EFFECTS OF RANDOM FACTOR ON AIRFOIL FLUTTERERS

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The two-dimensional airfoil with structural non-linearity of order five in turbulent flow is investigated. Both the improved averaging method and the stochastic averaging method are used to obtain the reduced stochastic differential equation. Then by using the singularity theory the explicit parameter conditions for P-bifurcation are deduced. Upon these results, the amplitude probability density and joint probability density of pitch and pitch rate are displayed for four qualitatively different cases. The effects of the random excitation strength and free-stream velocity of Stochastic P-bifurcation are discussed. All theoretical results are verified by Monte-Carlo simulations.

1. Introduction

Limit cycle oscillation induced airfoil flutter is one hot spot in nonlinear aeroelasticity. Numerous studies indicate the periodic solutions originated from Hopf bifurcation may result in the second bifurcation, which will lead the system to the multi-stable state. However, little research of bifurcation problems has been carried for nonlinear airfoil systems under random excitation.

Limit Cycle Oscillation (LCO), Hopf bifurcation and chaos are the major directions for nonlinear airfoil flutter analysis. LCO stability and amplitude are directly influenced by structural nonlinearities. Second bifurcation occurred in airfoil system with cubic nonlinearity has been intensively studied for flutter analysis. High order Harmonic Balance (HB) method is applied to study the second bifurcation after the supercritical Hopf bifurcation for cubic airfoil system[1]. Hysteresis property is discovered from second bifurcation. In addition, different initial conditions would result in different critical velocity for the occurrence of second bifurcation. Amplitude and frequency of LCO in subsonic flow of two-dimensional airfoil is obtained through nonlinear algebraic equations[2]. Poincare mapping is used to examine the multi-occurrence of bifurcation with periodic motion, subcritical bifurcation, supercritical bifurcation and coexistence of multiple LCOs[3]. Super and sub-critical Hopf bifurcation of cubic nonlinear airfoil dynamics in supersonic and hypersonic flow are studied in reference[4]. The influence of parameter varying over Hopf bifurcation is analyzed using the singularity theory. Local stability in the vicinity of critical aeroelastic values near bifurcation is compared for linear and nonlinear stiffness[5]. The dynamics of airfoil with three degrees of freedom and cubic nonlinearity is unveiled from the combination of numerical simulation and HB method, which indicates the parameter range of sub/super-critical and divergent bifurcation[6]. Other nonlinearities occurred in airfoil include piecewise linearity, freeplay and hysteresis, which are systematically being investigated in[7-10]. LCO with dual stable state is discovered in freeplay nonlinearity via numerical simulation[11].
On the other hand, experimental researches are also booming in related fields. Wind tunnel test and HB method are utilized to for two-dimensional airfoil LCO analysis. The author proposed a straightforward criterion of flutter detection. The effect of hystera nonlinearity on LCO appearance is confirmed experimentally by Berggren. Local bifurcation phenomenon of flutter is studied experimentally in low speed wind tunnel. Due to the existence of random excitation and aeroelastic load, subcritical bifurcation will occur which eventually result in the second bifurcation. This is the coexistence of multiple stable states.

Turbulence is usually considered as the origin of random excitation in stochastic flutter analysis. Mean Square Root (MSR) of responses is analyzed by taking the flow speed and turbulence strength into consideration. Stochastic P-bifurcation is investigated in reference by considering the turbulence strength and nonlinear stiffness. Meanwhile, it considers the D-bifurcation caused by external random excitation. Reference studied the stochastic bifurcation originated deterministic Hopf bifurcation from joint probability density function, marginal probability density function and the Lyapunov exponent. Parametric excitation with white noise is exerted on airfoil system to observe its flutter phenomenon in, using Monte-Carlo simulation. Adaptive spectrum for LCO study is conducted in for stochastic nonlinear aeroelastic systems. In general, only a handful researches have been carried out with both random excitation and multi-stable states taken into consideration.

The current paper studies the stochastic P-bifurcation of two-dimension airfoil flutter with quintic nonlinearity with the effect of random excitation and flow speed on system’s steady state taken into account.

2. **Governing equation and steady state probability density**

Dynamic model of a typical two-dimensional airfoil with pitch ($\alpha$) and plunge ($h$) degree of freedom is given as, the meaning of all physical parameters can be found in:

\[
\begin{align*}
M\ddot{h} + S\dot{\alpha} + Kh &= A - c_1\dot{h} \\
S\ddot{h} + J_\alpha \dot{\alpha} + K_\alpha \alpha &= B - c_2\dot{\alpha}
\end{align*}
\]

where $K_\alpha = k_1 + k_{a_1}\alpha^2 + k_{a_2}\alpha^3$, $k_1$ is the linear stiffness respectively, and $k_{a_1}, k_{a_2}$ are coefficients describing the nonlinearities in $K_\alpha$. $c_1$, $c_2$ are the damping coefficients along the pitch and plunge direction. $A = -0.1Q\alpha - 0.1n(t)$, $B = 0.04Q\alpha + 0.04n(t)$, $Q$ is the generalized flow velocity, $n(t)$ is the random excitation. First using the improved average method we could obtain the equation which is dimension reduction, and through the stochastic averaging method we get the steady state probability density distribution of the amplitude. The Eq.(1) can be rewritten as follows:

Rewrite Eq (1) in state space representation, we have,
\[
\begin{pmatrix}
\dot{\alpha} \\
\dot{\hat{h}} \\
\dot{\hat{h}}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
\frac{0.04MQ + 0.1SQ - Mk_a}{JE_M - S^2} & \frac{Mc}{JE_M - S^2} & \frac{0}{JE_M - S^2} \\
0 & 0 & 0 \\
\frac{-0.1JEQ - 0.04SQ + Sk_a}{JE_M - S^2} & \frac{Sc_1}{JE_M - S^2} & \frac{J\hat{c}_1}{JE_M - S^2}
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\hat{h} \\
\hat{h}
\end{pmatrix}
\]

(2)

Eq. (2) can be generally expressed as \( \dot{x} = F(x, Q, t) \), where \( X = (x_1, x_2, x_3, x_4)^T = (\alpha, \hat{h}, h)^T \). Nonlinear terms and random excitation are considered small in this study, which enables the usage of improved averaging method for dimension reduction. We assume \( Q \) is the bifurcation parameter, and the bifurcation point for deterministic system is \( (x_i, Q_0) \), where \( Q_0 \) is a supercritical bifurcation point acquired from its local Jacobian matrix. Making that \( Q = Q_0 + \mu \) and \( \mu \) is the small disturbance near the bifurcation point.

Assume the solution of system (2) near the bifurcation point is \( x = G(t)b \) , where \( G(t) = TE(t) \) is the fundamental matrix of the Eq. (2), and \( E(t) = diag(e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3}, e^{\lambda_4}) \), \( T = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \) \( \lambda_j \) and \( \varphi_j \) are the eigenvalues and eigenvectors of the linear part of (2) near the bifurcation point \( Q_0 \).

By substituting \( x = G(t)b \) into Eq(2), the govern equation of vector \( b \) can be obtained readily:

\[ \dot{b} = \varepsilon G^*(t)F(x, \varepsilon)|_{x=G(t)b} = \varepsilon H(b, \varepsilon, t) \]  

(3)

where \( G^*(t) \) is the inverse of the \( G(t) \). Assume the first two components of \( b \) are:

\[
\begin{align*}
\dot{b}_1 &= r e^{i\theta} \\
\dot{b}_2 &= r e^{-i\theta}
\end{align*}
\]

(4)

where \( r \) is the modal amplitude, and \( \theta \) is the phase angle. This assumption is based upon that \( \lambda_1 \) and \( \lambda_2 \) lay on the imaginary axis and \( \lambda_3 \) and \( \lambda_4 \) have negative real parts.

Substitute Eq. (4) into Eq.(3), we have,

\[
\begin{align*}
\dot{r} &= \varepsilon \left[ H_1 e^{-i\theta} + H_2 e^{i\theta} \right] \\
\dot{\theta} &= \frac{\varepsilon}{2r} \left[ -H_1 e^{-i\theta} + H_2 e^{i\theta} \right] \\
b_l &= \varepsilon H_l, \quad l = 3, 4
\end{align*}
\]

(5)
Note that \( \lambda_1 \) and \( \lambda_4 \) have negative real parts and for long term evolutions \( b \) and \( b_4 \) will eventually vanish, we can simply reduce the original dynamics as:

\[
\begin{align*}
\dot{r} &= \varepsilon [H_1 \cos \theta + H_2 \sin \theta] = \varepsilon \Phi(r, \theta, \varepsilon) \\
\dot{\theta} &= \varepsilon [-H_1 \sin \theta + H_2 \cos \theta] / r_j = \varepsilon \Phi^*(r, \theta, \varepsilon)
\end{align*}
\]  

(6)

Eq. (6) is the new standard equation after reducing the system dimension. Then through the stochastic averaging method the Itô stochastic differential equation corresponding to the reduced system is:

\[
\begin{align*}
\mathrm{d}r &= a_1 \mathrm{d}t + \sqrt{b_{11}} \mathrm{d}w_1(t) \\
\mathrm{d}\theta &= a_2 \mathrm{d}t + \sqrt{b_{12}} \mathrm{d}w_2(t)
\end{align*}
\]

(7)

where \( w_1(t) \) and \( w_2(t) \) represents independent normalized Wiener processes, \( D \) is the intensity of the random excitation. Because \( r \) is independent of \( \theta \), from the first equation of the Eq. (7), the FPK equation with respect to \( r \) can be derived:

\[
\frac{\partial p(r,t)}{\partial t} = - \frac{\partial}{\partial r} [a_1 p(r,t)] + \frac{1}{2} \frac{\partial^2}{\partial r^2} [b_{11} p(r,t)]
\]

(8)

The steady-state solution of the FPK equation is the stationary probability density function of \( r \):

\[
p_s(r) = 40.7766r \frac{N}{D} \exp \left( \frac{5.7189 \mu r^2 + 2.5485 r^4 - 3.3981 r^6}{D} \right)
\]

(9)

where \( N \) is the normalization factor. The procedure from Eq (7) - (9) can be found from literatures [23]. Eq.(9) shows the stationary PDF of modal amplitude \( r \), and from \( x = G(t)b \) the relationship exists between the steady-state response and the modal response can be got. Note that \( \varphi = \omega t + \theta \), where \( \omega \) is the LCO frequency that can be found in \( \lambda_1 \) and \( \lambda_2 \). The amplitude probability density of the pitch and plunge \( p_s(r_a) \) and \( p_s(r_h) \) are thus obtained:

\[
p_s(r_a) = 19.4493r_a \frac{N}{D} \exp \left( \frac{1.3011 \mu r_a^2 + 0.1319 r_a^4 - 0.0400 r_a^6}{D} \right)
\]

(10)

\[
p_s(r_h) = 7.2709r_h \frac{N}{D} \exp \left( \frac{0.1818 \mu r_h^2 + 0.0026 r_h^4 - 0.0001 r_h^6}{D} \right)
\]

(11)

3. The steady-state of the deterministic system

Fig1 gives the bifurcation diagram of the deterministic system corresponding to \( D=0 \). In Fig1 the red dotted line represents unstable limit cycle and unstable equilibrium point, the black solid line represents stable limit cycle and stable equilibrium point. Recall that \( \mu = 0 \), which associates \( Q_0 = 11.84493764 \) is the Hopf bifurcation point, this point corresponds to the first critical speed, namely, the linear critical speed. \( \mu = -0.1114082323 \) with \( Q_1 = 11.73085532 \) is the secondary bifurcation point, this point corresponds to the second critical speed, i.e. nonlinear critical speed. The linear critical speed can be determined by linear stability analysis, while the determination of nonlinear critical speed is obtained via nonlinear analysis. From Fig1 it can be found that high order nonlinear effect leads to secondary bifurcation and the dual stability phenomenon.
Parameter influence over system response is examined in this section. First rewritten the Eq.(9) as

\[ p_s(r) = R(r, D, \mu) \exp(Q(r, D, \mu)) \]

and rearrange the bifurcation equation as:

\[ g(r) = R(r, D, \mu) \exp(Q(r, D, \mu)) - p_s(r) \]  \hspace{1cm} (12)

According to the singularity theory, transition sets are acquired [24]:

\[ H = \{ p_s = R(r, D, \mu) \exp(Q(r, D, \mu)), R' + RQ = 0, R^{'} + 2RQ' + RQ'^2 = 0 \} \]  \hspace{1cm} (13)

\[ DL = \{ p_s = R(r, D, \mu) \exp(Q(r, D, \mu)), R' + RQ' = 0, y_i \neq y'_j \} \]  \hspace{1cm} (14)

As shown in Fig(2), in \((D, \mu)\) parameter plane the transition sets divide the parameter plane into 3 regions, black solid line is Hysteresis set and blue dot dash line is Double limit point set. Fig 3 gives the different PDF curves of aeroelastic modal amplitude correspond to these 3 sub-regions, blue stars of each case shown in Fig 3 represents the numerical solutions acquired from Monte-Carlo simulation. It can be seen that theoretical results match well with numerical ones.
When parameters are selected from region 1a from \((D, \mu)\) in Figure 2, only one peak occurs of the PDF of both pitch and plunge DOF, and their oscillations are near the origin thus the large oscillation is less likely to happen.

After one stochastic P-bifurcation when crossing region 1a to region 2, the PDF curves emerge the second peak, which is lower than the first one. Qualitative demonstration can be found in Figure 3(b), which implies the occurrence of LCO. However, the large amplitude LCO lasts relatively short in this case, which in general still belongs to small oscillations of the airfoil.

The second peak starts to exceed the first one when the parameter set crosses the DL set to region 3, as shown in Figure 3(c), large amplitude LCO will dominate the nonlinear motions. On the contrary of the previous case, while small oscillations last relatively short. Therefore, we encounter the second stochastic P-bifurcation.

The system meets another P-bifurcation when \(\mu\) continue increasing which drives the parameter set to region 1b crossing the right H set. Both PDF curves of pitch and plunge amplitude return from double peak to single peak. However, the peak is far from origin this time, which indicates the large amplitude LCO as shown in Figure 3(d).

With the increasing of flow velocity, the system evolves from small oscillation near the origin to large LCO through three stochastic P-bifurcations. The large LCO we discuss here resembles the dangerous large amplitude oscillation often occurs in deterministic systems.

Transition set disappears from the \((D, \mu)\) parameter space when \(Q = Q_2 = 11.696412\). Due to the presence of random disturbance, when \(Q\) is between the nonlinear critical speeds \(Q_1\) of deterministic system and \(Q_2\), the system will oscillate with large amplitude. Nevertheless, this will not happen in deterministic system as shown in the bifurcation diagram in Figure 1, where only stable equilibrium exists. Hence, in stochastic system, the introduction of random disturbance lower the flutter speed compare with its deterministic counterpart. Similar results can be found in [20].

In order to observe the stochastic P-bifurcation straightforwardly, Figure 4 given the joint PDF of all regions shown in Figure 2.
From Figure 4, similar joint PDF shape change from single peak to double peak can be observed after one P-bifurcation occurs (see Figure 4(a) and (b)). By increasing the flow velocity, the height of outer rim will grow and the single peak at the origin will decrease, the joint PDF of \( p_s(\alpha, \dot{\alpha}) \) will evolve to what is shown in Figure 4(c). By keep increasing the flow velocity, the single peak near the origin vanishes and there is only one ring exists in the joint PDF (see Figure 4(d)). Large amplitude LCO happens under this case with a dissipative shape in PDF.

### 5. Concluding Remarks

This paper studies the stochastic P-bifurcation for two-dimension airfoil with structural nonlinearity in turbulence flow. Singularity theory is applied for transition set, which gives the critical parametric conditions for stochastic P-bifurcation. Three types of steady states are obtained with their corresponding PDF and joint PDF. Monte-Carlo simulation verifies the feasibility of the singularity theory.

When flow speed reaches \( Q_2 = 11.696412 \), transition set disappears; when flow speed \( Q \) stays between \( Q_1 \) which is the nonlinear critical speed of deterministic system and \( Q_2 \), stochastic P-bifurcation occurs under random excitation, which results in large amplitude oscillation. And without the random excitation, the system stays at the equilibrium rather than the large amplitude LCO. Hence, the random effect has negative effect on the security of airfoil. So in order to avoid the harmful LCO, nonlinear critical speed of stochastic system should lower than that of the deterministic system.
REFERENCES