VIBRATION DAMPING OF THE CANTILEVER BEAM
WITH THE USE OF THE PARAMETRIC EXCITATION

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The effect of the parametric vibration damping is theoretically well described and the first experiments are published. The paper deals with the analysis of the parametric excitation of the vibration body and the cantilever beam which is modeled by the lumped parameter system. The principle of the parametric excitation is based on the use of piezoactuators as an element of the controlled stiffness which varies periodically in time according to a sinusoidal function. The excitation frequency is selected the frequency of the principle parametric resonances and combination parametric resonances. The main topic of the paper is focused on the analysis of the effect of these frequencies and the amplitude of excitation on the damping ratio. The paper presents a simulation study of this method of damping vibrations.

1. Introduction

A cantilever beam is a simple example of a mechanical structure that can serve as an object for testing the active vibration control. A promising way to reduce vibration of mechanical structures is parametric excitation, which is fundamentally different from the active vibration control systems developed by Krenk [1]. Unlike active vibration damping, which is based on the use of linear methods for control of the linear time invariant systems (LTI), the new way of damping uses a periodic change of a parameter, usually spring stiffness \( K_{n+1} \) as it is shown in Fig. 1. Of course the mean value of the stiffness \( K_{\overline{\delta}} \) is constant in time. Such a system becomes non-linear and potentially unstable for an interval of the frequency \( \omega_0 \) and \( 2\mu \) of the mentioned parameter variation.

\[
K_{n+1} = K_{\overline{\delta}}(1 + 2\mu \cos(\omega_0 t))
\]

**Figure 1.** Parametric excitation of a linear time-invariant (LTI) multibody system.

A fundamental research in the field of instability of the non-linear mechanical systems was done by Tondl [2, 3, 4]. The main conclusions were published many times. They state formulas for calculating the frequency of parametric resonances and describe the instability intervals for the frequency of parametric excitation if the mechanical system contains only one element with the periodic change of the parameter according to the sinusoidal function of time.
Now we will use the summary of a paper written by Petermeier and Ecker [5] who distinguish between Principal Parametric Resonances at frequencies $\omega_{j/n}^{p}$ and Combination Parametric Resonances at frequencies $\omega_{jnk/n}^{c}$. These frequencies are defined as follows

$$\omega_{j/n}^{p} = \frac{2\Omega_{j}}{n}, \quad \omega_{jnk/n}^{c} = \left| \frac{\Omega_{j} - \Omega_{k}}{n} \right|, \quad j, k = 1, 2, ..., n = 1, 2, ...$$

where $\Omega_{j}$ and $\Omega_{k}$ are the $j$-th and $k$-th natural frequency of the linear system. The denominator $n$ represents the order of the parametric resonances. Horst Ecker is developing methods for parametric damping of mechanical systems and carries out experiments in this field damping at the Vienna University of Technology.

Unlike Petermeier and Ecker [5], who use the cantilever beam model of the Timoshenko type, this paper examines the effect of parametric control based on the lumped parameter model of the Euler-Bernoulli type as the beam is thin and the Euler-Bernoulli theory is best to use for design the model. The simulation of the beam vibration decay and the random excitation is a tool for analyzing the mentioned effect. The main objective of the article is to demonstrate a positive effect of parametric excitation on damping of the cantilever vibration with the use of a piezoactuator patch.

2. Mathematical model of cantilever beams

The mathematical model should be simple enough that it could be created in the Matlab-Simulink environment as a lumped parameter model. The resonant frequency and deflection of the beam of the discrete elements should match the beam as a continuum. It is assumed that the beam is combined from rigid elements which are connected by flexible links formed by torsion springs as it is shown in Fig. 2. It is also assumed that the bending stiffness $K_{\delta} = \tau/\Delta\delta$ of the flexible links of the adjacent elementary beams relates the applied bending moment $\tau$ to the resulting relative rotation $\Delta\delta$ of the elementary beams and the corresponding potential energy. All the bending stiffness will be identical except for the one connection whose stiffness would be periodically changed. The periodic changes in the bending stiffness can be achieved by using the patch piezoactuator which is glued to the surface of the beam and is not connected with the rigid frame. The effect of this phenomenon will be discussed at the end of this chapter. This assumption will change the system on nonlinear and non-stationary.

![Figure 2. Coordinates of elements of a cantilever beam.](image)

The multibody system in Fig. 1 is associated with the Cartesians coordinates $x, y, z$. The cantilever beam is clamped at the $xy$-plane and its centerline is parallel to the $z$-axis. It is assumed only a planar motion of the cantilever beam in the $yz$-plane. The link of a pair of the adjacent beam elements is considered in the mentioned plane as free with the mentioned torsion spring. To avoid the additional set of constrains for the link of the individual beam elements in one point the coordinate system is chosen in such a way that describes motion of the meeting points of two adjacent elemen-
tary beams called nodes [6]. The vertical coordinates of these nodes are designated by \( y_1, y_2, \ldots, y_N \). The angle of rotation of the elementary beams with respect to the horizontal axis can be designated by \( \delta_1, \delta_2, \ldots, \delta_N \) and their measure in radians can be expressed by \( \delta_n = (y_n - y_{n-1})/\Delta L \) if all angles are small enough. For \( n = 1 \) it is valid \( \delta_1 = y_1/\Delta L \) and therefore it is assumed that \( \delta_0 = 0 \).

The coordinates of the beam equidistant points in the Cartesian coordinates and the independent generalized coordinates for Lagrange's equations of motion are identical. For further derivation it makes sense only the motion in the direction of the y-axis. Because they are assumed small deformations, the shifts of the nodes in the direction of the z-axis are neglected. The potential energy \( V \) of the cantilever beam is as follows

\[
V = \sum_{n=0}^{N-1} \frac{1}{2} K_n (\Delta \delta_n)^2 = \frac{1}{2 \Delta L^2} \sum_{n=0}^{N-1} K_n \left( y_{n+1} - 2y_n + y_{n-1} \right)^2. \tag{2}
\]

For the beam in the horizontal position, the force of gravity is simply added to the force acting at the element. The kinetic energy \( T \) of the cantilever beam as a continuum which is replaced by its lumped parameter model is as follows

\[
T = \sum_{n=1}^{N} \left[ \frac{1}{2} \Delta m \left( \frac{dV}{dt} \right)^2 + \frac{1}{2} \Delta J_n \left( \frac{d\delta_n}{dt} \right)^2 \right], \tag{3}
\]

where \( \Delta J_n \) is the moment of inertia [kg m^2] about the axis which is situated the horizontal and it is perpendicular to the centerline of the elementary beam. The cantilever beam is a conservative system. Lagrange's equations of motion of such a system are as follows

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{y}_n} \right) - \frac{\partial T}{\partial y_n} + \frac{\partial V}{\partial y_n} = 0, \quad n = 1, 2, \ldots, N. \tag{4}
\]

The first derivatives of the potential energy \( V \) in the Lagrange's equations with respect to the variables \( y_k, k = 1, 2, \ldots, N \) and \( \dot{y}_k, k = 1, 2, \ldots, N \) were used to create the stiffness matrix. The non-stationary stiffness \( K_{n+1} \) depends neither on \( y_k \) nor on \( \dot{y}_k \) which is missing in the formula (5)

\[
K_{n+1} = K_{0} (1 + 2\mu \cos(\omega_0 t)). \tag{5}
\]

The only problem is that three equations of motion contain the stiffness \( K_{n+1} \) as shown in the following formulas [6]. After the substitution from (5) it can be obtained

\[
\begin{align*}
n-1: \quad & \ldots + K_{n+1} \Delta L^2 \left( y_{n+1} - 2y_n + y_{n-1} \right) + \ldots = 0 \quad \ldots + 6K_{0} \Delta L^2 y_{n-1} + \ldots = -K_{0} \Delta L^2 \delta(t) \\
n: \quad & \ldots - 2K_{n+1} \Delta L^2 \left( y_{n+1} - 2y_n + y_{n-1} \right) + \ldots = 0 \quad \Rightarrow \quad \ldots + 6K_{0} \Delta L^2 y_{n} + \ldots = +2K_{0} \Delta L^2 \delta(t) \\
n+1: \quad & \ldots + K_{n+1} \Delta L^2 \left( y_{n+1} - 2y_n + y_{n-1} \right) + \ldots = 0 \quad \ldots + 6K_{0} \Delta L^2 y_{n+1} + \ldots = -K_{0} \Delta L^2 \delta(t)
\end{align*}
\]

where

\[
\delta(t) = 2\mu \left( y_{n+1} - 2y_n + y_{n-1} \right) \cos(\omega_0 t) = 2\mu \left( y_{n+1} - y_n - (y_n - y_{n-1}) \right) \cos(\omega_0 t). \tag{7}
\]

The derivation of the equations of motion has been published previously, so it will include only the most important formulas [7, 8] with the use of Lagrange's equation. After introduction symbols \( \mathbf{M} \) for a mass square matrix, \( \mathbf{K} \) for a stiffness square matrix and \( \mathbf{y} = [y_1, y_2, \ldots, y_N]^T \) for a coordinate column vector into the matrix equation of motion we obtain the equation for forced vibration

\[
\mathbf{M} \ddot{\mathbf{y}} + \mathbf{K} \mathbf{y} = \mathbf{F}(t), \tag{8}
\]
where
\[ F_e = 2K_\delta \Delta L \mu \left(y_{n+1} - 2y_n + y_{n-1}\right) \cos\left(\omega_0 t\right) \left[\cdots, 0, -1, +2, -1, 0, \cdots\right]^T. \] (9)

It is supposed that the cross section of the beam is a rectangular. The moment of inertia of the beam element about the horizontal x-axis and perpendicular to the centerline of the beam is calculated according to the formula
\[ J_x = \Delta m \left(\Delta L^2 + h^2\right)/12 \] where \( h \) is height and \( \Delta m \) is mass of the element. The Lagrange's equation gives the mass matrix \( M \) of the following form
\[
M = \begin{bmatrix}
B & A & A \\
A & B & A \\
\vdots & \vdots & \vdots \\
A & B & A \\
A & B/2
\end{bmatrix}
\]

Similarly as for the mass matrix \( M \) the stiffness matrix \( K \) is created with the use of the Lagrange's equations. Assuming that the bending stiffness of all the element joint of is the same, then we can write
\[
K = \frac{K_s}{\Delta L^2} \begin{bmatrix}
6 & -4 & 1 \\
-4 & 6 & -4 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & -4 & 6 & -4 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & -4 & 5 & -2 \\
1 & -2 & 1
\end{bmatrix}. \] (11)

Handbooks of mechanics \[9\] state formulas for the deflection of the cantilever beam of the length \( L \) at the free end. The beam is considered as a continuum. This deflection due to the force \( F_N \) acting at the free end in the direction of the z-axis is as follows
\[
y_N = \frac{F_N L^3}{3EI_z}, \] (12)

where \( E = 2.14 \times 10^{11} \) N/m\(^2\) is Young’s modulus of the beam material, \( I_z = bh^3/12 \) is the area moment of inertia of the beam cross-section about the horizontal x-axis.

The stationary bending stiffness of each flexible link in Fig. 2 is designated as \( K_n \), \( n = 1, 2, \ldots, N \). The stationary values of all the bending stiffness is the same and equal to
\[
K_n = \frac{\tau}{\Delta \delta} = \eta \frac{EI_z}{\Delta L}, \] (13)

where \( \eta \) is a multiplication factor which will be discussed in the next paragraph section. The value of the multiplicative factor in equation (13) was designed so that the beam deflection at the free end of the lumped-parameter model fits the deflection, which was calculated using the formula
\[
y_0 = K^{-1}F_0, \] (14)
where \( F_0 = [0, 0, \ldots, F_N]^T \) is a force acting at the center of gravity.

It was found out that the value of this factor depends on the number \( N \) of elementary beams. The results are shown in Table 1. Increasing the number of elements means that the factor \( \eta \) tends to one which means that the bending stiffness tends to \( K_\delta = EI_\delta / \Delta L \) and the discrete cantilever beam tends to a continuous beam. Selection of \( N = 5 \) will be used in the simulations and therefore the correction factor \( \eta \) plays an important role. The positive consequence of the introduction of the correction factor is also that the resonant frequencies of a beam calculated for the beam as a continuum coincide with frequencies which are calculated for the beam which is divided in the elements with the use of the eigenvalues \( \lambda = \Omega^2 \) of the matrix \( K^{-1} M \).

<table>
<thead>
<tr>
<th>Number ( N )</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor ( \eta )</td>
<td>3</td>
<td>1.8750</td>
<td>1.3200</td>
</tr>
<tr>
<td>Number ( N )</td>
<td>10</td>
<td>20</td>
<td>50</td>
</tr>
<tr>
<td>Factor ( \eta )</td>
<td>1.1550</td>
<td>1.0763</td>
<td>1.0302</td>
</tr>
</tbody>
</table>

As mentioned earlier, the part of the stiffness of the \((n+1)\)-th spring in Fig. 2 is a sinusoidal function of time. The angular frequency \( \omega_0 \) of the parametric excitation is given by the formulas (1). The parameter \( \mu \) which represents the amplitude of changes in stiffness is searched experimentally with the use of simulation.

The presence of viscous damping, such as a dissipative force, extends the left side of the equation of motion by an additional term which is proportional to velocity

\[
M \ddot{y} + C \dot{y} + K y = F_r(t), \quad C = \alpha M + \beta K,
\]

where the matrix of proportionality \( C \) for Rayleigh damping is a linear combination of the mass and stiffness matrices. The relationship to the damping ratio \( \xi \) can be seen using the formula \( \xi = \pi (\alpha f + \beta f) \), where \( f \) is the frequency in hertz [10], where the constant of proportionality are as follows \( \alpha = 0.159 \, [s^{-1}] \) and \( \beta = 0.0000411 \, [s] \).

### 3. Matlab-Simulink model

The equation of motion (14) is the second order ordinary differential equation. After the introduction of this substitution \( x_1 = y \), and \( x_2 = \dot{y} \), then the second order equation of motion is divided into two ordinary differential equations of the first order

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= M^{-1} F_e - M^{-1} C x_2 - M^{-1} K x_1,
\end{align*}
\]

where \( M^{-1} K \), \( M^{-1} C \) or \( M^{-1} \) are parameters in the form of matrices. An arrangement of the sub-LTI system which models the cantilever beam for an arbitrary number of elements in the Matlab-Simulink environment is shown in Fig. 3. Entering the simulation is complete with the initial conditions \( x_1(0) = y(0) \) and \( x_2(0) = \dot{y}(0) \). The blocks of the Gain type contain matrix and their input is a
vector, and therefore the output is a vector as well. The simulation model (16) of the parametric excitation of the beam which is divided into 5 elements and the patch piezoactuator is between the third and fourth element as is shown in Fig. 4 and 5.

The effect of active vibration control is often demonstrates on the vibration decay of the beam which is bent into a stationary deflected position by acting the force of 10 N and then is suddenly released. The initial conditions are as follows

\[
y(0)=0, \quad \dot{y}(0)=0.\tag{17}
\]

The simulation concerns the cantilever beam of the following parameters: \( L = 0.5 \) m, \( b = 0.04 \) m, \( h = 0.005 \) m. The beam is divided into 5 elements, and therefore has 5 resonant frequencies. The value of these frequencies in Hz and in radians per second are listed in Table 2.

<table>
<thead>
<tr>
<th>Table 2. Resonant frequencies.</th>
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<tbody>
<tr>
<td>( f_k ) [Hz]</td>
</tr>
<tr>
<td>( \Omega_k ) [rad/s]</td>
</tr>
</tbody>
</table>

**Figure 4.** An equivalent arrangement to the parametric excitation.

**Figure 5.** Matlab-Simulink model of the beam for five elements and feedbacks.

### 4. Simulation of vibration decay

The effect of parametric excitation will be evaluated on the base of the decay rate from the initial beam deformation. The decay of vibration at the free end of the cantilever beam with excitation switch-off is shown in Fig. 6. The decay of vibration is shown on the upper panel A) in Fig. 6. The lower panel B) shows the decay in dB. Example of the effect of the parametric excitation on the decay rate for the frequency \( \omega_{12} \) which is the Combination Parametric Resonance of the first order is shown in Fig. 7. The half the amplitude \( \mu \) of excitation is 0.23.

The effect of the parametric excitation on the decay rate for the frequency \( n = 1 \), \( \omega_{11} = 2\Omega_1 \) of the first and second order of the Principle Parametric Resonances was tested as well, but without positive results. The parametric excitation at the frequencies \( \omega_{3-1/1}, \omega_{3-2/1}, \omega_{4-1/1}, \omega_{4-2/1} \) and \( \omega_{2-3/2} \) does not also improve the damping.
5. Simulation of response to random force

The vibration decay from the deflected position assesses the efficiency of damping in the time domain. The damping effect cannot be observed in the frequency domain. We assume that a force with a frequency spectrum that is close to white noise acts at the free end of the beam. The frequency spectrum of the deflection free end of the beam is calculated for the identical configuration of the feedbacks. All the above mentioned spectra are shown in the panels of Fig. 8.

The parametric excitation frequency is set at the resonant frequency \( \omega_{2-1/1}^C = \Omega_2 - \Omega_1 \), i.e. 86.9 Hz, and half the amplitude \( \mu \) of excitation is 0.023. The best excitation amplitude was determined by simulation calculation which starts with zero amplitude.

Frequency spectrum of decaying vibration from panel A) of Fig. 6 and 7 are shown in Fig. 8. Increasing of the decay rate is due to increasing the 3 dB bandwidth of the spectrum dominant peak.

6. Conclusions

The parametric excitation is one of the tools to increase the efficiency of vibration damping. The paper examined the Principle and Combination Parametric Resonance frequencies and their effect on the decay rate for the cantilever beam. It was found that the greatest effect on vibration damping has the difference frequency between the first and second resonance frequency of the cantilever beam. This frequency difference is a Combination Parametric Resonance frequency of the
first order. Other parametric resonance frequencies are without effect on the vibration damping. The frequency spectra clearly explain why increases damping of the parametrically damped systems. The dominating peak in the spectrum splits into two adjacent peaks and their magnitude is reduced.

The objective of this paper was to demonstrate that parametric damping reduces vibration. The amplitude of the changes in stiffness was chosen with the use of simulation.

REFERENCES

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