Optimal Vibration Control for Structural Quasi-Hamiltonian Systems with Noised Observations

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A structural control system with smart sensors and actuators is considered and its basic dynamic equations are given. The controlled, stochastically excited, and dissipative Hamiltonian system with a noised observation is obtained by discretizing the nonlinear stochastic smart structure system. The estimated nonlinear stochastic system with control is obtained, in which the optimally estimated state is determined by the observation based on the extended Kalman filter. Then the dynamical programming equation for the estimated system is obtained based on the stochastic dynamical programming principle. From this, the optimal control law dependent on the estimated state is determined. The proposed optimal control strategy is applied to two nonlinear stochastic systems with controls and noised observations. The control efficacy for stochastic vibration response reductions of the systems is illustrated with numerical results. The proposed optimal control strategy is applicable to general nonlinear stochastic structural systems with smart sensors, smart actuators and noised observations.

1. INTRODUCTION

The vibration control for engineering structures subjected to strong random excitations or micro disturbances is a significant research subject.[1–3] Smart materials have been applied to the structural vibration suppression. For example, magneto-rheological materials are used for actuators and piezoelectric materials are used for sensors.[4–20] A structure system with distributed smart sensors, actuators, and controller is called smart structure, which can sense structural response to external excitations and produce action to control structural response.[21–23] The dynamics and controls of structures with piezoelectric and magneto-rheological materials have been studied extensively. However, the control performance of the smart structure system depends strongly on the used control strategy. The smart structure system control includes two parts: state estimation based on sensing data and response control based on an estimated state, which are coupled with each other.[24–28] The stochastic system control with a noised observation is called a partially observable control. Only linear control strategies for partially observable smart structures have been proposed presently.

The smart structure, including multi-degree-of-freedom systems, rods, beams, plates, and shells in modal vibration can be modeled as a controlled, excited, and dissipative Hamiltonian system (or quasi-Hamiltonian system) with observation.[29] The stochastic optimal controls for linear and nonlinear systems have been studied and many control strategies have been presented.[29–46] However, the stochastic optimal control for a nonlinear system with a noised observation was only considered in several studies.[43] Under a specified condition, the separation theorem was applied to convert the nonlinear stochastic system with a noised observation into a completely observable linear system for determining optimal control, but the application is strongly limited. Thus, an approximate estimation and separation strategy is the alternative in practice. The extended Kalman filter is an optimum feasible approximate filter[24,28] and can be applied to the nonlinear stochastic control system with a noised observation. Based on the extended Kalman filter, the nonlinear stochastic control system with a noised observation is converted into another completely observable nonlinear stochastic system, and then the optimal control law is determined according to the stochastic dynamical programming principle. However, the stochastic optimal control for the partially observable nonlinear stochastic smart structure system (or quasi-Hamiltonian system) has not been studied based on the extended Kalman filter.

In the present paper, the stochastic optimal control for the vibration response reduction of structural quasi-Hamiltonian systems with a noised observations is studied. A new optimal control law expressed by the estimated system state is obtained based on the extended Kalman filter and stochastic dynamical programming principle. Firstly, the differential equations for the structure system with smart sensors and actuators are given. The equations are simplified to ordinary differential equations of the controlled, stochastically-excited, and dissipa-

tive Hamiltonian system with a noised observation. Secondly, the extended Kalman filter is applied in order to convert the controlled quasi-Hamiltonian system with a noised observation into the nonlinear stochastic control system for the optimally estimated state. Thirdly, the stochastic dynamical programming principle is applied to derive the dynamical programming equation. The optimal control law based on the estimated state is determined by the programming equation. Finally, the proposed optimal control strategy is applied to two nonlinear stochastic systems with controls and noised observations. Numerical results are given to illustrate the control efficacy.

2. VIBRATION EQUATIONS OF A QUASI-HAMILTONIAN SYSTEM

Smart materials have been applied recently to the vibration control of engineering structures under strong random excitations. For example, magneto-rheological materials are used for smart actuators whereas piezoelectric materials are used for smart sensors. A structure that has smart devices is a smart system that has the ability to sense structural responses to external excitations and produce actions to control structural response. System state estimations and control strategies are two key problems for effective implementation. In general, the nonlinear dynamic, constitutive, and geometric equations of the main structure can be expressed as[47]

$$\sigma_{ij,j} + f_i - \rho\ddot{u}_i - c_d\dot{u}_i = 0; \tag{1}$$

$$\sigma_{ij} = C_{ijkl}\,\gamma_{kl}; \tag{2}$$

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}); \tag{3}$$

where σ is the stress tensor, f is the body force vector, u is the displacement vector, ρ is the mass density, c_d is the damping coefficient, γ is the strain tensor, C is the elastic constant tensor, and i, j, k, and l denote Cartesian coordinates. The linear dynamic, constitutive, and geometric equations of the sensor structure can be expressed as[48]

$$\sigma_{ij,j}^{\mathrm{se}} + f_i^{\mathrm{se}} - \rho^{\mathrm{se}}\ddot{u}_i^{\mathrm{se}} - c_d^{\mathrm{se}}\dot{u}_i^{\mathrm{se}} = 0; \tag{4}$$

$$\sigma_{ij}^{\mathrm{se}} = C_{ijkl}^{\mathrm{se}}\gamma_{kl}^{\mathrm{se}} + e_{kij}^{\mathrm{se}}\varphi_{,k}^{\mathrm{se}}; \tag{5}$$

$$\gamma_{ij}^{\mathrm{se}} = \frac{1}{2}(u_{i,j}^{\mathrm{se}} + u_{j,i}^{\mathrm{se}}); \tag{6}$$

where the superscript "se" denotes sensor, for example, piezoelectric sensor, ϕ is the generalized electric potential and e is the piezoelectric stress constant. The differential equations for the electric displacement vector D are

$$D_{i,i}^{\mathrm{se}} = \rho_{em}^{\mathrm{se}}; \tag{7}$$

$$D_i^{\mathrm{se}} = e_{ikl}^{\mathrm{se}}\gamma_{kl}^{\mathrm{se}} - \varepsilon_{ik}^{\mathrm{se}}\varphi_{,k}^{\mathrm{se}}; \tag{8}$$

where ρ_{em} is the free charge density and ε is the dielectric constant. The mechanical and electrical differential equations for the actuator structure (e.g., piezoelectric actuator) can be expressed as similar to Eqs. (4) to (8), where the superscript se is replaced by ac. Equations (1)–(8) give a basic description of the smart structure system. However, the direct use of the equations for optimal estimation and control is impossible and then their simplification is necessary.

Dominant modes of the main structure with sensors and actuators are used to discretize the structure system in Eqs. (1) to (8), and the Galerkin method is used to eliminate the spatial variables generally. The simplified main structure with actuators is converted into a controlled, excited, and dissipative Hamiltonian system and the sensor structure is converted into an observation system. The inertias of the sensors are small and neglected. The differential equation of the controlled Hamiltonian system with n degree-of-freedoms can be expressed as[44]

$$\dot{X} = J\frac{\partial H_a}{\partial X} + CX + B\Phi_a + FW(t); \tag{9}$$

where H_a is the Hamiltonian, X is the state vector, J is the unit symplectic matrix, C is the damping coefficient matrix, B is the control coefficient matrix, Φ_a is the n_a-dimensional control vector, F is the excitation coefficient matrix, and W is the m-dimensional unit intensity excitation vector that is considered to be Gaussian white noise. The equation of the observation system can be expressed as

$$\Phi_s = DX + EW_s(t); \tag{10}$$

where Φ_s is the n_s-dimensional observation vector, W_s is the m_s-dimensional Gaussian white noise vector with unit intensity, and D and E are coefficient matrices. The observation noise is independent of the white noise excitation.

Equations (9) and (10) describe a nonlinear stochastic quasi-Hamiltonian control system with a noised observation, which is derived from the smart structure Eqs. (1) to (8). The system state is estimated by using Eq. (10) and the estimated state is used to determine the feedback control for the system in Eq. (9). The optimal estimation and control need to be considered for the nonlinear stochastic system in Eqs. (9) and (10).

3. STOCHASTIC OPTIMAL ESTIMATION AND CONTROL

The optimal control of the nonlinear stochastic system (9) and (10) includes the optimal state estimation and the optimal control based on the estimated state, which is called the partially observable optimal control.[27] The optimal estimation is to find an accurate state by minimizing the estimated error. The error index for the observation in Eq. (10) is given by

$$J_F(\hat{X}) = E[l_F(X - \hat{X})\,|\Phi_s]; \tag{11}$$

where $\hat{X}$ is the estimated state, $E[\cdot]$ is the expectation operator, and $l_F(\cdot) \geq 0$ is a continuous differentiable function (e.g., a quadratic function). For the nonlinear stochastic system (9) and observation (10) with index (11), the optimally estimated state probability density with infinite dimensions is generally difficult to be obtained exactly, and then an approximate estimation is alternative. The extended Kalman filter is an optimum feasible approximate filter for the nonlinear stochastic system[28] and is applied to the stochastic optimal control of quasi-Hamiltonian systems with noised observations. Based on the extended Kalman filter, the differential equation for the

estimated state is obtained from Eq. (9) with observation (10) and index (11) as[28]

$$\dot{X} = J\frac{\partial H_a(\hat{X})}{\partial \hat{X}} + C\hat{X} + B\Phi_a + RD^{\mathrm{T}}R_s^{-1}W_I(t); \quad (12)$$

where W_I is the Gaussian white noise vector with covariance matrix $R_s = EE^T$, and R is the covariance matrix of the estimated error $\tilde{X} = X - \hat{X}$, which is determined by

$$\dot{R} = \{J\frac{\partial^2 H_a(\hat{X})}{\partial \hat{X}^2} + C\}R + R\{J\frac{\partial^2 H_a(\hat{X})}{\partial \hat{X}^2} + C\}^{\mathrm{T}}$$
$$+ FR_WF^{\mathrm{T}} - RD^{\mathrm{T}}R_s^{-1}DR; \quad (13)$$

where $R_W = I$ is the identity matrix. Thus, the optimal control of the nonlinear stochastic system in Eqs. (9) and (10) is converted into that of the estimated system in Eq. (12).

The optimal control is to find a control law by minimizing certain performance index. The performance index for the system in Eq. (12) is given by

$$J_C(\Phi_a) = E[\int_0^{t_f} L_C(\hat{X}(t), \Phi_a(t))\mathrm{d}t + \Psi(\hat{X}(t_f))]; \quad (14)$$

where $L_C(\cdot) \geq 0$ is a continuous differentiable function, $\Psi(t_f)$ is a terminal cost and t_f is the terminal time. Equations (12) and (14) describe the optimal control problem of the non-linear stochastic system with the estimated state. Based on the stochastic dynamical programming principle,[24,25] the dynamical programming equation for system (12) with index (14) is obtained as

$$\frac{\partial V}{\partial t} + \min_{\Phi_a}\{\frac{1}{2}\mathrm{tr}(RD^{\mathrm{T}}R_s^{-1}DR\frac{\partial^2 V}{\partial \hat{X}^2})$$
$$+ [J\frac{\partial H_a}{\partial \hat{X}} + C\hat{X} + B\Phi_a]^{\mathrm{T}}\frac{\partial V}{\partial \hat{X}} + L_C(\hat{X}, \Phi_a)\} = 0; \quad (15)$$

where V is the value function, and $tr(\cdot)$ is the trace operator. The minimization of the left side of Eq. (15) yields the algebraic equation for the optimal control Φ_a^*

$$\frac{\partial L_C(\hat{X}, \Phi_a^*)}{\partial \Phi_a^*} + B^{\mathrm{T}}\frac{\partial V}{\partial \hat{X}} = 0. \quad (16)$$

The optimal control law is obtained by Eq. (16). For example, for the function with quadratic control $L_C = g(\tilde{X}) + \Phi_a^{\mathrm{T}}R_C\Phi_a$, where R_C is a positive definite symmetric constant matrix and $g(\hat{X}) \geq 0$, it is

$$\Phi_a^* = -\frac{1}{2}R_C^{-1}B^{\mathrm{T}}\frac{\partial V}{\partial \hat{X}}. \quad (17)$$

By substituting Eq. (17) into Eq. (15), the value function equation is obtained as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\mathrm{tr}(RD^{\mathrm{T}}R_s^{-1}DR\frac{\partial^2 V}{\partial \hat{X}^2})$$
$$+ [J\frac{\partial H_a}{\partial \hat{X}} + C\hat{X} + B\Phi_a^*]^{\mathrm{T}}\frac{\partial V}{\partial \hat{X}} + L_C(\hat{X}, \Phi_a^*) = 0. \quad (18)$$

Equation (18) can be solved to obtain V and then the optimal control Φ_a^* can be determined by Eq. (17), which is based

on the estimated state $\hat{X}$. The optimally controlled system is determined by substituting control Φ_a^* into Eq. (12), and the controlled state $\hat{X}$ is obtained by solving the equation. The statistics of state X can be calculated by using the state $\hat{X}$ and covariance R, which are used for evaluating the control effectiveness.

4. EXAMPLES AND NUMERICAL RESULTS

4.1. Example 1: single-degree-of-freedom nonlinear stochastic control system

To illustrate the application and effectiveness of the proposed optimal control strategy, we first considered the control for a main mode vibration of geometric nonlinear beams with a piezoelectric sensor and actuator under stochastic excitation. The nonlinear stochastic system with a control and noised observation can be expressed as

$$\ddot{q} + c\dot{q} + k_1q + k_3q^3 = b\varphi_a + e_0W(t); \quad (19)$$
$$\varphi_s = dq + e_1W_s(t); \quad (20)$$

where q is the generalized displacement, φ_a is the control, c, k_1, and k_3 are respectively the damping, linear stiffness, and nonlinear stiffness coefficients, b is the control coefficient, e_0 is the excitation amplitude, W is the Gaussian white noise with unit intensity, φ_s is the observation, d is the observation coefficient, e_1 is the observation noise amplitude, and W_s is the unit Gaussian white noise. Equations (19) and (20) are rewritten as Eqs. (9) and (10), respectively, where $\Phi_a = \varphi_a$, $\Phi_s = \varphi_s$, $p = \dot{q}$, and

$$H_a = \frac{1}{2}p^2 + \frac{1}{2}k_1q^2 + \frac{1}{4}k_3q^4$$
$$X = \left\{\begin{array}{c} q \\ p \end{array}\right\}$$
$$C = \left[\begin{array}{cc} 0 & 0 \\ 0 & -c \end{array}\right]$$
$$B = \left\{\begin{array}{c} 0 \\ b \end{array}\right\}$$
$$F = \left\{\begin{array}{c} 0 \\ e_0 \end{array}\right\}$$
$$D = [\begin{array}{cc} d & 0 \end{array}]$$
$$E = e_1 \quad (21)$$

Applying the extended Kalman filter yields the differential Eq. (12) for the estimated state with the covariance Eq. (13). Equation (12) is converted into the Itô differential equation for the averaged Hamiltonian[29] by using the stochastic averaging method,

$$\mathrm{d}\hat{H}_a = [m_h(\hat{H}_a) + < b\varphi_a\frac{\partial \hat{H}_a}{\partial p} >]\mathrm{d}t + \sigma_h(\hat{H}_a)\mathrm{d}\xi(t); \quad (22)$$

where m_h and σ_h are the drift and diffusion coefficients, respectively, ξ is the unit Wiener process, and $< \cdot >$ is the averaging operator. According to the stochastic dynamical programming principle, the dynamical programming equation
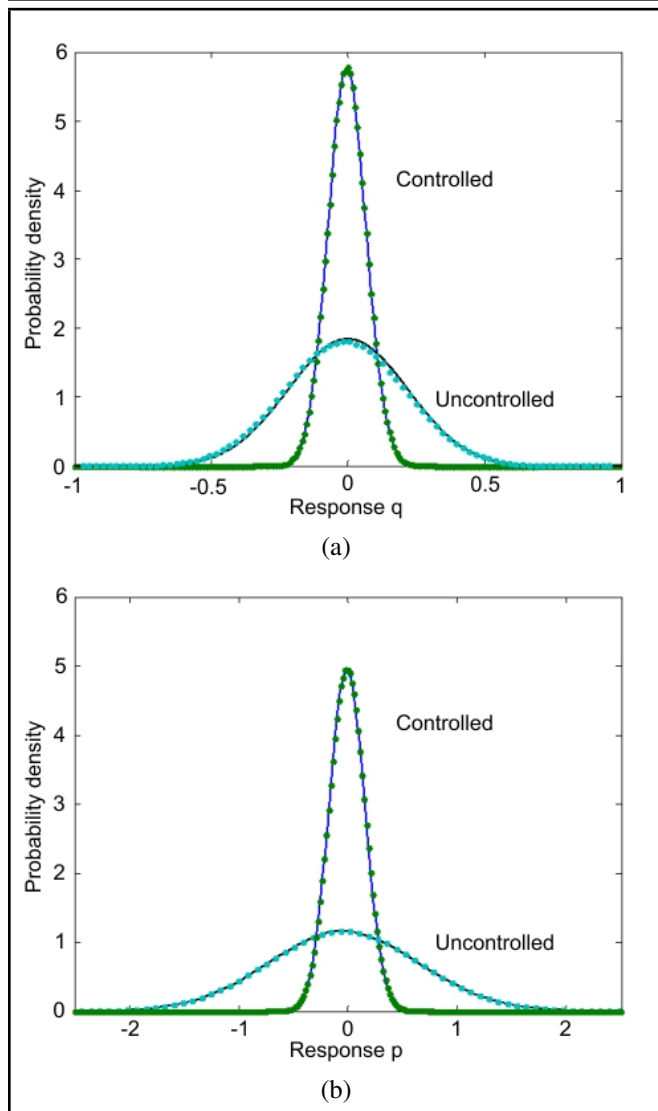
Figure 1. Controlled and uncontrolled probability densities (solid line: analytical; dot: simulated). The (a) Probability densities of displacement q and (b) Probability densities of velocity p.

similar to Eq. (15) is obtained. The optimal control law (17) leads to

$$\varphi_a^* = -\frac{bp}{2R_C}\frac{\mathrm{d}V}{\mathrm{d}\hat{H}_a}. \qquad (23)$$

The corresponding stationary value function equation is

$$\frac{1}{2}\sigma_h^2(\hat{H}_a)\frac{\mathrm{d}^2V}{\mathrm{d}\hat{H}_a^2} + m_h(\hat{H}_a)\frac{\mathrm{d}V}{\mathrm{d}\hat{H}_a}$$
$$-\frac{b^2}{4R_C}<p^2>(\frac{\mathrm{d}V}{\mathrm{d}\hat{H}_a})^2 + g(\hat{H}_a) = \gamma_0; \quad (24)$$

where γ_0 is a constant. The optimal control is determined finally by Eq. (23) with (24). Then the controlled response and its statistics can be obtained by solving Eq. (12) with Eq. (20) numerically. The analytical system responses can be evaluated by using the probability density, which is obtained by solving the Fokker-Planck-Kolmogorov equation associated with the Itô in Eq. (22).[29] The proposed optimal control efficacy for the nonlinear stochastic system with a noised observation is evaluated based on the response statistics.

The numerical results on the control system (19) and observation (20) with $c = 1.0$, $k_1 = 10.0$, $k_3 = 3.0$, $e_0 = 1.0$,
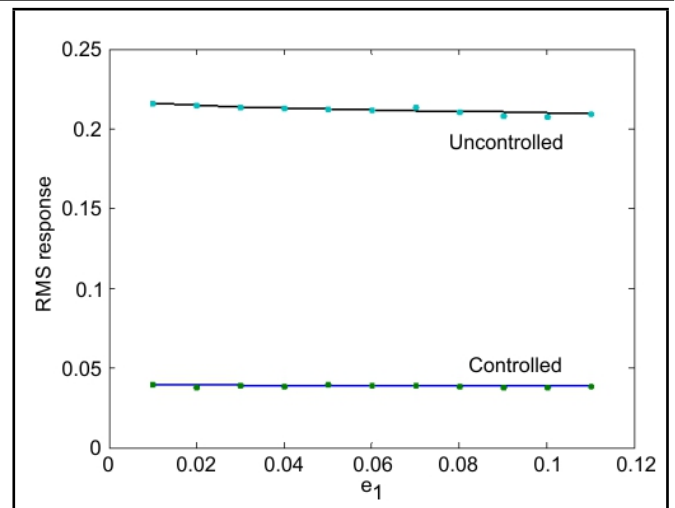

Figure 2. Controlled and uncontrolled RMS responses q for various observation noise amplitudes e_1 (solid line: analytical; dot: simulated).

$b = 1.0$, $d = 100.0$, $e_1 = 0.03$ and the quadratic control coefficient $S_{c2} = 1.0$ (i.e., coefficient of linear term of function g) are obtained and shown in Figs. 1 to 10. The probability densities of the controlled and uncontrolled generalized displacement q and velocity p responses are shown in Figs. 1(a) and 1(b), respectively. The controlled response probability density near zero is larger than the uncontrolled probability density so that the controlled response is reduced. Figure 2 illustrates that the controlled Root-Mean-Square (RMS) displacement response is smaller than the uncontrolled response and the controlled RMS response decreases slightly as the observation noise amplitude e_1 increases. The corresponding RMS optimal control φ_a^* varying with e_1 is shown in Fig. 10. Figure 3 illustrates the relative RMS response reduction $K > 80\%$ and the relative reduction per unit RMS control μ. The relative RMS response reduction K is the ratio of absolute difference of controlled and uncontrolled RMS responses to uncontrolled RMS response, and the relative reduction per unit RMS control μ is the ratio of the relative RMS response reduction to RMS control. The stochastic system response is reduced largely by using the proposed optimal control for various observation noise amplitudes. Figure 4 illustrates that the controlled RMS displacement response is smaller than the uncontrolled response for various observation coefficients d. Figure 5 illustrates the large relative RMS response reduction K and the small relative reduction per unit RMS control μ.

Figure 6 illustrates that the controlled RMS displacement response is smaller than the uncontrolled response, and the controlled RMS response has small increment relative to the uncontrolled response with the excitation amplitude e_0. Figure 7 illustrates the large relative RMS response reduction K and the small relative reduction per unit RMS control μ for various excitation amplitudes. Figure 8 illustrates that the controlled RMS displacement response decreases as the nonlinear stiffness coefficient k_3 increases and the controlled RMS response is smaller than the uncontrolled response. The corresponding RMS optimal control φ_a^* varying with k_3 is shown in Fig. 10. Figure 9 illustrates the large relative RMS response reduction K for various nonlinear stiffness coefficients. Therefore, the proposed optimal control can reduce largely the stochastic vi-
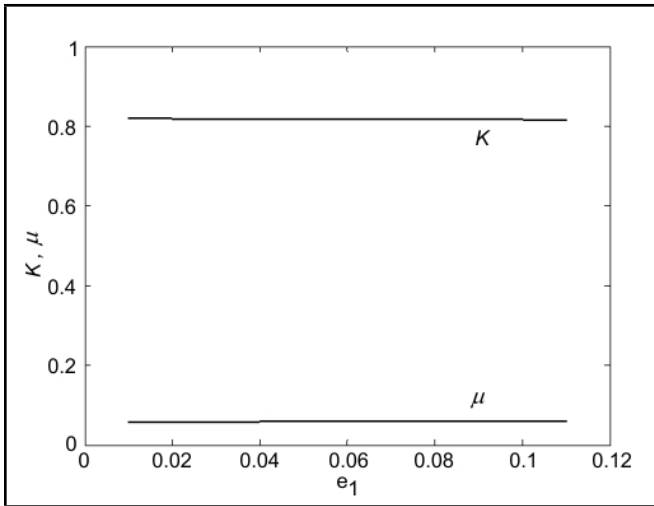
Figure 3. Relative response reduction K and its ratio to control μ for various observation noise amplitudes e_1.
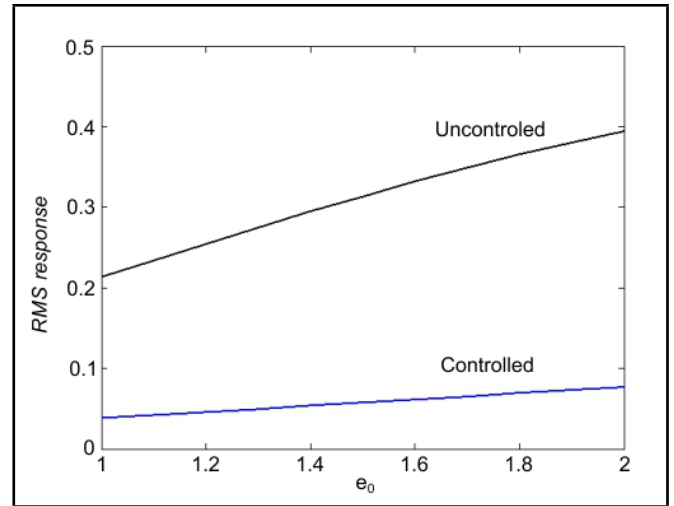

Figure 6. Controlled and uncontrolled RMS responses q for various excitation amplitudes e_0.
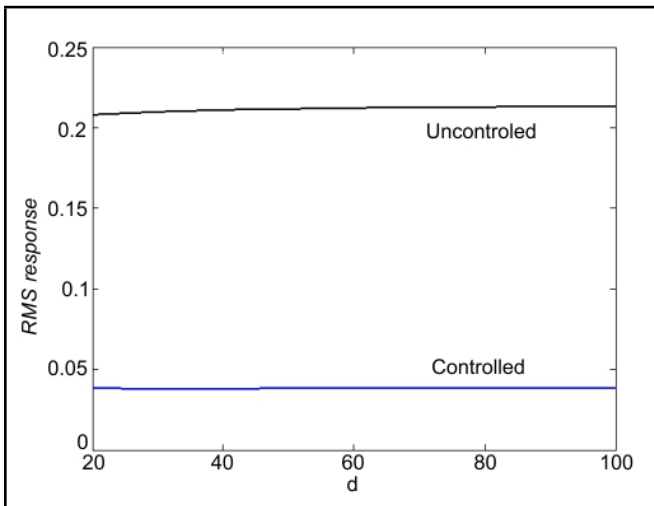

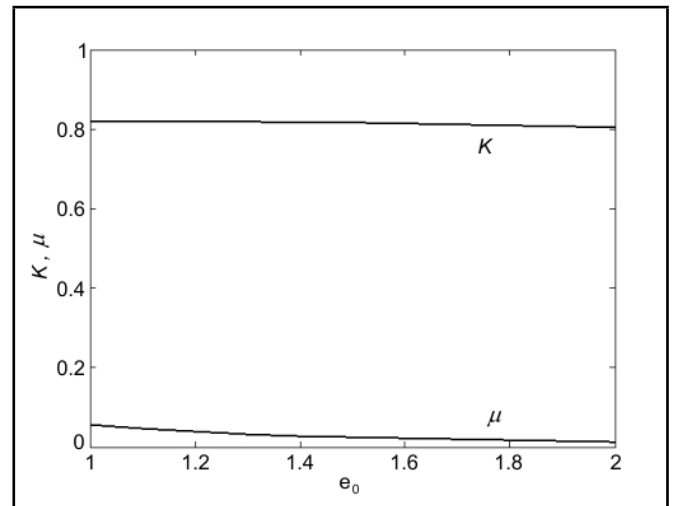**Figure 4.** Controlled and uncontrolled RMS responses q for various observation coefficients d.

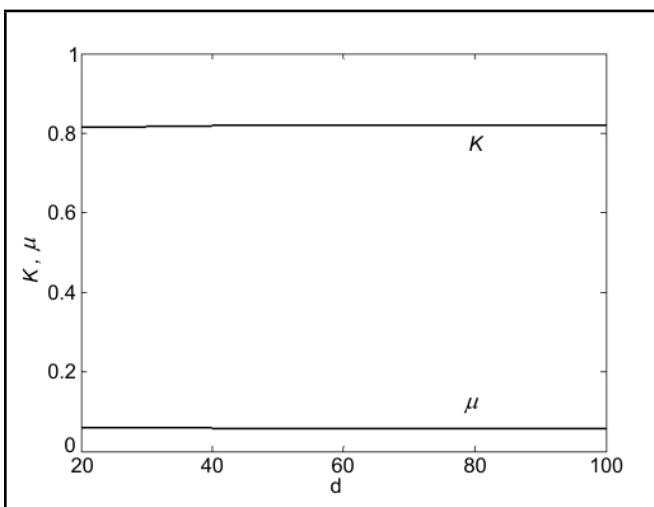**Figure 7.** Relative response reduction K and its ratio to control μ for various excitation amplitudes e_0.

Figure 5. Relative response reduction K and its ratio to control μ for various observation coefficients d.
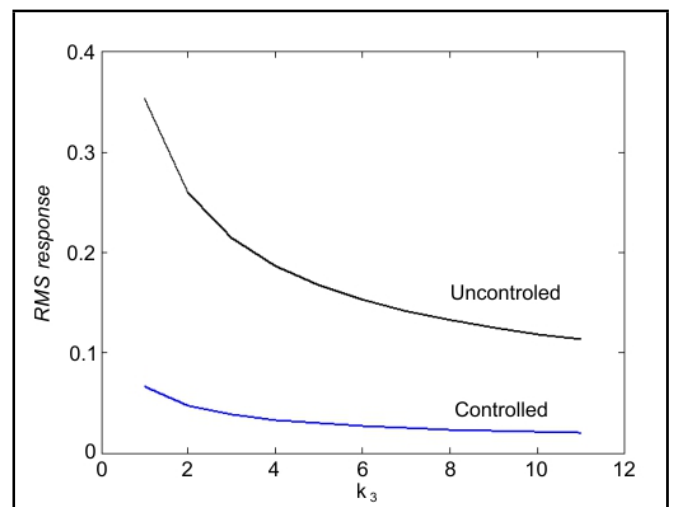

Figure 8. Controlled and uncontrolled RMS responses q for various nonlinear stiffness coefficients k_3.
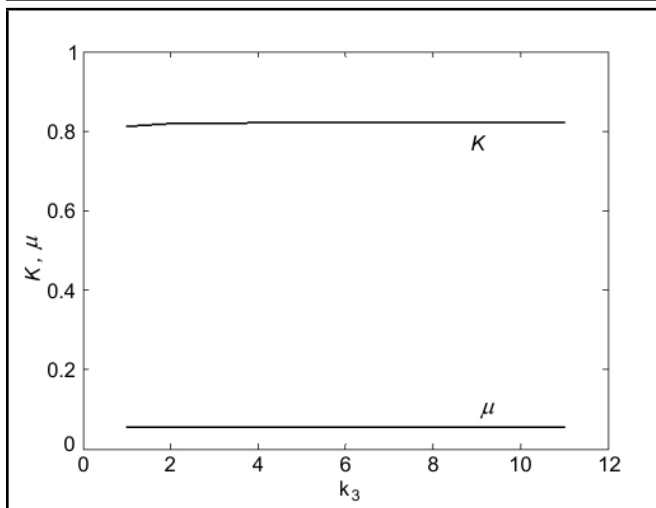
Figure 9. Relative response reduction K and its ratio to control μ for various nonlinear stiffness coefficients k_3.
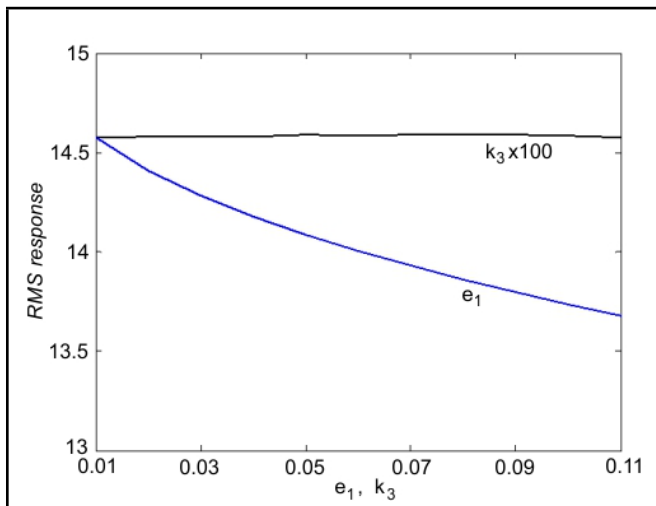

Figure 10. RMS controls φ_a^* for various observation noise amplitudes e_1 and nonlinear stiffness coefficients k_3 (note that $k_3 \times 100$ indicates that the value of k_3 is the product of horizontal coordinate and 100).

bration response of the nonlinear system (19) with a noised observation (20), and the control effectiveness has good robustness for various system parameters.

4.2. Example 2: two-degree-of-freedom nonlinear stochastic control system

To further illustrate the application and effectiveness of the proposed optimal control strategy, the control for a two-main-mode coupling vibration of geometric nonlinear beams with piezoelectric sensor and actuator under stochastic excitation was considered. The nonlinear stochastic system with a control and noised observation can be described by Eqs. (9) and (10), in which

$$H_a = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(k_{11}q_1^2 + k_{12}q_2^2) + k_2 q_1 q_2$$
$$+ \frac{1}{4}k_3(q_1 - q_2)^4,$$
$$X = [\ q_1 \quad q_2 \quad p_1 \quad p_2\ ]^{\mathrm{T}},$$
$$\Phi_a = [\varphi_{a1},\ \varphi_{a2}]^{\mathrm{T}},$$
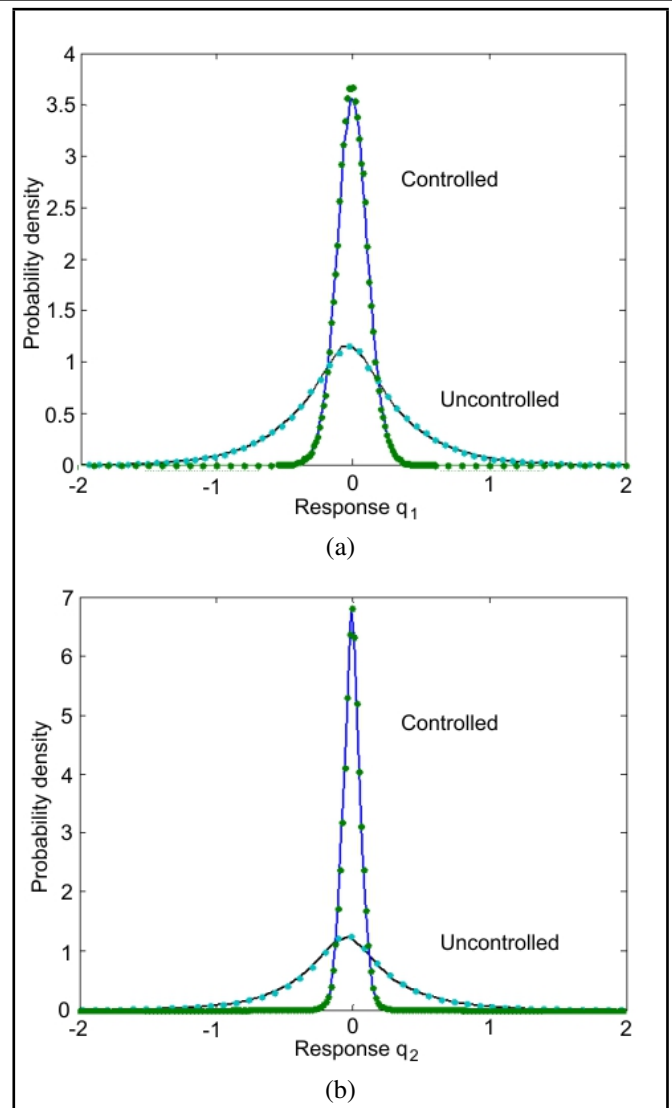

(a)

(b)

Figure 11. Controlled and uncontrolled probability densities (solid line: analytical; dot: simulated). The (a) Probability densities of displacement q_1 and (b) Probability densities of displacement q_2.

$$\Phi_s = [\varphi_{s1},\ \varphi_{s2}]^{\mathrm{T}},$$
$$C = \mathrm{diag}[0,\ 0,\ -c_1,\ -c_2],$$
$$F = [\ 0 \quad 0 \quad e_{01} \quad e_{02}\ ]^{\mathrm{T}},$$
$$B = \begin{bmatrix} 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \end{bmatrix}^{\mathrm{T}},$$
$$D = \begin{bmatrix} d_{11} & d_{12} & 0 & 0 \\ d_{12} & d_{22} & 0 & 0 \end{bmatrix},$$
$$E = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}; \qquad (25)$$

where q_i and p_i $(i = 1, 2)$ are respectively the generalized displacement and momentum, φ_{ai} $(i = 1, 2)$ is the control and φ_{si} $(i = 1, 2)$ is the observation, c_i, k_{1i}, k_2, and k_3 $(i = 1, 2)$ are the damping, linear stiffness, and nonlinear stiffness coefficients, respectively. On the other hand, b_i $(i = 1, 2)$ is the control coefficient, e_{0i} $(i = 1, 2)$ is the excitation amplitude, d_{ij} $(i, j = 1, 2)$ is the observation coefficient, and e_{1i} $(i = 1, 2)$ is the observation noise amplitude.

According to the above procedure, using the extended Kalman filter yields the differential Eq. (12) for the estimated

state. Using the stochastic averaging method converts Eq. (12) into the Itô differential Eq. (22) for the averaged Hamiltonian. Using the stochastic dynamical programming principle yields the dynamical programming equation similar to that in example 1. The optimal control law (17) leads to

$$\varphi_{ai}^* = -\frac{1}{2} R_{Cij}^{-1} b_j p_j \frac{\mathrm{d}V}{\mathrm{d}\hat{H}_a}. \tag{26}$$

The corresponding stationary value function equation is

$$\frac{1}{2}\sigma_h^2(\hat{H}_a)\frac{\mathrm{d}^2V}{\mathrm{d}\hat{H}_a^2} + m_h(\hat{H}_a)\frac{\mathrm{d}V}{\mathrm{d}\hat{H}_a}$$
$$- \frac{1}{4} < b_i p_i R_{Cij}^{-1} b_j p_j > (\frac{\mathrm{d}V}{\mathrm{d}\hat{H}_a})^2 + g(\hat{H}_a) = \gamma_0. \tag{27}$$

The optimal control is determined finally by Eq. (26) with (27). Then the controlled response and its statistics can be obtained by solving Eq. (12) with (20) or solving the Fokker-Planck-Kolmogorov equation associated with the Itô Eq. (22). The proposed optimal control efficacy for the nonlinear stochastic system with a noised observation is evaluated based on the response statistics.

Numerical results on the control system and observation with $c_1 = 1.0$, $c_2 = 5.0$, $k_{11} = 1.0$, $k_{12} = 4.0$, $k_2 = 0.01$, $k_3 = 1.0$, $e_{01} = 1.0$, $e_{02} = 0.3$, $b_1 = 1.0$, $b_2 = 0.5$, $d_{11} = 100.0$, $d_{12} = 10.0$, $d_{22} = 80.0$, $e_{11} = 0.03$, $e_{12} = 0.03$ and the quadratic control coefficient $S_{c2} = 1.0$ (i.e., coefficient of linear term of function g) are obtained and shown in Figs. 11 to 13. The probability densities of the controlled and uncontrolled generalized displacement responses (q_1 and q_2) are shown in Figs. 11(a) and 11(b), respectively. The controlled response probability density near zero is larger than the uncontrolled probability density. Figure 12 illustrates that the relative reduction K of the controlled RMS displacement response q_1 compared with the uncontrolled response and the relative reduction per unit RMS control μ vary with the observation noise amplitude e_{11}. Figure 13 illustrates that the relative reduction K of the controlled RMS displacement response q_1 and the relative reduction per unit RMS control μ vary with the observation coefficients d_{11}. It is seen again that the stochastic vibration response of the two-mode coupling nonlinear system with a noised observation is reduced largely by using the proposed optimal control.

5. CONCLUSIONS

The basic dynamic equations for a nonlinear stochastic control structure system with smart sensors and actuators have been given and simplified to the controlled, stochastically-excited, and dissipative Hamiltonian system with a noised observation. The optimally estimated nonlinear system with control and stochastic excitation has been determined based on the extended Kalman filter. The dynamical programming equation for the estimated system has been obtained based on the stochastic dynamical programming principle, and the optimal control law has been determined by the programming equation. The proposed optimal control strategy has been applied to two nonlinear stochastic systems with controls and noised
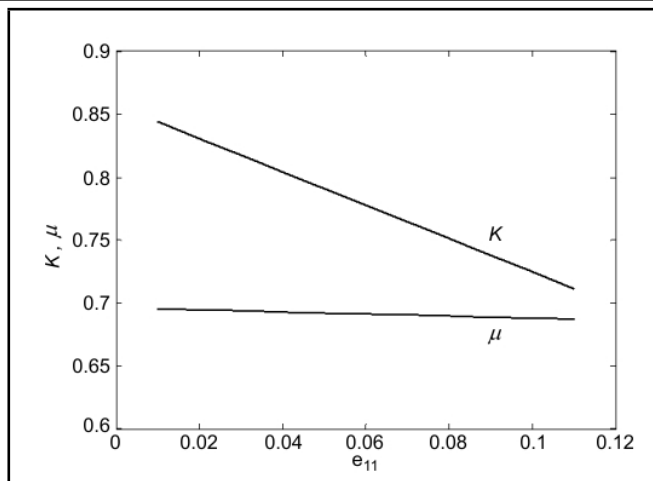

Figure 12. Relative response reduction K and its ratio to control μ for various observation noise amplitudes e_{11}.
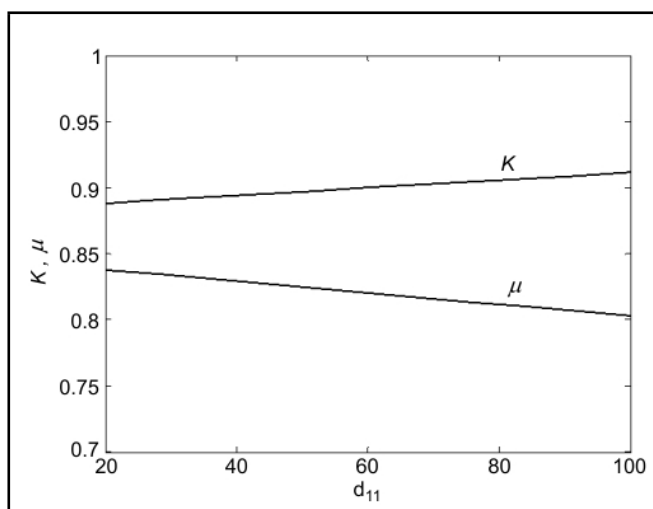

Figure 13. Relative response reduction K and its ratio to control μ for various observation coefficients d_{11}.

observations. Numerical results have illustrated that the proposed optimal control can reduce largely the stochastic vibration response of nonlinear systems with noised observations, and the control effectiveness is insensitive to varying system parameters such as observation and nonlinear coefficients. The proposed optimal control strategy is feasible and effective for the vibration response reduction of nonlinear stochastic smart structure systems with noised observations.

ACKNOWLEDGEMENTS

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