MODAL ANALYSIS AND FREQUENCY RESPONSE OF A FLEXIBLE PLATE WITH UNCERTAIN PROPERTIES USING POLYNOMIAL CHAOS EXPANSION

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The dynamic responses of engineering structures can be significantly influenced by structural uncertainties. This work examines the variability in the natural frequencies and frequency responses of a simply supported plate with uncertainty in its Young’s modulus using polynomial chaos expansion. The results obtained using the polynomial chaos expansion method are compared with Monte Carlo simulations. The polynomial chaos expansion method is then combined with the Arnoldi-based Krylov subspace technique to reduce the model order. By reducing the degrees of freedom in the numerical model, the computational efficiency is significantly increased while the physical content of the original system is preserved. The reduced order stochastic technique presented here provides a tool to assess a number of uncertain structural parameters for complex structures with significantly reduced computational effort.

1. Introduction

For nominally identical structures such as vehicles off the production line, great variability in the dynamic responses of each member occurs due to structural uncertainties arising from the manufacturing and assembly process\(^1\). The difficulty in attempting to predict the dynamic responses of engineering structures is immense, due to the large amount of physical variables which might be uncertain. Generally models of uncertainty are based on either a parametric or non-parametric description of uncertainty, or sometimes, on a combination of both. A parametric description of uncertainty means that the parameters of the dynamic system are taken to be uncertain variables that can be described statistically using various techniques, the perturbation method\(^2\), random factor method\(^3\) and the Polynomial Chaos Expansion\(^4\). Uncertainty is then propagated through the equations of mo-
tions. Non-parametric models of uncertainty assume that regardless of their detailed nature, the uncertainties in the system can be described using a universal model.\textsuperscript{5,7,16} Finally, one reference method is the statistical Monte Carlo (MC) method which generates a large number of samples \( N \) for the uncertain variables based on its probability density function and runs \( N \) simulations of the dynamic system to obtain the statistics of the output. For accurate results the sampling number \( N \) should be large enough, which results in high computational cost particularly for complex structures with many degrees of freedom and several uncertain properties.

The polynomial chaos expansion (PCE) was first introduced as the homogeneous chaos.\textsuperscript{8} In the PCE method, the stochastic system equations are transformed to a set of deterministic equations. Compared with MC simulation, the PCE method can obtain the statistical characteristics of the results with greatly reduced computational cost. However, as the order of the PCE and the degrees of freedom of the dynamic system increase, the number of deterministic equations in the PCE simulation increases exponentially.

This paper examines the variability in the natural frequencies and forced responses of a simply supported plate with uncertainty in its Young’s modulus using polynomial chaos expansion. To further improve the computational efficiency, the PCE method is combined with the Arnoldi-based Krylov subspace technique to reduce the model order, which involves fewer deterministic equations. In this combined technique only the order of the finite element model is reduced based on the considered frequency range. By reducing the degrees of freedom in the numerical model, the computational efficiency is significantly improved while the physical content of the original system is preserved.

2. Polynomial chaos expansion and model order reduction

2.1 Polynomial chaos expansion theory

The polynomial chaos expansion method involves transforming stochastic system equations to deterministic equations, by representing the uncertain variables using orthogonal polynomials of standard random variables. The first step is to project the uncertain variables onto a stochastic space spanned by a set of mutually orthogonal polynomials \( \Psi_i \), which are functions of a multi-dimensional random variable \( \xi = \{\xi_1, \xi_2, \ldots\} \). Every random variable has a corresponding random space \( \xi_i \in \Omega_i \ (i = 1, 2, \ldots) \). Then the uncertain quantity \( \chi \) can be expressed as\textsuperscript{2,10}

\[
\chi = \sum_{i=0}^{\infty} x_i \Psi_i(\xi)
\]

where \( x_i \) are the deterministic coefficients. The random base functions \( \Psi_i \) are a set of multi-dimensional polynomials in terms of \( \xi \) with the orthogonal relation of

\[
E[\Psi_i, \Psi_j] = \delta_{ij}E[\Psi_i^2]
\]

where \( \delta_{ij} \) is the Kronecker delta and \( E \) represents the expected value in the probability space. Selection of the random base function \( \Psi_i \) depends on the probability density function of random variables.\textsuperscript{10}

According to the orthogonal feature, the unknown coefficients \( x_i \) can be determined by stochastic Galerkin projection \textsuperscript{10}

\[
x_k = \frac{1}{E[\Psi_k^2]} \int_{\Omega} \chi \Psi_k(\xi) d\mu(\xi), \ k = 0, 1, 2, \ldots
\]

where \( d\mu(\xi) \) is the probability measure in the random space \( \Omega \). If the random variables \( \xi_i \) are continuous and mutually independent, then \( d\mu(\xi) \) can be expressed as

\[
d\mu(\xi) = \rho_1(\xi_1) \rho_2(\xi_2) \ldots \rho_n(\xi_n) d\xi_1 d\xi_2 \ldots d\xi_n
\]

where \( \rho(\xi) \) is the probability density function of the random variable.
2.2 Modal analysis and frequency response using the polynomial chaos expansion

The equation of motion for the dynamic system with proportional damping and in forced vibration is given by

\[-\omega^2 \mathbf{M} + j\eta \mathbf{K} + \mathbf{K})\mathbf{X} = \mathbf{F}\]  \hspace{1cm} (5)

where \(\eta\) is the proportional damping factor. To solve the system of nonlinear equations involved in the modal analysis, the Newton-Raphson algorithm is adopted. In the iterative process, the nominal values of the natural frequency and modeshapes of the dynamic system are set as the initial estimate.

In frequency response analysis, the uncertainty in force and displacement can be represented by the truncated PCE as

\[
\mathbf{X}(\xi) = \sum_{s=0}^{N_X} \mathbf{X}_s \psi_s(\xi)
\]  \hspace{1cm} (6)

\[
\mathbf{F}(\xi_f) = \sum_{u=0}^{N_f} \mathbf{F}_u \psi_u(\xi_f)
\]  \hspace{1cm} (7)

where \(N_X, N_f\) are respectively the number of polynomials to represent the displacement and force vectors. Substituting the expansion equations for the uncertain properties into Eq. (5) and projecting to the polynomial chaos basis using the Galerkin process yields

\[
(1 + j\eta) \sum_{p=0}^{N_k} \sum_{s=0}^{N_X} \mathbf{K}_p \mathbf{X}_s \mathbf{E}[\psi_p(\xi_k)\psi_s(\xi)\psi_t(\xi)] - \omega^2 \sum_{q=0}^{N_m} \sum_{s=0}^{N_X} \mathbf{M}_q \mathbf{X}_s \mathbf{E}[\psi_q(\xi_m)\psi_s(\xi)\psi_t(\xi)] = \sum_{u=0}^{N_f} \mathbf{F}_u \mathbf{E}[\psi_u(\xi_f)\psi_t(\xi)]
\]

\(t = 0,1,2,\ldots,N_t\)  \hspace{1cm} (8)

For each excitation frequency \(\omega\) and assuming \(N_X = N_t = w\), there are \(n \times (w + 1)\) deterministic PCE coefficients \([\mathbf{X}_0,\mathbf{X}_1,\ldots,\mathbf{X}_w]^T\) for the displacement which are solved simultaneously.

2.3 Model order reduction using Arnoldi-based Krylov subspace technique

The aim of model order reduction is to obtain a low-dimensional subspace \(\mathbf{T} \in \mathbb{R}^{n \times m}\) for the deterministic space. Then the original \(i^{th}\) PCE order coefficients \(\mathbf{X}_i\), can be approximated by the reduced-order \(\mathbf{X}_i^r\) as follows

\[
\mathbf{X}_i = \mathbf{T}\mathbf{X}_i^r + \mathbf{e}
\]  \hspace{1cm} (9)

\[
\mathbf{T} \times \mathbf{T}^T = \mathbf{I}^n
\]  \hspace{1cm} (10)

where \(\mathbf{X}_i \in \mathbb{R}^{n \times 1}\), \(\mathbf{X}_i^r \in \mathbb{R}^{m \times 1}\), \(m \ll n\), \(\mathbf{I}^n \in \mathbb{R}^{n \times n}\) is the identity matrix, the superscript \(r\) denotes the reduced-order matrix and \(\mathbf{e} \in \mathbb{R}^{n \times 1}\) is the negligible error. The transformation matrix \(\mathbf{T}\) is generated by the block Arnoldi algorithm. The reduced order number \(m\) is chosen based on the considered frequency range.

Introducing Eqs. (9) and (10) into the stochastic system and projecting to the Krylov subspace yields

\[
(1 + j\eta) \sum_{p=0}^{N_k} \sum_{s=0}^{N_X} \mathbf{T}^T \mathbf{K}_p \mathbf{T} \mathbf{T}^T \mathbf{X}_s \mathbf{E}[\psi_p(\xi_k)\psi_s(\xi)\psi_t(\xi)] - \omega^2 \sum_{q=0}^{N_m} \sum_{s=0}^{N_X} \mathbf{T}^T \mathbf{M}_q \mathbf{T} \mathbf{T}^T \mathbf{X}_s \mathbf{E}[\psi_q(\xi_m)\psi_s(\xi)\psi_t(\xi)] = \sum_{u=0}^{N_f} \mathbf{T}^T \mathbf{F}_u \mathbf{E}[\psi_u(\xi_f)\psi_t(\xi)],
\]

\(t = 0,1,2,\ldots,N_t\)  \hspace{1cm} (11)
By defining the reduced quantities as follows

\[ M_q^r = T^T M_q T, \quad M_q \in \mathcal{R}^{n \times n}, \quad M_q^r \in \mathcal{R}^{m \times m} \quad (12) \]

\[ K_p^r = T^T K_p T, \quad K_p \in \mathcal{R}^{n \times n}, \quad K_p^r \in \mathcal{R}^{m \times m} \quad (13) \]

\[ F_{u}^r = T^T F_u, \quad F_u \in \mathcal{R}^{n \times 1}, \quad F_{u}^r \in \mathcal{R}^{m \times 1} \quad (14) \]

the PCE system equations to be solved is given by

\[ (1 + j\eta) \sum_{q=0}^{N_q} \sum_{s=0}^{N_s} K_q^s \chi_q^s E[\psi_p(\xi_q)\psi_s(\xi)\psi_t(\xi)] - \]

\[ \omega^2 \sum_{q=0}^{N_q} \sum_{s=0}^{N_s} M_q^s \chi_q^s E[\psi_q(\xi_q)\psi_s(\xi)\psi_t(\xi)] = \sum_{u=0}^{N_f} F_u^s E[\psi_u(\xi_f)\psi_t(\xi)]. \]

\[ t = 0, 1, 2, \ldots, N_t \quad (15) \]

3. Numerical results

3.1 Simply supported plate with uncertain Young’s modulus

The dimensions and material properties for the simply supported steel plate are listed in Table 1. The frequency range of interest in this study is up to 100Hz. The element size is set to be 0.08 m and the element type is SHELL281 in ANSYS. The simply supported boundary condition is achieved by setting the displacement in the vertical direction at the four edges to be zero. The plate is excited by a point force of unity amplitude located at (0.6L_x, 0.6L_y). The mean and variance of the natural frequencies and the flexural displacement at a location (0.4L_x, 0.4L_y) are examined.

<table>
<thead>
<tr>
<th>Table 1. Parameters of the simply supported plate</th>
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</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>X-axis length L_x</td>
</tr>
<tr>
<td>Y-axis length L_y</td>
</tr>
<tr>
<td>Thickness h</td>
</tr>
<tr>
<td>Young’s modulus E</td>
</tr>
<tr>
<td>Density ( \rho )</td>
</tr>
<tr>
<td>Poisson ratio ( \nu )</td>
</tr>
<tr>
<td>Damping ratio ( \eta )</td>
</tr>
</tbody>
</table>

Variability in the material properties of the simply supported plate is generated using an uncertain Young’s modulus \( E \), which is assumed to follow a lognormal distribution with a mean and standard deviation given by Eqs. (16) and (17), respectively.

mean(\( E \)) = 210 GPa \quad (16) 

\[ \sigma_E = \frac{\sqrt{\text{var}(E)}}{\text{mean}(E)} = 5\% \quad (17) \]

The uncertain Young’s modulus is represented by the 3rd order Hermite PCE. The natural frequencies and frequency responses are represented by the 5th order Hermite PCE. Results obtained using the PCE method are compared with MC simulations using 5000 samples.
3.2 Natural frequencies for the simply supported plate

The original finite element model consists of 4852 DOFs. Considering a frequency range up to 100 Hz, the full model is reduced to an 80-DOF system using the Arnoldi-based Krylov subspace technique. The reduced mass and stiffness matrices are then combined with the PCE method to examine the dynamic characteristics. The mean and variance for the natural frequencies obtained from PCE and MC simulations for $\sigma_E = 5\%$ in the Young’s modulus are presented in Table 2. It is shown that in both the full and reduced models, the mean values of the natural frequencies are nearly identical to the deterministic values. Hence the natural frequencies are very well approximated by the 5th order PCE.

**Table 2.** Results for the mean and variance of the plate natural frequencies

<table>
<thead>
<tr>
<th>Order</th>
<th>Deterministic (Hz)</th>
<th>Full PCE</th>
<th>Full MC</th>
<th>Reduced PCE</th>
<th>Reduced MC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (Hz)</td>
<td>Variance (Hz$^2$)</td>
<td>Mean (Hz)</td>
<td>Variance (Hz$^2$)</td>
<td>Mean (Hz)</td>
</tr>
<tr>
<td>1</td>
<td>15.30</td>
<td>0.15</td>
<td>15.30</td>
<td>0.14</td>
<td>15.30</td>
</tr>
<tr>
<td>2</td>
<td>35.58</td>
<td>0.79</td>
<td>35.58</td>
<td>0.78</td>
<td>35.57</td>
</tr>
<tr>
<td>3</td>
<td>40.98</td>
<td>1.05</td>
<td>40.98</td>
<td>1.04</td>
<td>40.96</td>
</tr>
<tr>
<td>4</td>
<td>61.20</td>
<td>2.34</td>
<td>61.21</td>
<td>2.32</td>
<td>61.19</td>
</tr>
<tr>
<td>5</td>
<td>69.39</td>
<td>3.00</td>
<td>69.40</td>
<td>2.98</td>
<td>69.37</td>
</tr>
<tr>
<td>6</td>
<td>83.77</td>
<td>4.38</td>
<td>83.78</td>
<td>4.34</td>
<td>83.75</td>
</tr>
<tr>
<td>7</td>
<td>94.96</td>
<td>5.63</td>
<td>94.97</td>
<td>5.58</td>
<td>94.93</td>
</tr>
<tr>
<td>8</td>
<td>103.95</td>
<td>6.74</td>
<td>103.96</td>
<td>6.69</td>
<td>103.92</td>
</tr>
</tbody>
</table>

3.3 Frequency responses for the simply supported plate

The results for the frequency responses for $\sigma_E = 5\%$ in the Young’s modulus are presented in Figs. 1 and 2. It is shown that the reduced order stochastic model works very well in the results for both the mean and variance of the plate flexural displacement, compared with results obtained from MC simulations. The peak values in the mean and variance occur around the plate resonant frequencies. Compared with MC simulations, errors in the frequency responses around the peak value using the PCE method are attributed to the relative position of adjacent peaks as well as the level of uncertainty of the structural property. In Figs. 3 and 4, the level of uncertainty is increased by changing the standard deviation in the Young’s modulus to 10% and 20%, respectively. As the level of uncertainty increases, the error using the PCE method for the same Hermite order also increases.

For the simply supported plate examined in this work, the computational cost is significantly improved using the model reduction technique and is attributed to (1) the DOF of the dynamic system is greatly reduced from 4852 to 80 and hence fewer equations are involved in the PCE analysis, (2) the reduced mass and stiffness matrices are less singular compared with the original matrices, which makes the Newton-Ralphson algorithm easier to converge and the matrix calculations faster. For the MC simulations, the computational cost is reduced by 97%. In the PCE analysis, the reduced model only takes 0.01% of the computational time for the full model, which makes higher order PCE with several uncertain parameters practical to compute.
Figure 1. Mean and variance of the frequency response using reduced and full PCE models, $\sigma_E = 5\%$

Figure 2. Mean and variance of the frequency response using the reduced PCE model, $\sigma_E = 5\%$

Figure 3. Mean and variance of the frequency response using the reduced PCE model, $\sigma_E = 10\%$
4. Conclusions

The polynomial chaos expansion method can be used to investigate the influences of different structural properties with uncertainties on the dynamic responses. This paper examines a simply supported plate with uncertainties in its Young’s modulus. The natural frequencies and frequency response of the plate flexural displacement are obtained. To further improve the efficiency of the computational cost, the Arnoldi-based Krylov subspace technique is combined with polynomial chaos expansion method to reduce the model order. There is no loss of accuracy in the PCE method, no re-meshing of the structure and the transformation matrix only needs to be calculated once. This combined stochastic technique is shown to work very well with significantly reduced computational cost. The reduced order stochastic technique presented here provides a tool to assess the uncertain structural parameters for complex structures with acceptable computational cost.

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REFERENCES


