.14357	6.15713	6.17061	6.18399	6.19718	6.29282	6.43937	6.69385	9.08574
0.1	6.1299	6.14358	6.15715	6.17063	6.18401	6.19719	6.29284	6.4394	6.69387	9.08575
0.2	6.1299	6.14357	6.15715	6.17063	6.18401	6.19719	6.29284	6.4394	6.69387	9.08585
0.3	6.13	6.14377	6.15735	6.17083	6.18411	6.19739	6.29293	6.43959	6.69407	9.08732
0.4	6.13099	6.14467	6.15824	6.17172	6.1851	6.19838	6.29393	6.44059	6.69516	9.08865
0.5	6.13347	6.14725	6.16083	6.1742	6.18758	6.20076	6.29631	6.44286	6.69713	9.05246
0.6	6.13375	6.14733	6.16081	6.17419	6.18747	6.20055	6.29539	6.44075	6.69251	8.9186
0.7	6.11558	6.12896	6.14224	6.15532	6.1683	6.18117	6.27422	6.41659	6.66236	8.77277
0.8	6.05799	6.07127	6.08435	6.09743	6.11031	6.12309	6.21534	6.35641	6.60001	8.81281
0.9	5.96765	5.98192	5.996	6.00998	6.02386	6.03754	6.13658	6.28804	6.54981	9.05407
1	5.89884	5.91661	5.93419	5.95156	5.96873	5.98571	6.10702	6.28922	6.59581	8.95778

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