Developing Vibration Equations of an Orthotropic Wrapped Shell, Considering Residual Stress Effects; A Mathematical Approach

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In this paper, vibration equations of an orthotropic, thin rectangular plate wrapped around a porous drum are developed, considering residual stress effects. It is assumed that the plate is subjected to tension from both opposite sides and wrapped continuously around a cylindrical drum so that the wrapped portion behaves like a circular cylindrical shell. First of all, the Lame' parameters, required to constitute the geometry relations, are established for typical cylindrical shallow shell in cylindrical coordinate system. Then, the equations of motion are derived by utilizing the stored strain energy principle based on the Love assumptions. Finally, a set of more complete vibration equations is introduced by applying the simplifications of the Donnell-Mushtari-Vlasov theory. The equations derived under more stringent and precise assumptions are compared with those obtained and available in literature, and the discripancies are highlighted. The present study only aims to mathematically develop the governing relationships, where a numerical solution separately done by the authors can be found in other literature in which vibrational behavior has been completely discussed for moving and stable anisotropic wrapped plates.

NOMENCLATURE

A_1, A_2	gyroscopic inertia matrix
C_{ij}	Stiffness matrix corresponding to extension
D_{ij}	Stiffness matrix corresponding to bending
e_1, e_2, e_3	Unit vectors
g^1, g^2	Decomposed variables
h	Thickness of plate
K_1 to K_4	Symmetric stiffness matrix
k	Curvature
L_1, L_3	Length of flat segments
L_2	Length of wrapped or curved segment
M	Resultant moment, mass matrix
m	Number of terms in approximation function, longitudinal
n	Number of terms in approximatio function, lateral
P_0	Air pressure
P, P'	Typical vector points
P_{ij}	Dummy coefficients or stiffness ratios
Q, $\overline{Q}$	Stiffness matrices principal & material direction
$Q(t)$, $\dot{Q}(t)$	Reduced order matrix of spatial vector q
$Q_{\theta z}$, Q_{yz}	Transverse shear forces
q	Time dependent variable(vector)
q^r	Static load
$R(x, y)$	Radius of the drum or wrapped region
R	Redial direction, Residual index
$\overline{r}$	Displacement vector
T	Tension
U	Potential energy
u	Displacement in x direction
v	Displacement in y direction
W	Spatial or time independent variable (matrix)
w	Transverse displacement in z direction
X, Y, Z	Cartesian directions
α_1, α_2, α_3	Orthogonal curvilinear coordinates
ε	Strain
$\Phi(\theta, y)$	Spatial or time independent variable of Airy Function
ϕ	Airy Function
φ	Direct angle with X direction
θ	Tangential direction
θ_w	Wrapping angle
ρ	Mass density
σ^r	Residual stress
υ	Poisson's ration
ω	Excitation or response frequency
Ψ^1_{ij}, Ψ^2_{ij}	Approximation functions
ξ	Local coordinate in longitudinal direction
$\nabla^2_k(\cdot)$, $\nabla^2_r(\cdot)$	Second order Laplace operators

1. INTRODUCTION

Wrapped plates are widely used in many industries, such as manufacturing of papers, foils, and magnetic films; conveyer belt systems and band saw blades.[1–12] The general schematic of the application is shown in Fig. 1. The safety of relatively thin products, such as newspapers and webs, as well as their manufacturing appliance, has drawn the attention of engineers in recent years to make a design mechanism that is more efficient and versatile.

Vibration analysis of plates and shells during translation has been turned to an essential process in order to extract the modal properties and to prevent possible damages or failure. As shown in Fig. 1, because of the shape of the drum circumference, the mid part of the plate behaves like a cylindrical shell. It is also supposed that the pressurized air exits from the

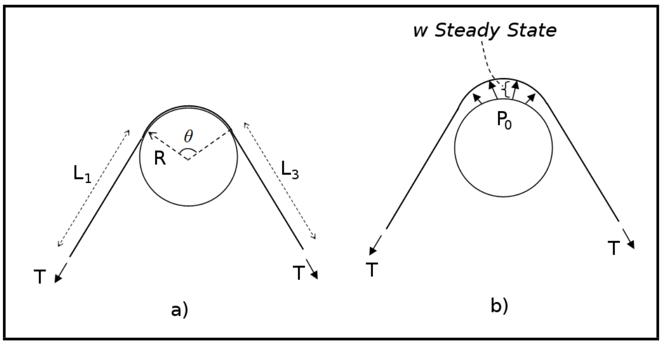

Figure 1. Lateral view of tensioned plate wrapped around a porous drum, (a) without air pressure, (b) with air pressure P_o after reaching to the steady state, θ angle of wrapped segment, R radius of drum, L_1, L_3 flat segments length.

holes around a drum and makes an air cushion in order to mitigate the contact between the plate and the drum and to avoid damage specially to considerably thin plates.

Fluid-structure interaction between the plate and ejected air from a porous drum has been studied by Müftü and Cole.[8] The governing equations, with a relatively high accuracy rate, have been derived based on the Donnel theory, which considers non-linear terms in strains and then is solved by the finite difference method and then by simplifying the equilibrium equation. The helical wrapping case has also been investigated separately by Müftü.[12]

Lopez and Müftü[13] analysed vibration of a thin tensioned web that had been helically wrapped around a turn-bar based on Kirchhoff-Love assumptions. They extracted dynamic properties including natural frequencies and mode shapes in the stable case by finite element method, and later, Sadeqi *et al.*,[1,14,15] followed and extended their investigations to orthotropic and anisotropic moving web, utilizing numerical Rayleigh-Ritz solution. Conflict between fiber orientation and helical wrapping angle, as well as critical speed limitation, was reported in their study.

The present work, aims to introduce a new set of equations, which governs the wrapping process with higher accuracy compared to the previous ones, considering the effect of stored stress, called residual stress, in the structure during the translation and wrapping.

First, in the next section, the general geometric aspect of the issue, as well as Lame' parameters, is surveyed, and the strains are obtained for a cylindrical shell that is continuously joined to rectangular flat plates from two sides based on Love assumptions in the cylindrical coordinate system.

Then, in Section 3, the governing equations of motion of the orthotropic tensioned shell- plate wrapped around a drum are derived based on the Donnell-Mushtari-Vlasov theory, considering the uniformly exhausted air pressure. The effects of air pressure and tension are separately considered as initial stress (residual stress) in the static equation set, which appears in the dynamic set of equations or in the same equations of motion. It is shown that the in- plane forces can be analyzed for both orthotropic and isotropic cases utilizing the Airy Stress Function coupled with the bending equations. Heaviside function is also used to present curvature distribution along the plate.

It is supposed that the tensioned plate encompasses the drum completely and is continuously bent around it. The exhausted air acts as a uniform distributed load on the shell segment and raises the plate till it reaches a steady state (Fig. 1 b)). Hence, the final transverse displacement in the gap between the shell and drum is the summation of the radial and lateral displacements with respect to the origin.

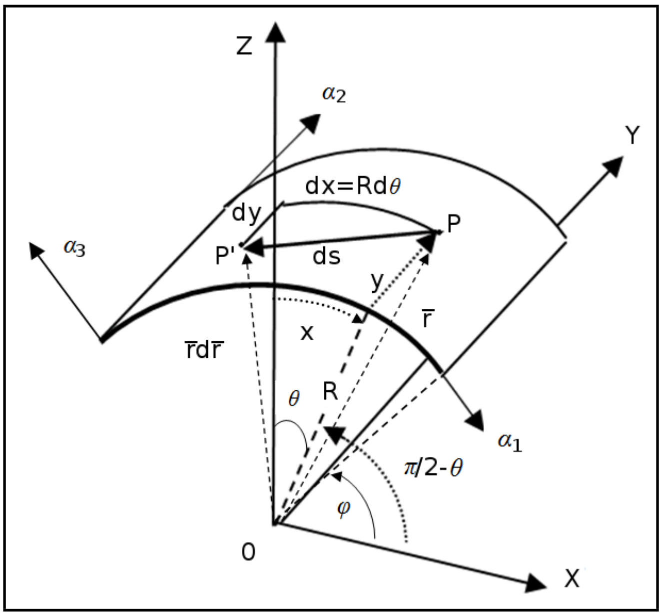

Figure 2. The neutral surface of a cylindrical shell in triple Cartesian (X, Y, Z), cylindrical (r, θ, y), and orthogonal curvilinear (α_1, α_2, α_3) coordinate system.

It is also assumed that the magnitude of the tension is constant during the translating process, and the supports at two opposite edges (where tension acts on) are considered a simple hinge. Therefore, the air pressure that raises the plate (separation between the plate and drum) causes an increase in the plate length and, consequently, would increase its stiffness.

In Section 4, as a result, obtained equations are compared to other references and newly appearing terms and discrepancies are clarified and discussed, but more numerical results can be found in Sadeqi *et al*[1].

2. GENERAL EQUATION OF CYLINDRICAL SHELL

Figure 2 shows a cylindrical non-closed shallow shell element illustrated in three coordinate systems: Global Cartesian, Cylindrical and Orthogonal Curvilinear.

Using Love assumptions together with a new coordinate system (Fig. 2), the governing equations can be derived.[13] Using a reference neutral surface as shown in Fig. 2, the differential variation of the displacement vector $\bar{r}$ between neighboring points P and P' is[16,17]

$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2; \tag{1}$$

and the magnitude of ds becomes

$$(ds)^2 = A_1^2 (d\alpha_1)^2 + A_2^2 (d\alpha_2)^2. \tag{2}$$

This equation is the first fundamental form, and the coefficients A_1, A_2 are the Lame' parameters. These parameters are determined with respect to the chosen system of coordinates. If the cylindrical system is selected as reference coordinates, the position vector can be rewritten in terms of θ and y as:

$$\bar{r} = R\sin\theta \bar{e}_1 + y\bar{e}_2 + R\cos\theta \bar{e}_3; \tag{3}$$

where $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ are the unit vectors in X, Y, Z direction, respectively. Considering $\alpha_1 = \theta$, and $\alpha_2 = y$ results in

$$A_1 = \left|\frac{\partial \bar{r}}{\partial \theta}\right| = |R\sin\theta \bar{e}_1 - R\cos\theta \bar{e}_3| = = R\sqrt{\cos^2\theta + \sin^2\theta} = R; \tag{4a}$$

$$A_2 = \left|\frac{\partial \bar{r}}{\partial y}\right| = |\bar{e}_2| = 1. \tag{4b}$$

In the present study the cylindrical coordinates for either flat or wrapped (curved) parts is used. In this coordinate system, the radial, rotational, and longitudinal components are in y, θ, z, directions, respectively. Also, in order to present the curvature distribution along the plate, use Heaviside step function is introduced as

$$\frac{1}{R(\theta)} = \begin{cases} 0 & \tan^{-1}\left(\frac{L_1}{R}\right) + \frac{\pi+\theta_w}{2} < \theta < \frac{\pi+\theta_w}{2} \\ \frac{1}{R} & \frac{\pi+\theta_w}{2} \le \theta \le \frac{\pi-\theta_w}{2} \\ 0 & \frac{\pi-\theta_w}{2} < \theta < \frac{\pi-\theta_w}{2} - \tan^{-1}\left(\frac{L_3}{R}\right) \end{cases}. \tag{5}$$

which helps to define the curvature effects in terms of the wrapping angle θ_w and the flat segment length L_1, L_3.

2.1. Strain-Displacement Relations

Using assumptions of the Love theory, strain-displacement relations can be expressed as:

$$\begin{Bmatrix} \varepsilon_\theta \\ \varepsilon_y \\ \varepsilon_{\theta y} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_\theta^0 \\ \varepsilon_y^0 \\ \varepsilon_{\theta y}^0 \end{Bmatrix} + z \begin{Bmatrix} k_\theta \\ k_y \\ k_{\theta y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{A_1}\frac{\partial u}{\partial \theta} + \frac{w}{R(\theta)} + \frac{1}{2A_1^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \\ \frac{1}{A_2}\frac{\partial v}{\partial y} + \frac{1}{2A_2^2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{1}{A_1}\frac{\partial v}{\partial \theta} + \frac{1}{A_2}\frac{\partial u}{\partial y} + \frac{1}{A_1 A_2}\left(\frac{\partial w}{\partial \theta}\right)\left(\frac{\partial w}{\partial y}\right) \end{Bmatrix} + z \begin{Bmatrix} \frac{1}{A_1}\frac{\partial}{\partial \theta}\left(\frac{u}{R(\theta)} - \frac{1}{A_1}\frac{\partial w}{\partial \theta}\right) \\ \frac{1}{A_2}\frac{\partial}{\partial y}\left(-\frac{1}{A_2}\frac{\partial w}{\partial y}\right) \\ \frac{1}{A_2}\frac{\partial}{\partial y}\left(\frac{u}{R(\theta)} - \frac{1}{A_1}\frac{\partial w}{\partial \theta}\right) + \frac{1}{A_1}\frac{\partial}{\partial \theta}\left(-\frac{1}{A_2}\frac{\partial w}{\partial y}\right) \end{Bmatrix}. \tag{6}$$

2.2. Stress-Strain Relationship & Resultant Loads

For an orthotropic element, the stress strain relationship is:

$$\begin{Bmatrix} \sigma_\theta \\ \sigma_y \\ \sigma_{\theta y} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\theta \\ \varepsilon_y \\ \varepsilon_{\theta y} \end{Bmatrix};$$

$$Q_{11} = \frac{E_1}{1 - \upsilon_{12}\upsilon_{21}}, \quad Q_{12} = \upsilon_{21} Q_{11}, \quad Q_{22} = \frac{E_2}{E_1} Q_{11}, \quad Q_{66} = G_{12}; \tag{7}$$

where E_1, E_2 are elastic modules, G_{12} indicates shear modules and υ_{12}, υ_{21} show Poisson's ratio in the material principal directions. Resultant force due to normal stress in θ direction will be

$$\begin{Bmatrix} N_\theta + N_\theta^r \\ N_y + N_y^r \\ N_{\theta y} + N_{\theta y}^r \\ M_\theta + M_\theta^r \\ M_y + M_y^r \\ M_{\theta y} + M_{\theta y}^r \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_\theta + \sigma_\theta^r \\ \sigma_y + \sigma_y^r \\ \sigma_{\theta y} + \sigma_{\theta y}^r \\ z(\sigma_\theta + \sigma_\theta^r) \\ z(\sigma_y + \sigma_y^r) \\ z(\sigma_{\theta y} + \sigma_{\theta y}^r) \end{Bmatrix} dz = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\theta^0 \\ \varepsilon_y^0 \\ \varepsilon_{\theta y}^0 \\ k_\theta \\ k_y \\ k_{\theta y} \end{Bmatrix};$$

$$C_{ij} = \bar{Q}_{ij} h, \quad D_{ij} = \bar{Q}_{ij}\frac{h^3}{12}; \tag{8}$$

where $\overline{Q}_{ij}$ is the same Q_{ij} in Eq. (7), after transformation and h thickness of the plate. Also, superscript r, indicates the residual effects.

2.3. Equation of Motion

Using the Minimum Potential Energy Principle, the stored strain energy in an infinitesimal element, which is under stress σ (due to transverse pressure) and residual or pre-stress σ^r (due to tension and bending) will be[17]

$$dU = [\frac{1}{2}(\sigma_\theta \varepsilon_\theta + \sigma_y \varepsilon_y + \sigma_{\theta y}\varepsilon_{\theta y} + \sigma_{\theta z}\varepsilon_{\theta z} + \sigma_{yz}\varepsilon_{yz}) + +\sigma_\theta^r \varepsilon_\theta + \sigma_y^r \varepsilon_y + \sigma_{\theta y}^r \varepsilon_{\theta y} + \sigma_{\theta z}^r \varepsilon_{\theta z} + \sigma_{yz}^r \varepsilon_{yz}]dV. \tag{9}$$

Substituting the stresses in terms of resultant loads, and finally rearranging the expressions in terms of displacements (u, v, w) , five relationships coupled of residual and non-residual terms are achievable and can be divided into two sets: one static set due to the static load q^r resulting in initial stresses as

$$-A_2\frac{\partial N_\theta^r}{\partial \theta} - A_1\frac{\partial N_{\theta y}^r}{\partial y} - A_1 A_2 \frac{Q_{\theta z}^r}{R(\theta)} = A_1 A_2 q_\theta^r; \tag{10a}$$

$$-A_2\frac{\partial N_{\theta y}^r}{\partial \theta} - A_1\frac{\partial N_y^r}{\partial y} = A_1 A_2 q_y^r; \tag{10b}$$

$$-A_2\frac{\partial Q_{\theta z}^r}{\partial \theta} - A_1\frac{\partial Q_{yz}^r}{\partial y} + A_1 A_2 \frac{N_\theta^r}{R(\theta)} = A_1 A_2 q_z^r; \tag{10c}$$

$$A_2\frac{\partial M_\theta^r}{\partial \theta} + A_1\frac{\partial M_{\theta y}^r}{\partial y} - A_1 A_2 Q_{\theta z}^r = 0; \tag{10d}$$

$$A_2\frac{\partial M_{\theta y}^r}{\partial \theta} + A_1\frac{\partial M_y^r}{\partial y} - A_1 A_2 Q_{yz}^r = 0; \tag{10e}$$

where $Q_{\theta z}$ and Q_{yz}, according to the plane stress assumptions, are the transverse shear forces as

$$Q_{\theta z} = \int_{z=\frac{-h}{2}}^{z=\frac{h}{2}} (\sigma_{\theta z} + \sigma_{\theta z}^r) dz, \quad Q_{yz} = \int_{z=\frac{-h}{2}}^{z=\frac{h}{2}} (\sigma_{yz} + \sigma_{yz}^r) dz; \tag{11}$$

and another, dynamic set due to the dynamic loads q, demonstrating the vibration behavior of the problem. Disregarding the resultant forces N_θ, N_y, $N_{\theta y}$ compared to the residual resultant forces N_θ^r, N_y^r, $N_{\theta y}^r$, these equations can be rearranged as:

$$-A_2\frac{\partial N_\theta}{\partial\theta}-A_1\frac{\partial N_{\theta y}}{\partial y}-A_1A_2\frac{Q_{\theta z}}{R(\theta)}+A_1A_2\rho h\ddot{u}=A_1A_2q_\theta; \tag{12a}$$

$$-A_2\frac{\partial N_{\theta y}}{\partial\theta}-A_1\frac{\partial N_y}{\partial y}+A_1A_2\rho h\ddot{v}=A_1A_2q_y; \tag{12b}$$

$$-A_2\frac{\partial Q_{\theta z}}{\partial\theta}-A_1\frac{\partial Q_{yz}}{\partial y}+A_1A_2\frac{N_\theta}{R(\theta)}-\frac{A_2}{A_1}\left(\frac{\partial N_\theta^r}{\partial\theta}\right)\left(\frac{\partial w}{\partial\theta}\right)-\frac{A_2}{A_1}N_\theta^r\left(\frac{\partial^2 w}{\partial\theta^2}\right)-\frac{A_1}{A_2}\left(\frac{\partial N_y^r}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)-\frac{A_1}{A_2}N_y^r\left(\frac{\partial^2 w}{\partial y^2}\right)-\left(\frac{\partial N_{\theta y}^r}{\partial\theta}\right)\left(\frac{\partial w}{\partial y}\right)-\left(\frac{\partial N_{\theta y}^r}{\partial y}\right)\left(\frac{\partial w}{\partial\theta}\right)-(N_{\theta y}^r)\left(2\frac{\partial^2 w}{\partial\theta\partial y}\right)+A_1A_2\rho h\ddot{w}=A_1A_2q_z; \tag{12c}$$

$$A_2\frac{\partial M_\theta}{\partial\theta}+A_1\frac{\partial M_{\theta y}}{\partial y}-A_1A_2Q_{\theta z}=0; \tag{12d}$$

$$A_2\frac{\partial M_{\theta y}}{\partial\theta}+A_1\frac{\partial M_y}{\partial y}-A_1A_2Q_{yz}=0; \tag{12e}$$

In the present paper, the Donnell-Mushtari-Vlasov theory is used to simplify and reach a solvable form rather than facing difficulties caused by coupling that exists between the in-plane forces (Eqs. (12a) and (12b)) and the bending moments (Eqs. (12c)-(12e)).

3. DONNELL-MUSHTARI-VLASOV THEORY

The theory is applicable for shells under transverse loads, and in contrast to other theories, the bending and in-plane effects are also considered. The fundamental assumption is to neglect the effect of in-plane deflection in the bending strains but not in the plane strains.

The second assumption is to neglect the in-plane inertial effects ($\ddot{u}$, $\ddot{v}$), and the last, the term $Q_{\theta z}/R$ must be eliminated in Eq. (12a). Applying these assumptions as well as substituting the values of in-plane and bending strains into the two moment Eqs. (12d) and (12e), then substituting shear forces in terms of related moments in Eq. (12c), five Eqs. (12a)-(12e) are reduce to three. Finally, by introducing the Airy stress function ϕ as

$$N_\theta=\frac{1}{A_2^2}\frac{\partial^2\varphi}{\partial y^2}; \tag{13a}$$

$$N_y=\frac{1}{A_1^2}\frac{\partial^2\varphi}{\partial\theta^2}; \tag{13b}$$

$$N_{\theta y}=-\frac{1}{A_1A_2}\frac{\partial^2\varphi}{\partial y\partial\theta}; \tag{13c}$$

three equations can be diminished again to a single equation but with two unknowns as

$$D_{11}\frac{1}{A_1^4}\frac{\partial^4 w}{\partial\theta^4}+2(D_{12}+2D_{66})\frac{1}{A_1^2}\frac{\partial^4 w}{\partial\theta^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+\nabla_k^2\varphi-\nabla_r^2 w+\rho h\ddot{w}=q_z; \tag{14}$$

where $\nabla_k^2(\cdot)$ and $\nabla_r^2(\cdot)$ are as in Eqs. (15a) and (15b) (see on the next page).

Now, the deflection components u, v are eliminated and one equation remains with two unknowns w and ϕ. To find the second equation, the compatibility equation is necessary, which can be written as[17]

$$\frac{k_\theta}{R(\theta)}+\frac{1}{A_1^2}\left(\frac{\partial^2(\varepsilon_y^0)}{\partial\theta^2}\right)+\frac{1}{A_1^2}\left(\frac{\partial^2(\varepsilon_\theta^0)}{\partial y^2}\right)-\frac{1}{A_1A_2}\frac{\partial^2(\varepsilon_{\theta y}^0)}{\partial\theta\partial y}=0. \tag{16}$$

Substituting the strains with terms of the axial forces given by

$$\left\{\begin{array}{c}\varepsilon_\theta^0\\ \varepsilon_y^0\\ \varepsilon_{\theta y}^0\end{array}\right\}=\left[\begin{array}{ccc}P_{11}&-P_{12}&0\\ P_{12}&-P_{22}&\\ 0&&P_{66}\end{array}\right]\left\{\begin{array}{c}N_\theta\\ N_y\\ N_{\theta y}\end{array}\right\};$$

$$P_{11}=\frac{C_{22}}{C_{11}C_{22}-C_{12}^2},\quad P_{22}=P_{11}\frac{C_{11}}{C_{22}},\quad P_{12}=P_{11}\frac{C_{12}}{C_{22}},\quad P_{66}=\frac{1}{C_{66}}; \tag{17}$$

and using Eqs. (13a)–(13c) results in

$$P_{22}\frac{1}{A_1^4}\frac{\partial^4\varphi}{\partial\theta^4}+\frac{1}{A_1^2A_2^2}(P_{66}-2P_{12})\frac{\partial^4\varphi}{\partial\theta^2\partial y^2}++\frac{1}{A_2^4}P_{11}\frac{\partial^4\varphi}{\partial y^4}-\left(\frac{1}{R(\theta)}\frac{\partial^2 w}{\partial y^2}\right)=0; \tag{18}$$

The previous equation together with Eq. (14), are the equations of motion of an orthotropic shell wrapped around a porous pressurized drum. These fourth order equations are coupled, and four boundary conditions are needed at each edge (two for w, and two for ϕ) to solve them. A numerical approach has been taken to solve the typical simplified equations for the anisotropic moving case in Sadeqi *et al.*[1] Assuming harmonic time dependence and separation of variables as

$$w(\theta,y,t)=W(\theta,y)e^{j\omega t}=\sum_{i=1}^{m}\sum_{j=1}^{n}g_{ij}^1\Psi_{ij}^1(\theta,y)e^{j\omega t}; \tag{19a}$$

$$\varphi(\theta,y,t)=\Phi(\theta,y)e^{j\omega t}=\sum_{i=1}^{m}\sum_{j=1}^{n}g_{ij}^2\Psi_{ij}^2(\theta,y)e^{j\omega t}; \tag{19b}$$

and also by decomposing in terms of variables g^1 and g^2, the following matrix can be defined as

$$\left[\begin{array}{cc}K_1-M\omega^2&K_2\\ K_3&K_4\end{array}\right]\left\{\begin{array}{c}g^1\\ g^2\end{array}\right\}=0; \tag{20}$$

where g^1 and g^2 are of rank$(m\times n)\times 1$, vectors and K_1 to K_4, and M are symmetric $(m\times n)\times(m\times n)$ rank matrices in terms of approximation functions given by Ψ_{ij}^1, Ψ_{ij}^2 (see[1]). For the isotropic case, Eq. (14) becomes

$$D\nabla^4 w+\nabla_k^2\varphi-\nabla_r^2 w+\rho h\ddot{w}=q_z; \tag{21}$$

where

$$\nabla^4(.)=\left(\frac{1}{A_1^4}\frac{\partial^4(.)}{\partial\theta^4}+\frac{2}{A_1^2A_2^2}\frac{\partial^2(.)}{\partial\theta^2}\frac{\partial^2(.)}{\partial y^2}+\frac{1}{A_1^4}\frac{\partial^4(.)}{\partial y^4}\right); \tag{22}$$

and other operators, compatibility equation, boundary conditions, and Airy function ϕ remain the same as for the orthotropic case. Following a similar procedure leads to

$$Eh\nabla_k^2 w-\nabla^4\varphi=0; \tag{23}$$

$$\nabla_k^2(\cdot) = \frac{1}{A_1 A_2}\frac{\partial}{\partial y}\left(\frac{1}{R(\theta)}\frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial y}\right); \tag{15a}$$

$$\nabla_r^2(\cdot) = \frac{1}{A_1^2}\left(N_\theta^r\frac{\partial^2(.)}{\partial\theta^2} + \left(\frac{\partial(N_\theta^r)}{\partial\theta}\right)\left(\frac{\partial(.)}{\partial\theta}\right)\right) + \frac{1}{A_2^2}\left(N_y^r\frac{\partial^2(.)}{\partial y^2} + \left(\frac{\partial(N_y^r)}{\partial y}\right)\left(\frac{\partial(.)}{\partial y}\right)\right) + \frac{1}{A_1A_2}\left[2\left(N_{\theta y}^r\frac{\partial^2(.)}{\partial\theta\partial y}\right) + \left(\frac{\partial(N_{\theta y}^r)}{\partial\theta}\right)\left(\frac{\partial(.)}{\partial y}\right) + \left(\frac{\partial(N_{\theta y}^r)}{\partial y}\right)\left(\frac{\partial(.)}{\partial\theta}\right)\right]. \tag{15b}$$

$$N_x + N_x^r = C\left[\frac{1}{A_1(x)}\frac{\partial u}{\partial x} + \frac{w}{R(x)} + \frac{1}{2A_1^2(x)}\left(\frac{\partial w}{\partial x}\right)^2 + v\left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)\right], \quad C = \frac{Eh}{(1-v^2)}; \tag{28}$$

substituting w and ϕ from Eqs. (19a) and (19b) into Eqs. (21) and (23) and then applying operators ∇^4 and ∇_k^2, respectively, and subtracting each other gives a single but higher order equation achieved as[17–20]

$$D\nabla^8 W + Eh\nabla_k^4 W - \nabla^4\nabla_r^2 W - \rho h\omega^2\nabla^4 W = 0; \tag{24}$$

which is the appropriate form of vibration equation for the isotropic case.

4. COMPARISON WITH OTHER REFERENCES

Substituting $\theta = x/R$ and transforming to the Cartesian coordinates with $A_1 = A_2 = 1$, the equations obtained in the present work are compared to the simplified cases available in other references. In the absence of dynamic load q_z and acceleration term $\ddot{w}$, for isotropic case, Eq. (22) becomes:

$$D\nabla^4 w - \nabla_r^2 w + \nabla_k^2\varphi = 0; \tag{25}$$

where operators from Eqs. (15a) and (15b) can be expressed as

$$\nabla_r^2 w = N_x^r\frac{\partial^2 w}{\partial x^2} + 2N_{xy}^r\frac{\partial^2 w}{\partial x\partial y} + N_y^r\frac{\partial^2 w}{\partial y^2} + \frac{\partial N_{xy}^r}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial N_{xy}^r}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial N_y^r}{\partial y}\frac{\partial w}{\partial y}; \tag{26}$$

$$\nabla_k^2\varphi = \frac{N_x}{R(x)}. \tag{27}$$

Substituting the values of in-plane forces from Hooke's law in terms of strains is presented in Eq. (28) (see on the top of the page).

Considering the upward direction of air pressure as positive, and since the tension in x direction and the distributed air pressure normal to shell are the only external loads, solution of in-plane force N_θ^r in Eq. (10c), results in

$$N_x^r = T + R(x)q_z^r. \tag{29}$$

Neglecting the in-plane strains and substituting for N_x from static solution leads to:

$$D\nabla^4 w - (T + R(x)q_z^r)\frac{\partial^2 w}{\partial x^2} - 2N_{xy}^r\frac{\partial^2 w}{\partial x\partial y} - N_y^r\frac{\partial^2 w}{\partial y^2} - \frac{\partial N_{xy}^r}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial N_{xy}^r}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial N_y^r}{\partial y}\frac{\partial w}{\partial y} + \frac{C}{R^2(x)}w = q_z^r + \frac{T}{R(x)}. \tag{30}$$

4.1. Equilibrium Equation in Reference[8]

The governing equation for the non-helical wrapping case of plate, has been developed by Müftü and Cole[8] based on Donnel theory as

$$D\nabla^4 w - T\frac{\partial^2 w}{\partial x^2} - N_y\frac{\partial^2 w}{\partial y^2} - 2N_{xy}\frac{\partial^2 w}{\partial x\partial y} + \frac{C}{R^2(x)}w = p - \frac{T}{R(x)}; \tag{31}$$

where p is the air pressure so that the positive direction is assumed to be downward. The basic differences between this equation and one obtained here (Eq. (30)) can be expressed as:

1. The effect of exhausting air pressure on increasing tension is considered in Eq. (3). The air pressure (assuming upward direction as positive) causes the wrapped part to rise up, and also causes increasing elongation from boundaries; consequently, the value of effective tension increases. In Müftü and Cole,[8] it is assumed that the tension is constant during the process; therefore, the plate together with the supports could move in a rigid manner as long as it would reach a steady state.
   On the other hand, the effect of tension on the air pressure is the same for both equations because the tension T causes a pressure equal to T/R from the wrapped region to the drum surface. Hence, the minimum pressure required to separate the wrapped part from the drum is T/R.
2. In Eq. (31), there are no residual resultant forces N_y^r and N_{xy}^r. These can be neglected in the absence of distributed in-plane forces and small thickness. Otherwise, the static set Eqs. (10a)-(10e), should be independently solved using the Airy function ϕ.

4.2. Equations for Non-Helical Wrapping[12,13]

The equilibrium equation for wrapped plate in the presence of the air pressure exhausted from the turn bar has been developed as[12]

$$D\nabla^4 w - T\frac{\partial^2 w}{\partial x^2} + \frac{C}{R^2(x)}w = p + p_c - \frac{T}{R(x)}; \tag{32a}$$

where p_c, the air pressure, is considered the contact pressure. The in-plane forces are also neglected in this equation. In the absence of in-plane forces ($N_y = N_{xy} = 0$), and for static load, Eq. (30), can be simplified to

$$D\nabla^4 w - (T + q_z^r R(x))\frac{\partial^2 w}{\partial x^2} + \frac{C}{R^2(x)}w = q_z^r + \frac{T}{R(x)}; \tag{32b}$$

Note that if the positive direction of air pressure in both Eqs. (32a) and (32b) are assumed to be the same, the right-hand-side terms would be equal. Also, the vibration equation of the mentioned plate in Müftü[12] is presented as

$$D\nabla^4 w - T\frac{\partial^2 w}{\partial x^2} + \frac{C}{R^2(x)} w + \rho h \ddot{w} = 0. \tag{33a}$$

In the absence of air pressure ($q_z^r = 0$) and in-plane forces ($N_y = N_{xy} = 0$), Eq. (21), could be simplified to

$$D\nabla^4 w - \left(T + q_z^r R(x)\right)\frac{\partial^2 w}{\partial x^2} + \frac{C}{R^2(x)} w + \rho h \ddot{w} = 0. \tag{33b}$$

For constant tension during the process of wrapping and also after reaching steady state, Eqs. (33a) and (33b) would be equal.

5. SUMMARY AND CONCLUSION

In this study, general equations of wrapped orthotropic plate were developed by defining new coordinates and virtual work approach. It was shown that in/out plane forces affect the bending moments and vice versa, so that the constitutive equations appear as coupled form. Any change in the tension leads to change in the air pressure required to separate the plate of the cylindrical drum, as well as stiffness of the plate and natural frequencies.

First, two residual and dynamic sets of quintuplet equations were obtained, then by using the Donnell-Mushtari-Vlasov theory, they were simplified and reduced to three equations in terms of second-order derivation of transverse displacement w and, finally, by defining Airy functions, reduced to two equations of forth order. In other words, the five second-order equations were diminished to the three equations of higher order and could even decrease for isotropic case to a single equation of eighth order.

It was also shown that the obtained equations had additional terms than available ones in other references, because of contributing the in/out of plane resultant force in bending equation which causes a higher accuracy in analysis. Moreover, interaction between the tension and air pressure as a transverse load was considered and kept in the equations.

A numerical solution to find frequencies and mode shapes for a typical simplified equations of anisotropic moving wrapped plate studied in a separate paper, which has been addressed in the context.

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