PERFORMANCE AND STABILITY CONSTRAINTS OF AN ACTIVE ACOUSTIC METAMATERIAL

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Metamaterials are the subject of significant interest due to their ability to reproduce behaviour beyond what is possible with naturally occurring media such as the appearance of high levels of wave attenuation at specified frequencies, known as band gaps. These band gaps can be tuned to appear at low frequencies, providing isolation performance at long wavelengths where the performance of traditional passive isolation techniques is generally poor. However, due to the resonant nature of the band gaps they tend to only extend over a very narrow frequency range.

The application of active control within a metamaterial structure can provide an effective solution to this since the natural material response can be modified to enable attenuation over a much broader frequency region. This paper presents an active acoustic metamaterial, consisting of an array of active Helmholtz resonators. An actuator embedded within the resonators is used with a feedback controller to regulate the velocity of the air within the neck of the resonator elements, thereby enhancing their natural performance. The attainable performance of such an active metamaterial, however, is constrained by the requirement to maintain closed loop stability and formal constraints are presented and illustrated in the paper. Constrained non-linear optimisation is then employed to design a practical feedback controller consisting of an FIR filter, and its performance analysed.

1. Introduction

The novel behaviour that can be produced using acoustic metamaterials has led to significant research interest over the last decade. In his landmark paper, Vesalago described the potential behaviour of a ‘left handed’ electromagnetic material with simultaneously negative electromagnetic permeability and permittivity, however it was not until Smith et al. combined locally resonant elements within a periodic structure many years later that such a material could be realised. More recently the same principle has been applied to viscoelastic waves. Metamaterials can have a negative refractive index, where Snell’s law still applies however the path of the reflected wave lies to the opposite side of the incident normal than one would expect in a regular medium. Band gaps appear in the dispersion characteristics where low frequency resonances occur leading to high levels of wave attenuation in the material. These novel properties mean that metamaterials are of particular interest and have been proposed as a potential solution to achieve acoustic cloaking, transmission blocking and subwavelength acoustic lenses. However, since the behaviour of metamaterials is reliant on local resonances, the desirable effects only occur over narrow frequency bands. To address this, more recent research has been published where the possibility of incorporating active control within a metamaterial struc-
Figure 1: (a) A SHS based metamaterial sample. (b) The transmission coefficient of a SHS metamaterial tuned to 1145Hz, consisting of 1 (blue), 2 (green) and 3 (red) layers. (c) The effective bulk modulus of the sample. (d) The transmission coefficient of a metamaterial with mixed resonator elements.

This paper presents an investigation into the stability and performance limits of a metamaterial consisting of active Helmholtz resonators. The work builds on the initial investigation of Reynolds et al\(^{11}\) where a passive metamaterial consisting of an array of split hollow spheres (SHS) was produced and validated experimentally. To enhance the band gap region of the original metamaterial, the prospect of adding active functionality to the resonator units is explored. The active resonator units would consist of a split hollow sphere with an actuator embedded into the base of the resonator cavity, a microphone to measure neck velocity, and a feedback control system. For any active system to be practically realisable then closed loop stability must be ensured, often at the expense of system performance.

Here, the absolute formal stability limits of the active Helmholtz resonator model are derived and performance limits calculated and compared to the passive case. Next, a practical feedback controller is designed. An FIR filter is employed, ensuring open loop stability, and the coefficients are selected using an optimisation routine. The cost function and constraints of the optimisation are chosen to achieve a performance goal whilst retaining closed loop stability and remaining within practical power limits. The results demonstrate that an active architecture can greatly enhance the width and depth of the band gap region, and provide a path toward truly broadband, controllable metamaterials.

2. The Acoustic Metamaterial

The metamaterial considered here consists of an array of split hollow sphere (SHS) units suspended within an acoustically transparent mesh. The SHS act as Helmholtz resonators and provide a
mechanism for creating a low frequency band gap and dispersive material parameters\textsuperscript{12,13}. In previous work by Reynolds et al\textsuperscript{11} passive metamaterials of this kind were fabricated using additive layer technology and tested within an impedance tube to assess their performance. A photograph of a metamaterial sample is shown in Figure 1a. It was shown that a band gap appears as expected around the region of the SHS natural frequency, as shown in Figure 1b. By creating a metamaterial layer that contained a mixture of resonators, band gaps appear at each of the respective natural frequencies, and it is possible to get these band gaps to overlap, effectively creating a wider regions of attenuation at the expense of band gap depth. An example of this is shown in Figure 1d.

3. An Active Helmholtz Resonator

Whilst the use of passive mixed resonators provides a potential solution to overcome the narrow-band nature of metamaterial behaviour, the width of the resonant region is enhanced at the expense of its magnitude, and therefore it requires a large number of resonators to achieve a desirable level of attenuation at the band gap\textsuperscript{8}. It has been shown that the addition of active components to metamaterials the band gap attenuation can be achieved over a much wider frequency range\textsuperscript{9}.

For simplicity the material developed here will consist of a single resonator unit per layer. Consider an active Helmholtz containing an acoustic actuator capable of producing a pressure within the resonator cavity. The actuator is controlled using a microphone to detect the pressure at the resonator neck, and a control loop, $G = HK$, where $H$ represents the passive dynamics and $K$ the controller. Such a structure is shown in Figure 2a and is represented by the circuit shown in Figure 2b. The work of Yuan\textsuperscript{14} demonstrates how to calculate the acoustic impedance of such an active Helmholtz resonator with an inertance $L_h$, resistance $R_h$ and compliance $C_h$. $Z_{m+r}$ represents the impedance of the resistance and inertance terms, and $s = j\omega$.

\begin{equation}
Z_a = \frac{P_h}{u_h} = \frac{(sC_hZ_{m+r} + 1)}{sC_h + G} = \frac{(sC_h(sL_h + R_h) + 1)}{sC_h + G}
\end{equation}

The transmission coefficient, $T$, of an active metamaterial consisting of a single active resonator within a section of tube can now be calculated by combining this expression with a transfer matrix model of the SHS based metamaterial\textsuperscript{11}

\begin{equation}
T = \frac{P_t}{P_{t0}} = \frac{2}{T_{1,1} + \frac{T_{1,2}}{z_{N+1}} + z_{N-1}T_{2,1} + \frac{z_{N-1}}{z_{N+1}} + T_{2,2}}
\end{equation}
where $T = \begin{bmatrix} \cos(kd) + \frac{z_d \sin(kd)}{Z_a} & j z_d \sin(kd) \\ \frac{j \sin(kd)}{z_d} + \frac{N_r \cos(kd)}{Z_a} & \cos(kd) \end{bmatrix}$ \hspace{1cm} (3)

where $z_{N-1} = z_{N+1} = z_d$, the acoustic impedance of the duct and $kd$ is the wavenumber in air multiplied by the material thickness respectively. Substituting in Equation 1, neglecting the static delay term $1/(\cos kd + j \sin kd)$ and rearranging this expression gives the closed-loop transfer function

$$T = \frac{s^2 2 L_h C_h^2 + s 2 R_h C_h + 2}{s^2 L_h C_h + s(2 R_h C_h + z_d C_h) + (2 + z_d G)} \hspace{1cm} (4)$$

4. Performance and Stability Constraints of the Active Helmholtz Resonator

Using the impedance representation of an active Helmholtz resonator means that the transmission performance and effective material parameters can be determined using the transfer matrix and equivalent circuit model respectively. Careful design of the controller $K$ can then improve the performance of the material. As with any feedback system stability is the utmost importance and, along with the physical limits of the components used when building an active system, will be the aspect that limits the practical performance of the system.

The Routh-Hurwitz stability criterion can be used to show that a closed-loop transfer function with a second order denominator will be stable if all of the coefficients of the characteristic equation are greater than zero. The transfer function of the active Helmholtz resonator in a duct has a second order denominator providing that the control loop dynamics, $G$, are second order or lower. By considering a simple second order control loop, $G = g_2 s^2 + g_1 s + g_0$, therefore, it is possible to define by inspection the following inequalities which define the stability limits of the closed-loop system

$$-\frac{2}{z_d} < g_0 < \infty \hspace{1cm} (5)$$
$$-\frac{(2 R_h C_h + z_d)}{z_d} < g_1 < \infty \hspace{1cm} (6)$$
$$-\frac{M_h C_h}{z_d} < g_2 < \infty \hspace{1cm} (7)$$

Note that for the purposes of this investigation $R_h$ is assumed to be constant; a valid assumption if thermoviscous losses are negligible and radiation resistance is the dominant source of loss. In the examples presented below $R_h = 2e - 6/S n^2$, the other physical variables are based on a SHS of 12.5mm radius with a 4mm diameter neck aperture.

To determine the limits of band gap performance determined by the stability bounds calculated above, the transmission coefficient of the metamaterial was calculated for a zero order, a purely first order controller and a purely second order controller, where in each case the controller coefficient is set first to the lower stability bound and then to the upper stability bound described above (i.e. 6 controllers in total). Figure 3a shows the Transmission coefficient for the three different controllers when they are set to their lower frequency bound. The results show that for the zeroth and second order controller, at the lower stability bound the depths and width of the band gap is considerably improved, at the expense of enhancing the response in low and high frequency regions respectively. This suggests that the zeroth order controller would suit a situation where there is no low frequency content in the disturbance, and the second order no high frequency content. For the first order controller the band
Figure 3: The transmission coefficient of the closed loop acoustic metamaterial with a zeroth (blue), first (red) and second (red) order controller, at the lower (a) and upper (b) stability bounds. Compared to the passive metamaterial (blue dotted) gap is replaced by a region of enhancement. Note that the zeroth and second order controllers are purely real, whilst the first order is purely imaginary. Figure 3b shows the results for a positive coefficient value for \( g_0, g_1 \) and \( g_2 \). A value of \( 10^{-5} \) is used, although note that since \( G \) incorporates actuator dynamics, the controller gain \( K \) is actually several orders of magnitude larger. The results show that for the zeroth order controller the transmission performance mirrors that of the first order controller at its lower bound from Figure 3a. The first and second order controllers in this instance provide significant levels of broadband attenuation.

It is clear from the results above that significant performance gains can theoretically be achieved using the active Helmholtz resonator structure compared to a passive resonator, whilst retaining closed loop stability. However, practically realising such a system will require imposing further constraints. The control loop must be causal, there will be physical limits on the capabilities of the hardware used, and the actuator dynamics will be of consequence. Therefore whilst the controllers above demonstrate the potential of the approach they are not practically realisable.

5. Constrained Nonlinear Optimisation

This section describes using Constrained Nonlinear Optimisation (CNO) to iteratively design a control loop filter to achieve the design objectives as an alternative to explicitly defining the dynamics of the control loop \( G \). The optimisation routine attempts to minimise the output of a cost function \( f(x) \) whilst satisfying a non-linear constraint.

\[
\min_x f(x) \text{ such that } c(x) \leq 0 \tag{8}
\]

In the case of the active Helmholtz resonator the algorithm implemented to optimise the coefficients of an FIR feedback filter. An FIR filter is used to ensure open loop stability, simplifying the design process. The output of the function \( c(x) \) is set to ensure closed loop stability and ensure that the power output of the active system is constrained. The system, consisting of a microphone, amplifier, control filter and loudspeaker, is inherently open loop stable, and as such closed loop stability is ensured using the simple Nyquist criterion, such that

\[
-1 \leq \text{Re}(G) \\
\therefore c(x) = -\text{Re}(G(x, \omega)) - 1 \leq 0 \tag{9}
\]
Figure 4: The transmission performance (a) and effective bulk modulus (b) of the active resonator system when the feedback filter has been optimised to minimise transmission from 800-1400Hz. (d) The transmission performance if the filter is optimised over 400-500Hz.

The actuator dynamics, $H$, are represented by a simple second order model of a loudspeaker, as suggested by Yuan\textsuperscript{14}. It was assumed that the loudspeaker has a continuous power handling rating of 0.1W, the microphone has a sensitivity of -60dB relative to 1V/Pa, the microphone amplifier provides 60dB of gain. Therefore, presuming a maximum disturbance pressure of 20Pa (approx 120dB SPL) the power output of the control filter, $K$, must be less than 0.1W.

The optimisation routine terminates when it reaches a point at which the gradient of the solution is below a pre-defined tolerance level. This is an effective global strategy if the solution to the cost function is convex, where a single minimum exists. Several starting points were tested for each case and similar results were achieved in each case, however this is not a conclusive proof that the solutions are convex and a global minimum has been achieved.

5.1 Minimising Transmission

In order to optimise a filter to minimise the transmission coefficient of the active Helmholtz resonator within a duct, the function to be minimised was set to

$$f(x) = \frac{1}{\omega_{ul} - \omega_{ll}} \sum_{\omega_{ul}}^{\omega_{ll}} T(x, \omega)$$

(10)

where the subscripts $ul$ and $ll$ represent the upper and lower frequency limits respectively over which the optimisation takes place, and the closed loop transfer function is now a function of the FIR filter coefficients $x$.

Figure 4a shows the transmission response of an active single resonator within a duct when the optimisation has been set to minimise the transmission between 800Hz and 400Hz. The results show that transmission is reduced across the target bandwidth, with a rippling effect seen out of band. In the upper frequency range there are spikes in the response above 0dB, where sound transmission would be enhanced. The effective bulk modulus (not shown here) that results from this active system is practically zero within the optimisation band, whilst the out of band performance consists of several positive and negative spikes in both the imaginary and real parts. From an impedance standpoint large levels of transmission loss coinciding with a bulk modulus of zero makes sense, since the acoustic impedance of the system $z_m = \rho_s c_m = \sqrt{\rho_c B_c}$, where $c_m$ is the effective sound speed within the system. A bulk modulus of zero leads to an acoustic impedance of zero; a short circuit in the transmission line, leading to high levels of reflection. This reflection was observed in the results although not presented here.

Figure 4b shows the transmission performance of the system when the filter has been optimised to minimise transmission between 400-500Hz. The transmission has been effectively suppressed.
Figure 5: (solid) The real part of $B_e$ of the active resonator system when the feedback filter has been optimised to minimise $\text{Re}(B_e)$ from (a) 800-1400Hz, and (b) from 400-500Hz, compared to the passive case (dotted) within the optimisation band, as well as, to a lesser extent, the out of band region. In addition there is a significant dip in transmission in the region of the natural frequency of the passive resonator. This demonstrates one of the advantages of using resonant metamaterials over traditional active isolation solutions. The passive band gap due to the resonant inclusions is enhanced by the active functionality such that performance is gained from both the active and passive elements.

5.2 Minimising the Effective Bulk Modulus

The optimisation routine was then applied to try to manipulate the effective material parameters. To attempt to create a broad region of negative effective bulk modulus, the CNO function was set to minimise the maximum value of the real part of the bulk modulus using the cost function

$$f(x) = \sum_{\omega_{nl}} \max(\text{Re}(B_e(x, \omega)))$$

When a broad frequency range is targeted (800-1400Hz) to reduce the real part of the bulk modulus, as shown in Figure 5a, the net effect is to suppress the effective bulk modulus over the entire frequency range to zero, but not achieving negative values. There is also a small ripple effect seen out of band, similar to that seen on previous results. When a narrower frequency band of 400-500Hz is targeted away from the passive band gap of the Helmholtz resonator, as shown in Figure 5b, negativity is achieved but only over a very small frequency range and the magnitude of the effect is very modest. Note the scale on Figure 5b is different to the other graphs, highlighting the small magnitude of the negative effect. The limits of performance here are unknown, and further investigation is needed to improve these results using this design method.

6. Conclusion

Two methods have been presented to simulate the potential performance of an active Helmholtz resonator within a duct that could be used as the basis for an active acoustic metamaterial. An active Helmholtz resonator structure is proposed and it is shown that if a feedback controller is designed to operate on the margins of stability then very high levels of performance can be achieved with respect to transmission loss. The effective material parameters are also affected significantly. These results show that when using the architecture suggested closed loop stability will not realistically be the limiting factor on performance. However, these results do not take into account open loop stability, nor energy limits that must be satisfied when using physical equipment and transducers.
The second method presented, constrained non-linear optimisation, provides a more realistic design method for a feedback controller whilst ensuring both closed loop stability and power output limits are met. The method is particularly effective for minimising sound transmission over a specified frequency range, either to widen the range and depth of the passive band gap, or to create a band gap in a different frequency region. The results here suggest that even under these constraints significant levels of transmission loss can be achieved. The width and depth passive band gap can be enhanced, and additional regions of attenuation can also be created in absence of a passive band gap. Importantly in this second scenario, even if the filter is targeted at frequencies away from the passive band gap there is an additional significant dip in transmission in the region of the natural frequency of the passive resonator. The passive band gap due to the resonant inclusions is enhanced by the active functionality such that performance is gained from both the active and passive elements, demonstrating an advantage of using resonant metamaterials over traditional active isolation solutions. However, the method is not very effective at controlling the effective material parameters, and in most instances did not achieve negativity.

This project was carried out with funding from the Engineering and Physical Sciences Research Council (EPSRC). This support is gratefully acknowledged.

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