METHODS FOR ANALYZING AND SOLVING DISPERSION CHARACTERISTICS OF LEAKY MODES OF LAYERED HALF-SPACE WITH FLUID LOADING

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A systematic scheme for solving complex roots of leaky waves is described in this paper. Firstly, the secular equation of fluid loading planar layered structure is derived by using the global matrix method. Secondly, the complex range of the roots of the secular equation is determined by Local Summit Method, then it is refined through minima searching and a robust bisection method, and is further examined through the moving phase method. Finally, the whole dispersion curve for given mode is traced from extrapolation algorithm. For fluid loading layered structures with energy leaking into the half-space, the calculation of the square root of the wave number results in multi-values of the characteristic function, only specific combination of signs of wave-number corresponding to the correct modes with physical rationality. In order to obtain the correct dispersion curves, a criterion for selecting the right combination of signs of the half-space wave-number is proposed.

1. Introduction

In NDT and NDE areas, the leaky waves are very promising components for characterizing layered structures, as the leaky modes carry information of the structural properties, such as thickness, bonding, and elastic constants, etc. New applications using leaky waves are continuously being developed and often each new application calls for a detailed analysis on the nature of wave propagation in each specific case. Methods for solving dispersion characteristics of leaky modes is a difficult problem, especially for fluid loading layered half-spaces when energy leaking into both the half-spaces.

During the past several decades, many methods have been proposed to obtain the accurate or estimation of the complex roots of the characteristic equation. Brazier-Smith and Scott\textsuperscript{2} firstly introduce the Winding Number Integral method, which is based on the foundation of generalized logarithmic residue theorem. The complex plane was divided into many rectangle grids, the number of roots in these grids was determined at the first step, then the complex roots of the equation was obtained combining with algebraic theorem. Barshinger and Rose\textsuperscript{3} proposed a minimization routine with the real roots as the starting values, and a simplex type of searching algorithm was used to find the accurate complex roots. In addition to an excellent summary of the use of matrix techniques to model multi-layered structures, Lowe\textsuperscript{4} presents some insights on performing the numerical solution of the dispersion equation. The core of his solving process involves the functional minimization routine, which consists of a coarse search and a fine search, and the whole curve is traced with quadratic extrapolation. For fluid loading cylindrical layered or planar layered half-spaces, multi-
values of the characteristic function will be resulted from the calculation of the square root of the wave number, then the searching process should be conducted at different Riemann Sheets. Zinin and Cheng directly figure out their results, while Zhang concludes that only the multi-values of the wave number corresponding to the waves propagating in the outermost medium which extending to the infinity result in the real poles.

In this study, the complex range of the roots of the secular equation is determined with the help of the Local Summit Method, combined with Lowe’s functional minimization method, aimed at setting up a solving routine which will be physically meaningful, easily understood and conveniently implemented. A criterion for selecting the right combination of signs of the half-space wave number is summarized with respect to fluid-loaded planar layered half-space, and an example of ‘Fluid-Steel-Cement Half-space’ structure validates the correctness of this method.

2. Planar layered models

It is usually assumed in planar layered models that the wavelengths are significantly smaller than the width of the planar plane and therefore that a plane strain analysis is valid. The coordinate system may then be reduced to the plane defined by the direction of propagation of the waves (X) and the norm to the planar plane (Z), shown in Fig. 1. The global matrix method firstly proposed by Knopoff and implemented as described by Lowe is used to calculate dispersion curves in this paper, which uses a single matrix to assemble all the equations from every different interface. The wave propagation model can handle leaky waves (complex wave numbers) without modification and is diverge free from high frequencies and large thickness values in contrast with the Transfer Matrix Method.

After Lowe, the harmonic scalar potential and vector potential of displacement of the layer (i) can be expressed as,

\[ \tilde{\phi}_i(k, z, \omega) = (A_{L_i}e^{-jk_{L_i}z} + A_{L_i}e^{jk_{L_i}z})e^{j(\omega t - kx)} \]  \hspace{1cm} (1)

\[ \tilde{\psi}_i(k, z, \omega) = (A_{S_i}e^{-jk_{S_i}z} + A_{S_i}e^{jk_{S_i}z})e^{j(\omega t - kx)} \]  \hspace{1cm} (2)

where, \( k = \omega / c \), \( k_{L_i}^2 = \omega^2 / V_{L_i}^2 - k^2 \), \( k_{S_i}^2 = \omega^2 / V_{S_i}^2 - k^2 \), wave number (k), circular frequency (\( \omega \)), \( V_{L_i} \) and \( V_{S_i} \) represent the longitudinal and shear wave speed of Layer (i) respectively, (i=1,2,…,n). Through some mathematical operation with these potentials, one can get the displacements and stresses at any location of layer (i). The bond between different layers is assumed to be perfect, so the displacements and stresses must be continuous across the interface. All layer interface matrices (D) which describes the displacement and stress fields along the planar layered structure, are assembled to obtain the system matrix (S), which then multiplying the partial wave ampli-
tude \((A_i)\) results in the equation describing the whole planar layered models without traction from outside as follows

\[
\begin{bmatrix}
[D_1b][-D_{2c}] \\
[D_2b][-D_{3c}] \\
\vdots \\
[D_{(n-1)b}][-D_{nc}]
\end{bmatrix}
\begin{bmatrix}
\{A_1 \} \\
\{A_2 \} \\
\vdots \\
\{A_{(n-1)} \}
\end{bmatrix}
=[S]
\begin{bmatrix}
\{A_1 \} \\
\{A_2 \} \\
\vdots \\
\{A_{n} \}
\end{bmatrix}=0.
\]

(3)

Each interface matrix \((D_i)\) contains all the layer material parameters: \(V_{Li}\), \(V_{Si}\), thickness \((d_i)\), bulk density \((\rho_i)\), circular frequency \((\omega)\), and wave number \((k)\). As every point on a dispersion curve represents a solution to \(S\) where all boundary conditions are satisfied simultaneously, in order to have nontrivial solutions, the determinant of the system matrix \((S)\) must be set equal to zero, which produces the secular equation

\[
f(\omega, k) = \text{det}[S] = 0.
\]

(4)

The solutions to above equation are usually called modal solutions, if only energy radiation exist from the structure, one have to search the solutions in the complex wave number domain. Thus the numerical solution is in three variables: \(\omega, k_{\text{real}}\) and \(k_{\text{imag}}\).

3. Dispersion curves of leaky modes

Dispersion curves are plots describing the variation of velocity (or real wave number) and attenuation (or imaginary wave number) with frequency for leaky modes. From the theoretical point of view they are plots of the locus of the solutions of the secular function (Eq. (4)). Firstly, the complex range of the solution will be located through the Local Summit Search method, and the accurate roots can be obtained by Lowe’s \(4^\text{th}\) method, then for any given mode the whole curve will be traced using extrapolation algorithm. During the process, the frequency will be held constant while solving for the wave numbers.

3.1 Local Summit Search method

For complex function \(f(z)\), which is continuously analytic in the complex plane except for finite number of poles, the absolute value \(|f(z)|\) approaches infinite at the point of the pole \(z^*\), and the value decreases as the variable leaving away from the pole, the further the absolute value becomes less. So, each pole will result in a Local Summit in the absolute value domain. Through searching the location of the Local Summit in the complex plane, one can get the complex space where the poles exist, then the accurate pole of the \(f(z)\) can be obtained using iterative algorithm within above complex space.

With the same consideration, the zeroes of the complex function \(f(z)\) can be obtained by searching the Local Summit. Assume

\[
g(z) = \frac{1}{1+|f(z)|},
\]

(5)

it is easy to say that the location of the zeroes of \(f(z)\) coincident with the Local Summit of \(g(z)\). Such that, one can determine the zeroes of \(f(z)\) by searching the Local Summit of \(g(z)\), so the method is called Local Summit Search method.
When searching the Local Summit for given frequency, the real and imaginary part of the velocity (or wave number) should be discretized and the increment should be small enough, to ensure that all the Local Summits resulting from all the zeroes could be found, as shown in Fig. 2 and Fig. 3. The absolute values of the determinant changing drastically within a very small complex space in Fig. 2, while the corresponding Local Summit highlight itself in Fig. 3. So it will be much easier to determine the location of the Local Summit by direct searching method.

### 3.2 Fine Search

After the approximate solutions have been located by the Local Summit Search method, the fast-converging Muller or iterative method can be used to improve the values to acceptable accuracy. But for modes that exist at very close or even coincident locations, fast-converging methods always tend to the wrong mode. The fine search algorithm conducted by Lowe is robust even if slower to converge.

The core of the Fine Search is a single-variable searching. For given frequency, the real velocity (or real wavenumber) is kept constant, while changing the imaginary velocity (or imaginary wavenumber) until finding the local minimum. A check is then made to determine whether the minimum is a solution of the function, and, if so, the search is complete. If not, the single-variable search is repeated with the imaginary velocity (or imaginary wavenumber) kept constant, while changing the real velocity (or real wavenumber) until finding a new local minimum. Again the check is performed, if not again, the real input is varied, and so on until convergence is achieved.

A robust bisection method which converges unconditionally is employed at this point, retaining the same input parameter as the variable. Three samples are chosen at the beginning of an iteration cycle of this method: a current minimum, the other two samples obtained with a lower and a higher value of the variable, which are named the central sample, the left sample, and the right sample respectively. The size of the sampling interval between right and central is the same as between central and left. The characteristic function is then sampled at the two mid-points of these intervals so that a total of five equi-spaced samples are known. The sample with the lowest absolute value out of these five becomes the new central for the next iteration cycle, and its neighbours become the new left and right. When the sampling interval has been reduced to an acceptable tolerance, the minimum is deemed to have been obtained.

After each iteration of the single-variable search the converged minimum is examined using the moving phase method to see whether it is a valid solution. Theoretically, the real and imaginary part of the value of the characteristic function approach zero simultaneously when the variable tending to the minimum. Since the function is smooth, the angle calculated between the vectors in complex space from the origin to the sampling points just before and just after the converged minimum approaches 180 degrees. Practically, if the angle is greater than 90 degrees, the converged minimum is considered as the valid solution.
3.3 Curve tracing

For many cases of strongly attenuating waves, it is time consuming to perform local summit searching in order to detect the minima as the imaginary part of the velocity (or wavenumber) increasing drastically. In wavenumber-frequency space, the dispersion curves are generally closer to straight lines. So the whole curve for given mode can be generated using the curve tracing algorithm, which starts from several solution points and incrementing the wavenumber slowly and the new point being refined by the fine search.

Figure 4. Curve generation using curve tracing. Figure 5. The scheme for quadratic extrapolation.

For given mode, two solution points are calculated by combining the Local Summit Search method (sweeping the real velocity or real wave number, as shown in Fig. 4) and the Fine Search with the frequency interval of ∆f, the linear extrapolation is used to predict the following four solution points with frequency increasing or decreasing. Every new solution point must be refined through the Fine Search before being chosen to predict the next solution. After six points have been obtained the algorithm switches to a quadratic extrapolation scheme, which employs 2∆f as the equally spaced input and is used to extrapolate both the real (k) and imaginary (Atten) part of the wavenumber, as shown in Fig. 5.

3.4 Selection of the signs of the wave numbers

For simple cases, such as plate or multi-plates immersed in fluid, when evaluating the characteristic function (Eq. (4)) all the longitudinal and shear wave numbers of the structure only need to be kept positive, but for complex situations, e.g. fluid-loading layered half-space structures, the sign of the wave numbers should be chosen correctly. Here, a criterion for choosing the signs is summarized with mathematical explaining.

In their study, Zhang\(^7\) presents that only the multi-values of the wave number with respect to the waves propagating in the outermost medium which extending to the infinity result in the real poles. So for a fluid loading n-layered half-space structure, only the wave number in the half-space medium will result in the multi-values of the function. The wave number of the longitudinal and shear waves in the half-space goes as follows:

\[
\begin{align*}
    k_{Ln} &= \pm \sqrt{\frac{\omega^2 - \omega^2}{V_{Ln}^2 - c^2}} = \pm \sqrt{\Delta_L}, \\
    k_{sn} &= \pm \sqrt{\frac{\omega^2 - \omega^2}{V_{sn}^2 - c^2}} = \pm \sqrt{\Delta_s}.
\end{align*}
\]

So, there are four different combinations of signs of wave number, corresponding to Riemann Sheet R1 (‘++’), R2 (‘+’), R3 (‘-+’) and R4 (‘--’) respectively. As the solution with physical meaning is needed in practice, from the mathematical view of point, the exponential component in the scalar and vector potential must not diverge with \(z\) leaving away from the interface. Consequently, the signs of the wave number of the bulk waves in the half-space medium should be chosen as follows: (1) for \(c > V_{Ln}\).
\[ k_{Ln} = \sqrt{\Delta_L}, \] positive is selected, thus the scalar potential in the half-space becomes,
\[ \hat{\varphi}_n = A_{Ln} e^{i k_{Ln} z} = A_{Ln} e^{i \sqrt{\Delta_L} z}, \]
(the harmonic component \( e^{i(\omega t - kx)} \) is omitted), for any value of \( z \) leaving away from the interface the amplitude of the exponential component equals to unit, and will not diverge. From the same consideration, positive is also chosen for the sign of shear wave number. So, for this circumstance, the roots-finding algorithm is conducted in Riemann Sheet R1 (’++’).

(2) for \( V_{Ln} > V_{Sn} > c \),
\[ k_{Ln} = -j \sqrt{\Delta_L}, \] negative is selected, thus the scalar potential in the half-space becomes,
\[ \hat{\varphi}_n = A_{Ln} e^{i k_{Ln} z} = A_{Ln} e^{-\sqrt{\Delta_L} z}, \]
with \( z \) leaving away from the interface the amplitude of the exponential component decrease to infinitesimal, and will not diverge.

\[ k_{Sn} = \sqrt{\Delta_S}, \] positive is chosen, thus the vector potential in the half-space becomes,
\[ \hat{\psi}_n = A_{Sn} e^{i k_{Sn} z} = A_{Sn} e^{i \sqrt{\Delta_S} z}, \]
with \( z \) leaving away from the interface the amplitude of the exponential component equals to unit, and will not diverge either.

So, for this case, the roots-finding algorithm is conducted in Riemann Sheet R3 (’-+’).

(3) for \( V_{Sn} > c \),
\[ k_{Sn} = -j \sqrt{\Delta_S}, \] negative is selected, thus the vector potential in the half-space becomes,
\[ \hat{\psi}_n = A_{Sn} e^{i k_{Sn} z} = A_{Sn} e^{-\sqrt{\Delta_S} z}, \]
for any value of \( z \) leaving away from the interface the amplitude of the exponential component decrease to infinitesimal, and will not diverge. From the same consideration, negative is chosen for the sign of longitudinal wave number. So, for this circumstance, the roots-finding algorithm is conducted in Riemann Sheet R4 (’-‘).

Then a criterion for selecting the right combination of signs of the half-space wave number is concluded: the sign of the wave number of the bulk waves propagating in the outermost half-space medium depends on the relative magnitude between the bulk velocity and the phase velocity, if the bulk velocity is bigger the negative is selected for the wave number, while positive is chosen as the bulk velocity is smaller.

4. Dispersion curves for ‘Water-Steel-Cement Half-Space’ structure

![Figure 6](image6.png)  ![Figure 7](image7.png)

Figure 6. The phase velocity curves.

The phase velocity (see Fig. 6) and attenuation (see Fig. 7) curves for fluid-loading Steel-Cement half-space structure are obtained using method mentioned above. The flexural wave mode is divided into two modes (Pseudo-A0-1 mode and Pseudo-A0-2 mode) by the line representing the
shear velocity of the cement, as they are solved in Riemann Sheet R4 (‘--’) and R3 (‘-+’) respectively. The extensive wave mode consisting of Pseudo-S0-1 mode and Pseudo-S0-2 mode is much more obviously divided by the line representing the longitudinal velocity of the cement, and these two different modes are traced in Riemann Sheet R1 (‘++’) and R3 (‘-+’) respectively. Although the phase velocity curves of Pseudo-A1 mode and Pseudo-S1 mode seem to be continuous in the frequency range of interest, they are also traced in different Riemann Sheets, as depicted in attenuation curves by two dotted rectangles which is shown in Fig. 7.

Table 1. Materials acoustic properties.

<table>
<thead>
<tr>
<th>Material</th>
<th>Density (kg/m³)</th>
<th>Longitudinal Velocity (m/s)</th>
<th>Shear Velocity (m/s)</th>
<th>Quality Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1000</td>
<td>1500</td>
<td>-</td>
<td>20</td>
</tr>
<tr>
<td>Steel</td>
<td>7800</td>
<td>5930</td>
<td>3250</td>
<td>1000</td>
</tr>
<tr>
<td>G-class Cement</td>
<td>1800</td>
<td>3700</td>
<td>2000</td>
<td>95</td>
</tr>
</tbody>
</table>

5. Conclusion

The task of finding the complex roots of the characteristic functions for leaky waves is substantially more difficult than for the real roots and although several methods have been published, some details have never been mentioned, especially for fluid-loading layered half-space structures. A systematic routine is described in this paper, combining the advantages of different solving methods. The coarse search achieved through the Local Summit Search method, which owns explicit physical meaning and is easier to understand. The fine search employing the five point bisection is severely robust, to the maximum extent avoiding tracing into the wrong mode. For layered half-space structures with energy leaking into the half-space, the calculation of the square root of the wave number results in multi-values of the characteristic function. In order to find the modes with physical rationality, a criterion for selecting the right combination of signs of the half-space wave-number is proposed with mathematical explaining. The method proposed in this study is validated by comparing the results with those given by Zinin⁵, Cheng⁶ and Froelich¹⁰, and the dispersion curves of ‘Water-Steel-Cement Half-space’ is given as an example.

REFERENCES

