COLOR SYMMETRY EMBEDDED IN RESONANT VIBRATION OF NONLINEAR SOLIDS

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The free-vibration acoustic resonance (FVAR) of two-dimensional St. Venant–Kirchhoff hyperelastic materials has been investigated within the framework of nonlinear elasticity and the calculus of variations. Variational analysis revealed that the displacement satisfies the time reversal symmetry, time periodicity, periodic vanishing of displacement velocity and surface traction free conditions. Numerical analysis based on the Ritz method revealed the existence of four types of nonlinear FVAR modes which can be classified on the basis of a single colour and three types of bicolour magnetic point groups, rather than the conventional irreducible representations. This result indicates that the colour symmetry is embedded in the finite amplitude nonlinear FVAR mode.

1. Introduction

Free-vibration acoustic resonance (FVAR) of a solid has been investigated since the era of Rayleigh and Ritz [1, 2, 3, 4]. In 1977, Demarest investigated the phenomena and pointed out that the second-order elastic constants tensor $C_{ijkl}$ can be obtained from the FVAR frequency $\omega_i$ [5]. Soon after, Ohno conducted the ultrasound spectroscopy experiment and determined the $C_{ijkl}$ tensor from $\omega_i$ [6]. Several researchers contributed to improve the method and, consequently, it is now established as resonant ultrasound spectroscopy (RUS) [7, 8, 9, 10, 11]. This method enables us to determine the complete set of $C_{ijkl}$ tensor from one single-crystal specimen. It is applicable to a mm-ordered small-sized specimen with a sufficient accuracy: inaccuracy of $C_{ijkl}$ is generally less than 0.1%. To the author’s knowledge, RUS is the state-of-the-art method for determining the $C_{ijkl}$ tensor of solids.

Theory of RUS is based on linear elasticity and the calculus of variations. Although the linear approximation greatly simplify the analysis, it prevents us to analyze the nonlinear-related phenomena. For instance, we fail to determine the third-order elastic constants $C_{ijklmn}$ from resonance frequency $\omega_i$. Obviously, FVAR of nonlinear solids is less understood and further investigation is required. Recently, the author investigated FVAR of a nonlinear solid and revealed that (i) resonance frequency $\omega_i$ depends on the vibration amplitude, (ii) there exists a new-type of symmetry, color symmetry, embedded in the FVAR and (iii) nonlinear FVAR modes can be classified on the basis of magnetic point group, rather than the conventional irreducible representations [12]. In the present study, we first provide the variational formulation for FVAR of two-dimensional St. Venant–Kirchhoff hyperelastic material. Then, we demonstrate the existence and structure of the color symmetry embedded in FVAR. We also reveal that vibration symmetry of the nonlinear FVAR mode can be reasonably explained from a magnetic point group.
2. Variational formulation

2.1 Euler–Lagrange equation

We consider a two-dimensional St. Venant–Kirchhoff hyperelastic material defined by \( \Omega = \{ x_i | -L_i < x_i < L_i, i = 1, 2 \} \). The domain is isotropic so that it belongs to the point group \( C_{2v} = \{ E, C_2, \sigma_x, \sigma_y \} \). It is subjected to a displacement \( u_i = u_i(x_i, t) \) which is assumed to be continuously differentiable with respect to the all arguments. Let \( u_{i,j} = \partial u_i / \partial x_j \) and \( u_{i,t} = \partial u_i / \partial t \) be the displacement gradient and material time derivative and let \( \rho \) be the mass-density which is assumed to be a constant in the reference configuration \( \Omega \). Then, the action integral of the hyperelastic material is given by

\[
I(u_i) = \int_0^{2\pi/\omega} \int_\Omega \mathcal{L}(u_{i,t}, u_{i,j})dV dt, \quad \mathcal{L} = \frac{1}{2} \rho (u_{1,t}^2 + u_{2,t}^2) - \frac{1}{2} \lambda \eta_{i,i} - \mu \eta_{i,j} \eta_{j,i}. \tag{1}
\]

Where \( \mathcal{L} \) is the Lagrangian density and \( \eta_{ij} = (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})/2 \) the Green–Lagrange strain tensor which includes the geometrical nonlinearity. \( \lambda \) and \( \mu \) are called the Lamé constants. According to the principle of stationary action, the displacement \( u_i \) due to FVAR satisfies the condition \( \delta I = 0 \). To this end, we consider the following transformations:

\[
t \to t + \alpha \varphi + o(\alpha), \quad u_i \to u_i + \alpha \psi_i(x_i, t, u_i, u_{i,j}) + o(\alpha). \tag{2}
\]

where \( \varphi = \text{const.} \), and \( \psi_i \) is an arbitrary but continuously differentiable function. In (2), the first transformation comes from the fact that the condition \( \delta I = 0 \) poses a variable-end-point type variational problem. The second is an ordinary variation of the function \( u_i \). Under the transformations of Eq. (2), the first variation \( \delta I \) of the action integral ends up with the following form [13]:

\[
\delta I = - \int_0^{2\pi/\omega} \int_\Omega \left[ \left( \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial u_{i,t}} + \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial u_{i,j}} \right) \tilde{\psi}_i - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial u_{i,t}} \tilde{\psi}_i + \mathcal{L} \varphi \right) - \frac{\partial}{\partial x_j} \left( \frac{\partial \mathcal{L}}{\partial u_{i,j}} \tilde{\psi}_i \right) \right] dV dt, \tag{3}
\]

where \( \tilde{\psi}_i = \psi_i - u_{i,t} \varphi \). To satisfy the condition \( \delta I = 0 \) for all \( \varphi \) and \( \psi_i \), the Euler–Lagrange equation

\[
\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial^2 W}{\partial u_{i,j} \partial u_{k,l}} \frac{\partial^2 u_k}{\partial x_i \partial x_j} = 0, \tag{4}
\]

as well as the natural boundary conditions, \( u_{1,t}|_{t=0} = u_{1,t}|_{t=2\pi/\omega} = 0, u_{2,t}|_{t=0} = u_{2,t}|_{t=2\pi/\omega} = 0, \mathcal{L}|_{t=0} = \mathcal{L}|_{t=2\pi/\omega} \) and \( T_i \eta_{ij} = 0 \) (on \( \partial \Omega \)), should be satisfied.

Throughout this study, we assume that the displacement gradient is fairly small so that the coefficients \( \partial^2 W / \partial u_{i,j} \partial u_{k,l} \) appeared in Eq. (4) satisfies the strong ellipticity condition for all \( x_i \in \Omega \) and \( t \in (0, 2\pi/\omega) \). In that case, Eq. (4) implies a system of quasilinear wave equations and whose solution is free from singularities such as a shock. To solve the problem, we need an initial condition \( u_i(x_i, 0) \) for all \( x_i \in \Omega \). However, this is one of the unknown quantities that we would like to determine throughout the analysis. Hence, we solve the variational problem \( \delta I = 0 \) by a direct method.

2.2 Numerical analysis by the Ritz method

To obtain the stationary solution of the action integral of Eq. (1), we approximate the displacement function \( u_i \) by the following two-dimensional Fourier series expansion:

\[
u_1 = \sum_{n=0}^{N} \sum_{m=1}^{M} (a_{n,m} \phi_m + b_{n,m} \varphi_m + c_{n,m} \chi_m + d_{n,m} \psi_m) \cos(n \omega t), \tag{5}
\]

\[
u_2 = \sum_{n=0}^{N} \sum_{m=1}^{M} (e_{n,m} \phi_m + f_{n,m} \varphi_m + g_{n,m} \chi_m + h_{n,m} \psi_m) \cos(n \omega t), \tag{6}
\]
where

\[ \phi_m = A_{m_1,m_2} \sin \left( \frac{(2m_1 + 1)\pi x_1}{2L_1} \right) \sin \left( \frac{(2m_2 + 1)\pi x_2}{2L_2} \right), \]

\[ \varphi_m = A_{m_1,m_2} \sin \left( \frac{(2m_1 + 1)\pi x_1}{2L_1} \right) \cos \frac{m_2 \pi x_2}{L_2}, \]

\[ \chi_m = A_{m_1,m_2} \cos \frac{m_1 \pi x_1}{L_1} \sin \left( \frac{(2m_2 + 1)\pi x_2}{2L_2} \right), \]

\[ \psi_m = A_{m_1,m_2} \cos \frac{m_1 \pi x_1}{L_1} \cos \frac{m_2 \pi x_2}{L_2}. \]

Here, \( A_{m_1,m_2} = 1/\sqrt{4L_1L_2} \) (for \( m_1 = m_2 = 0 \)), \( A_{m_1,m_2} = 1/\sqrt{2L_1L_2} \) (for \( m_1 = 0 \) or \( m_2 = 0 \)) and \( A_{m_1,m_2} = 1/\sqrt{L_1L_2} \) (otherwise). Eqs. (5) and (6) satisfies the time reversal symmetry and the NBCs. To meet the time periodicity of Lagrangian density, we set the order of harmonic \( n \) in Eqs. (5) and (6) to be an integer quantity. Inserting the displacement into the action and integrating it over the domain \( \Omega \times (0, 2\pi/\omega) \), we obtain the action integral \( I(a_{n,m}, \ldots, h_{n,m}) \) as a function of the coefficients \( a_{n,m} \) to \( h_{n,m} \). The stationary condition is then given by the nonlinear equations:

\[ \frac{\partial I}{\partial a_{n,m}} = \cdots = \frac{\partial I}{\partial h_{n,m}} = 0. \]  

(11)

### 2.3 Subsidiary condition

The condition given by Eq. (11) provides \( X \)-th system of nonlinear equations, where \( X \) is the total degree of freedom of the Fourier series expansion. However, the number of unknown quantities is \( X + 1 \): the \( X \) coefficients of displacement (\( a_{n,m} \), \ldots, \( h_{n,m} \)) and a resonance frequency \( \omega \). Therefore, one equation is missing. To implement and close the nonlinear equations, we introduce the \( L^2 \) norm of the displacement function \( u_i \) at \( t = 0 \) by

\[ \|u_i\|_{L^2}^2 = \left[ \int_{\Omega} \left\{ u_i^2(x_1, x_2, 0) + u_i^2(x_1, x_2, 0) \right\} dV \right]^{1/2}, \]

(12)

and impose a subsidiary condition \( \|u_i\|_{L^2}^2 = \text{const} \). There are several advantages for introducing the condition. First, it compensate for the missing equation. Second, it avoids the trivial solution \( a_{n,m} = \cdots = h_{n,m} = 0 \). Finally, and perhaps most importantly, the low amplitude limit \( \|u_i\|_{L^2}^2 \to 0 \) is equivalent to the conventional RUS theory.

The \( (X + 1) \)-th system of simultaneous equations is nonlinear, and hence, we fail to solve it in an analytical form. Therefore, we first linearize the hyperelastic material and calculate a linearized FVAR mode. The nonlinear equations (11) with respect to \( \|u_i\|_{L^2}^2 = \text{const} \) are then solved numerically around the linearized solution by a convergent calculation based on the Newton method. Further details on the numerical analysis can be seen in our previous study [12].

### 3. Results and discussion

#### 3.1 Vibration pattern

For numerical analysis, we set \( \lambda = 1, \mu = 0.33, L_1 = 1.1 \) and \( L_2 = 0.9 \). Figure 1 shows nonlinear FVAR patterns of the \( A_2' - 2 \) mode obtained at \( \|u\| = 0.05 \). The notation \( A_2' - 2 \) implies that the result is obtained from a convergent calculation around the linearized \( A_2 - 2 \) mode. Although only a half period is shown here, the remaining half would be obvious considering the time reversal symmetry of displacement with respect to the \( t = 0 \) axis. Figure 1(b) to (e) shows the displacement components for \( n = 0 \) to 3 in Fig. 1(a), thereby their summation yields (a). From the figures, we can
see that \( n = \) odd components have \( A_2 \) symmetry, as they are invariant for the symmetry operations of \( E \) and \( C_2 \) as well as \(-\sigma_x\) and \(-\sigma_y\). To our surprise, however, the symmetry of \( n = \) even components are \( A_1 \) because the patterns of Fig. 1(b) and (d) are invariant under all symmetry operations in the point group \( C_{2v} = \{E, C_2, \sigma_x, \sigma_y\} \). We also confirmed this feature in other modes that belong to \( A'_2 \). Essentially the same feature has been observed in \( B'_1 \) and \( B'_2 \) modes. These result indicates that the symmetry of a nonlinear FVAR mode depends on the parity of the order of harmonics \( n \). For the case of \( A'_1 \) mode, however, the symmetry is always \( A_1 \) irrespective to the parity.

![Figure 1](image)

**Figure 1.** (a)FVAR pattern of \( A'_2 - 2 \) mode obtained at \( ||u_i||_{L_2} = 0.05 \). (b) to (e) are displacement components for \( n = 0 \) to 3. To display low amplitude components, the displacements are magnified by (a) 5×, (b) 100×, (c) 5×, (d) 100× and (e) 30×, respectively.

### 3.2 Magnetic point group

As mentioned, FVAR of a linearized solid has been classified on the basis of irreducible representations [14]. However, this classification is no longer applicable for nonlinear FVAR mode because (i) it predicts the symmetry of \( n = 1 \) (or harmonic vibration) mode exclusively and (ii) the symmetry depends on the parity of the order of harmonics \( n \).

The exotic connection between the parity \( n \) and vibration symmetry can be explained from magnetic point group. Let \( \hat{T} \) be a time reversal operator that reverses the sign of time with respect to the \( t = \pi/2\omega \) axis. Then, the operator transforms the time-dependent part of the displacement function such that

\[
\hat{T} \cos(n\omega t) = (-1)^n \cos(n\omega t).
\]

According to group theory, there exist four types of magnetic point groups related to the point group \( C_{2v} \) [15]. These are

\[
\begin{align*}
C_{2v}(C_{2v}) &= \{E, C_2, \sigma_x, \sigma_y\}, \\
C_{2v}(C_2) &= \{E, C_2\} + \hat{T}\{\sigma_x, \sigma_y\}, \\
C_{2v}(C_{1h}) &= \{E, \sigma_y\} + \hat{T}\{C_2, \sigma_x\}, \\
C_{2v}(C'_{1h}) &= \{E, \sigma_x\} + \hat{T}\{C_2, \sigma_y\}.
\end{align*}
\]
Here, \( C_{2v}(C_{2v}) \) is the point group \( C_{2v} \) itself and is called the *single color* (black or white) magnetic point group. The remaining three groups include the time reversal operator, and are therefore called the *bicolor* (black and white) magnetic point groups. To simplify the notation, we express \( C_{2v}(C_{2v}) = A'_1, \ C_{2v}(C_2) = A'_2, \ C_{2v}(C_{1h}) = B'_1 \) and \( C_{2v}(C'_{1h}) = B'_2 \).

The four magnetic point groups successfully describe the vibration symmetry of a nonlinear FVAR mode. For instance, we consider the \( A'_2 \) mode. According to Eq. (15), it predicts that the symmetry of \( n = \text{Odd} \) components are invariant for the operations of \( \{ E, C_2, -\sigma_x, -\sigma_y \} \), whereas that of \( n = \text{Even} \) components are \( \{ E, C_2, \sigma_x, \sigma_y \} \). Obviously, the former implies the \( A_2 \) symmetry and the latter \( A_1 \). This is nothing but the symmetry observed in Fig. 1. Essentially the same result can be obtained from Eqs. (16) and (17), and which explain the symmetry of \( B'_1 \) and \( B'_2 \) modes. On the other hand, for the case of \( A'_1 \) mode, Eq. (14) predicts that the symmetry is invariant only for \( \{ E, C_2, \sigma_x, \sigma_y \} \), irrespective to the parity of \( n \). Hence, the magnetic point groups explain the result obtained in the present numerical analysis.

### 4. Conclusions

In this study, we investigated FVAR of a two-dimensional St.Venant–Kirchhoff hyperelastic material within the framework of nonlinear elasticity and the calculus of variations. The results obtained in this study are summarized as follows:

1. FVAR of two-dimensional St.Venant–Kirchhoff hyperelastic material is formulated as a variable domain type variational problem. The variational analysis revealed that the stationary solution should satisfy the time reversal symmetry, time periodicity, periodical vanishing of displacement velocity and surface normal traction free conditions.

2. Numerical analysis revealed that vibration symmetry of nonlinear \( A'_2 \) mode depends on the parity of \( n \): the symmetry is \( A_2 \) for \( n = \text{Odd} \) components whereas it becomes \( A_1 \) whenever \( n = \text{Even} \). We observed essentially the same result in \( B'_1 \) and \( B'_2 \) modes. For the case of \( A'_1 \) mode, however, the symmetry is \( A_1 \) irrespective to the parity. Conventional irreducible representations fail to explain the parity-related vibration symmetries.

3. Group theoretical analysis revealed that there exists one type of *monocolor* magnetic point groups, \( C_{2v}(C_{2v}) \), and three types of *bicolor* magnetic point groups, \( C_{2v}(C_2) \), \( C_{2v}(C_{1h}) \) and \( C_{2v}(C'_{1h}) \), corresponding to the point group \( C_{2v} \). These magnetic point groups successfully explain the vibration symmetry obtained in the numerical analysis. This result indicates that the color symmetry is embedded in the nonlinear FVAR modes.

### REFERENCES


