Implementation of a Boundary Element Method for High Frequency Scattering by Convex Polygons with Impedance Boundary Conditions

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Many acoustic and electromagnetic wave scattering problems can be formulated as the Helmholtz equation. Standard finite and boundary element method solution of these problems becomes expensive, as the frequency of incident wave increases. On going research has been devoted to finding methods that do not loose robustness when the wave number increases. Recently, Chandler-Wilde et al. have proposed a novel Galerkin boundary element method to solve the problem of acoustic scattering by a convex polygon with impedance boundary conditions. They applied approximation spaces consisting of piecewise polynomials supported on a graded mesh with smaller elements adjacent to the corners of the polygon and multiplied by plane wave basis functions. They demonstrated via rigorous error analysis that was supported by numerical experiments that the number of degrees of freedom required to achieve a prescribed level of accuracy need only grow logarithmically as frequency increases. In this paper, we discuss issues related to detail implementation of their numerical method.

1. INTRODUCTION

We consider the two-dimensional problem of scattering of a time-harmonic acoustic incident plane wave:

$$u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}, \text{ in } D := \mathbb{R}^2\backslash\bar{\Omega}; \tag{1}$$

by a convex polygon Ω, with impedance boundary Γ. Here $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{d} = (\sin\theta, -\cos\theta) \in \mathbb{R}^2$ is a unit vector representing the direction of the incident field, θ is the incidence angle, and the frequency of the incident wave is proportional to the wavenumber $k > 0$. The scattered field $u^s := u^t - u^i \in C^2(\bar{D})$ (where u^t and u^i denote the total and incident field respectively) satisfies the Helmholtz equation:

$$\Delta u^s + k^2 u^s = 0, \text{ in } D. \tag{2}$$

We shall consider the impedance boundary condition here:

$$\frac{\partial u^t}{\partial \mathbf{n}} + \mathrm{i}k\beta u^t = 0, \text{ on } \Gamma; \tag{3}$$

(where $\mathbf{n} = (n_1, n_2)$ denotes the outward unit normal vector to Γ, as depicted in Fig. 1 and $\beta \in L^\infty(\Gamma)$ and $\mathrm{Re}\beta > 0$ is relative surface admittance), and is supplemented with the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0; \tag{4}$$

where $r := |\mathbf{x}|$ and the limit holds uniformly in $\mathbf{x}/|\mathbf{x}|$. The Sommerfeld radiation condition is essential to scattering problems because it ensures that the scattered field is not reflected back from infinity.

As k increases, the incident field oscillates more rapidly, and so the complexity of the solution of Eq.(2) increases. As a result, the computational cost of standard schemes, such as the finite element or boundary element methods will grow in direct proportion to k, leading to large computing times for large k. It has been shown that in order to accurately model a wave, a fixed number of degrees of freedom M are needed per wavelength, with a rule of thumb in the engineering literature of 6 to 10 degrees of freedom per wavelength needed to maintain accuracy.[1,2] The price to pay for fixing M is that the number of degrees of freedom will be proportional to $(kL)^{d-1}$ in case of boundary element methods, where L is the linear dimension of the scattering obstacle and $d = 2$ or 3 is the dimension of the problem. Thus, as either k or the size of the scatterer grows, so does the number of degrees of freedom (at least $\mathcal{O}(k)$) in two-dimensional, hence the computational cost of numerical schemes increases. The previous and the current development on this active field of scattering problems is outlined explicitly at length in.[3]

For this paper we begin in Section 2 by discussing the boundary integral method we are going to apply. We describe the approximation space for the problem in Section 3. We proceed in Section 4 by presenting the implementation of our Galerkin scheme. We present formulas for the Galerkin scheme and describe how to evaluate oscillatory and non oscillatory integrals. In Section 5, we discuss how to solve non-overlapping integrals, a detail explanation of Gaussian quadrature rule is also explained in this section. In Section 6 we choose an example for our numerical experiment and present some results, whereas most of them can be found in.[4,5] We discuss our conclusion and some recommendations in Section 7.

2. BOUNDARY INTEGRAL EQUATION METHOD

The boundary value problem Eqs. (2) - (4) can be reformulated into boundary integral equation by applying Green's rep-