A decade ago transformational cloaking was discovered by Greenleaf, Lassas, and Uhlmann. Soon to follow were extensions of the approach to electrodynamics, elastodynamics, and acoustics. Early on, it was suggested that cloaking could be utilized to protect structures against seismic surface waves, by placing a suitable cloaking layer around the structure. Apart from the graded properties of the cloaking layer, the boundary conditions between cloak and surrounding soil, and the boundary condition between cloak and the structure are of importance for the performance of such a cloak. This is not surprising, as already the ancient Greeks discovered that, by arranging the interface between important structures and the underlying ground, it was possible to achieve some protection against earthquake waves, that might otherwise damage the structures. In the present paper we explore theoretically and numerically, how the inner boundary condition between cloak and the protected structure may affect the performance of some suggested modern elastodynamic cloaks.

1. Introduction

Figure 1: Illustration of the correlation between earthquake zones and the location of nuclear power plants around the world.

Ancient civilizations discovered that by arranging the interface between important structures (like temples) and the underlying ground, it was possible to achieve some protection against earth-
quake waves that might otherwise damage the structures. By placing the buildings on layers of animal skin and carbon gravel, it could apparently survive many earthquakes with comparatively little damage.

A decade ago transformational cloaking, for electric impedance tomography, was discovered by Greenleaf, Lassas, and Uhlmann\(^1\). Soon to follow were extensions of the approach to electrodynamics\(^2\), elastodynamics\(^3\) and acoustic\(^5\). Early on, it was suggested that cloaking could be utilized to protect structures against seismic surface waves by placing a suitable cloaking layer around the structure one wants to protect. A large scale example could be protection of nuclear power-plants built on earthquake zones, see Figure 1\(^6\). To put the protecting layer on the surface may be motivated by the fact that at least at some distance from the epicenter of the seismic event causing the seismic waves, the harmful waves are to a large extent surface waves, Rayleigh waves, rather than bulk waves. (The geometric attenuation in 2D being slower than that in 3D.)

2. **Restricted micropolar elasticity**

![Schematic representation of micropolar micro-structure.](image)

In order to allow for the possibility to cloak or deflect vibrations of different kinds, specific material characteristics must be designed. As will be presented in the subsequent section, the proposed cloak will require that the constitutive stiffness tensor \(C\) in the conventional elastic relation

\[
\sigma = C : \epsilon
\]

has to have major symmetry but not minor symmetry. One material that satisfies this requirement is micropolar material, see Figure 2. A general (hemitropic) micro-polar medium satisfies the constitutive equations\(^7,8\)

\[
\begin{align*}
\sigma & = C : (\nabla \otimes u - \epsilon \cdot \phi)^T + B : (\nabla \otimes \phi)^T \\
\mu & = B^T : (\nabla \otimes u - \epsilon \cdot \phi)^T + A : (\nabla \otimes \phi)^T
\end{align*}
\]

and the equations of motion

\[
\begin{align*}
\nabla \cdot \sigma^T & = \rho \ddot{u} \\
\nabla \cdot \mu^T + \epsilon : \sigma & = j \cdot \dot{\phi}
\end{align*}
\]
The totally non-symmetric third order tensor in three dimensions may be defined as $\epsilon = I \times I$, where $I$ is the second order unit tensor. The double contraction is defined so that $X \cdot Y = X_{ij} Y_{ij}$ for $X = X_{ij} e_i \otimes e_j$ and $Y = Y_{ij} e_i \otimes e_j$. Similarly $K \cdot Y = K_{ijkl} Y_{k\ell} e_i \otimes e_j \otimes e_k \otimes e_\ell$. $C, A$ are major symmetric tensors of order 4. The symmetry is under transpose of the first and last index pairs. Boldface superscript $T$ has been used to denote this transposition, and ordinary sans serif $T$ for transpose of second-order tensors. The major symmetry requirement can thus be stated as

$$A^T = A_{k\ell ij} e_i \otimes e_j \otimes e_k \otimes e_\ell = A_{ij k\ell} e_i \otimes e_j \otimes e_k \otimes e_\ell = A$$

and similarly for $C$. Note that neither of these tensors need satisfy the minor symmetries, whereby e.g. $C_{ijkl} \neq C_{ijlk}$ in general. $B$ is a tensor of order 4, with no assumptions on symmetry.

We now assume three things. First, that the material is centro-symmetric so that $B = 0$. Second, that time-harmonic conditions prevail, with time factor $\exp(-i \omega t)$. And third, that the curvature stiffness is much higher than the stiffness with respect to strains. Then the tensor $A$ in some suitable sense becomes very large, while the micro-moment tensor $\mu$ remains finite.

To be slightly more specific regarding the last point, let’s say that we e.g. have some scalar parameter $a$ that we let tend to positive infinity, and that for some $\epsilon > 0$

$$A = a M \otimes M + O[a^{1-\epsilon}] \text{ as } a \to +\infty$$

where $M = M_{ij} e_i \otimes e_j$ is some second order tensor. (We also assume that the matrix formed from its coefficients $M_{ij}$ is invertible.) Here the introduced non-standard open product of two second order tensors is defined as

$$X \otimes Y = X_{ik} Y_{j\ell} e_i \otimes e_j \otimes e_k \otimes e_\ell$$

Then, if $\mu$ is to remain finite, we must have that $\nabla \otimes \phi \to 0$ as $a \to +\infty$, and $\phi \to$ constant throughout the body. If the boundary condition is $\phi \big|_{\Gamma} = 0$, and the limit of $\phi$ is uniform, then in the limit $\phi$ must vanish throughout $\mathcal{B}$.

Under these three assumptions, the set of four equations implies

$$\nabla \cdot \sigma^T + \rho \omega^2 u = 0 \text{ in } \mathcal{B}$$

$$\sigma = C : (\nabla \otimes u)^T$$

This means that, given suitable boundary conditions on the displacement and/or traction vector, the assumption leads to a boundary value problem for the displacement field alone. The field $\mu$ may subsequently be retrieved. Equations [5] and [6] must be supplemented by a boundary condition, e.g.,

$$\hat{n} \cdot \sigma^T \big|_{\Gamma} = h$$

where $h$ is a prescribed vector field defined on $\Gamma$, and $\hat{n}$ is the outward-pointing unit normal.

Finally, it can be concluded that Equations [5] and [6] pertain to classic linear elastodynamics, without micropolar effects, whenever $C$ is minor symmetric.
3. Cloaking transformation

Figure 3: Cloaking transformation in 2D. a) Reference configuration b) Cloaked configuration.

As found in a special case in \(^9\), and more generally in \(^3\), a body consisting of this kind of micropolar material can mimic a homogeneous isotropic elastic body in the following sense: If the body inside \(B\) is micro-polar with a stiffness and density that varies in a suitable manner, the Traction-to-Displacement (TtD) map, that maps \(h \mapsto u\bigg|_\Gamma\) for the micro-polar solid, may be identically the same as that of a homogeneous, isotropic solid occupying the same region \(B^o\).

Consider a diffeomorphism \(\psi : B \to B^o\) such that the limit of \(\psi\) on \(\Gamma\) is the identity map. Let \(C^o = C^o_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l\) denote the elasticity tensor in a homogeneous isotropic elastic material, and let \(\rho_0\) be its constant mass density, and put

\[
C_{ijkl}(x) = J(x) \frac{\partial x_j}{\partial x^o_p} C^o_{pq} \frac{\partial x_q}{\partial x^o_L}
\]

\[
\rho(x) = J(x) \rho_0
\]

where

\[
x^o = \psi(x), \quad J(x) = \det \left( \frac{\partial \psi(x)}{\partial x} \right), \quad \frac{\partial x}{\partial x^o} = \left( \frac{\partial \psi(x)}{\partial x} \right)^{-1}
\]

Note that the elasticity tensor of the micro-polar material, with components given by Equation \([8]\) does indeed satisfy the major symmetry condition. Then we may consider the \(x^o_j\) as Cartesian coordinates in the homogeneous isotropic solid, and the \(x_j\) as Cartesian coordinates in the inhomogeneous, anisotropic micro-polar material, each occupying \(B\). If the displacement field in the homogeneous material is \(u^o(x^o)\), satisfying

\[
\frac{\partial}{\partial x_j^o} \left( C^o_{ijpq} \frac{\partial u^o_p(x^o)}{\partial x^o_q} \right) + \rho_0 \omega^2 u^o_i(x^o) = 0
\]

then \(u(x) = u^o(\psi^{-1}(x))\) satisfies

\[
\frac{\partial}{\partial x_j} \left( C_{ijpq}(x) \frac{\partial u_p(x)}{\partial x_q} \right) + \rho(x) \omega^2 u_i(x) = 0
\]
Even without special assumptions on the normal derivative of $\psi$ on $\Gamma$, it may be verified that both tractions and displacements on the boundary $\Gamma$ coincide in the two cases. For any surface excitation of the micro-polar body, the response at the boundary is the same as for a homogeneous body. The TtD maps of the two bodies are identical.

The idea is by blowing up a point in the homogeneous material model into the anisotropic (cloaking) material model by means of the transformation, one can then hide an object inside the new domain, see Figure 3. The cloak is primarily constructed for bulk waves, but also works well with Rayleigh waves. This because the Rayleigh wave effects are isolated to the surface and will thus mimic bulk waves and the boundary conditions on the surface will be satisfied because of symmetry.

4. The importance of the inner boundary condition

![Figure 4: Acceleration of the rigid-cylinder; comparison of different boundary conditions.](image)

The requirement to achieve perfect cloaking is that the transformation between the reference body and cloaked configuration is identical at the inner boundary. The cloaked body need to rotate freely as in the reference configuration and therefore we need a "slider" boundary condition on the inner boundary. In the numerical simulation, the difference between this kind of boundary condition versus welded boundary condition is very small, see Figure 4, and depends on the accuracy of the transformation, which is approximated in the finite element software by the element size. Because of the fact that the numerical results are approximations, only a partial cloak can be achieved and therefore the inner boundary condition is of less importance.

5. Numerical results by means of COMSOL Multiphysics™

The model implemented into COMSOL Multiphysics™ consists of a sub-surface cylinder subjected to surface waves (Rayleigh waves). In order to benchmark and compare the effects of the cloaking material, two cases are simulated in COMSOL Multiphysics™; one with the cloaking configuration and one without, see Figure 2. The results are presented as an acceleration plot of the rigid-cylinder and a surface displacement plot when subjected to Rayleigh waves after long time, see Figure 5.
The results conclude that the micropolar cloak performs well and effectively lowers the acceleration of a rigid-cylinder by at least half. In order to lower the acceleration even more, the computational accuracy should be increased. If the element size in the finite element software would be infinitely small together with the “slider” boundary condition, a perfect cloak could be achieved and the acceleration of this cylinder would go to zero.

6. Concluding remarks

In this contribution, it has been concluded that the considered restricted micropolar continuum can theoretically be used for elastodynamic cloaking from Rayleigh waves. However, this should come as no great surprise, as it has already been shown that cloaking against bulk waves is possible in that manner. In fact, the Rayleigh wave at a horizontal plane may be represented as a linear combination of such waves, though with imaginary wave numbers in the vertical direction.

COMSOL Multiphysics™ proved to be useful, after some modification, to perform the modeling of this type of cloak. The numerical results show that the inner boundary condition is of less importance than the approximation made on the transformation in the finite element code.

Bibliography

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