SIMPLE ANALYTICAL EXPRESSIONS FOR THE NATURAL FREQUENCIES OF AN EULER-BERNOULLI BEAM

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Simple analytical expressions that approximate the roots of a frequency equation are proposed for free vibration of an Euler-Bernoulli beam with various end conditions. The roots are used to calculate the natural frequencies of the beam. The frequency equation is transformed such that it is expressed in terms of the exponential function and trigonometric functions by means of Euler’s formula to avoid dealing with complex quantities. The transformed frequency equation is converted to another equation that includes the tangent function and the Gudermannian function using the relationship between the exponential function and the Gudermannian function. The asymptotic convergence of the Gudermannian function is used to unveil the asymptotic characteristics of the roots. The fact that the Gudermannian function asymptotically converges to $\pi / 2$ leads to simple analytical expressions that approximate the roots. Since the analytical expressions are attained on the basis of the asymptotic behavior of the Gudermannian function, they are particularly accurate in determining the roots for the higher modes. The significance of the analytical expressions is understood by comparing them with traditional expressions proposed by Timoshenko. Although the Timoshenko expressions have been used widely for several decades in calculating the roots, these expressions have never been validated because it is difficult to verify their accuracy involved in determining the roots for the higher modes. It is demonstrated that the roots estimated by the analytical expressions are the same as those determined by the Timoshenko expressions. Consequently, the Timoshenko expressions are corroborated because they have the same properties as the analytical expressions proposed in this paper and determine the roots in a precise fashion for the higher modes.

1. Introduction

The roots of a frequency equation for free vibration of an Euler-Bernoulli beam are studied in this paper. The characteristics of the roots are analyzed for various end conditions including fixed-fixed and fixed-hinged, and these roots are utilized to calculate the natural frequencies. The asymptotic behaviour of the roots is investigated and simple analytical expressions are derived to approximate the roots. The analytical expressions are referred to as Frequency Root and have turned out to determine the roots in an accurate manner particularly for the higher modes.

The objective of this work is to transform the frequency equation into another form of the equation that reveals the asymptotic properties of the roots and to provide the Frequency Root to validate analogous expressions proposed by Timoshenko et al. which are referred to as Timoshenko Root.
Traditional frequency equations shown in most classical textbooks contain nonlinear functions. Therefore, their roots should be determined with the aid of numerical root finding methods. It is practically impossible to decide all the roots because each of them has to be collected separately while its order is being tracked. The transformed frequency equation proposed in this paper enables understanding how the roots behave for large mode numbers and leads to the Frequency Root that supplies correct roots for the higher modes. Although the Timoshenko Root has been popular in determining the roots since they appeared in his textbook, there has been no reasonable argument that supports them because it is not feasible to verify their accuracy considering the frequency equation involves an infinite number of roots. The Timoshenko Root is corroborated in comparison with the Frequency Root. To this end, it is demonstrated that the Timoshenko Root provides the same roots as the Frequency Root, which means they share the same asymptotic characteristics.

This paper is divided into several sections. The frequency equation is transformed into another equation to characterize the roots for a fixed-fixed beam in Section 2. The properties of the roots are examined for a fixed-hinged beam in Section 3. The contributions of this work are summarized in Section 4.

2. Fixed-Fixed Beam

This section concerns a frequency equation for free vibration of an Euler-Bernoulli beam with a fixed-fixed end condition. The motion of the beam is decided according to the Euler-Bernoulli beam theory and is represented as the sum of the normal modes. The eigenvalue problem for the fixed-fixed beam of length \( L \) is

\[
\frac{d^4 Y(x)}{dx^4} - \beta^4 Y(x) = 0 \quad \text{for} \quad 0 \leq x \leq L
\]

where \( Y(x) \) is the normal mode and \( \beta \) is the eigenvalue which satisfies \( \beta^4 = \left( \frac{\rho A \omega^2}{EI} \right) \). \( E \) is the modulus of elasticity and \( I \) is the area moment of inertia of the cross section. \( \rho \) refers to the density per unit length and \( A \) refers to the cross sectional area per unit length. \( \omega \) is the circular natural frequency. The eigenvalues of Eq. (1) are used to calculate the natural frequencies.

\[
f_i = \frac{\omega_i}{2\pi} = \frac{\beta_i^2}{2\pi} \sqrt{\frac{EI}{\rho A}} = \frac{\lambda_i^2}{2\pi} \sqrt{\frac{EI}{\rho AL^4}}, \quad i = 1, 2, 3, \ldots
\]

where \( f_i \) and \( \omega_i \) are the natural frequency and the circular natural frequency for the \( i \)th mode, respectively. Note that the fact that \( \omega \) is greater than zero leads to \( \lambda = \beta L > 0 \). Equation (1) that meets the fixed-fixed end condition leads to the frequency equation.

\[
\cos(\lambda) \cosh(\lambda) = 1, \quad \lambda > 0
\]

\( \cosh(\lambda) \) in Eq. (3) is replaced with \( (e^\lambda + e^{-\lambda})/2 \) by means of Euler’s formula so that complex quantities do not appear in solving Eq. (3).

\[
\cos(\lambda)(e^{2\lambda} + 1) - 2e^\lambda = 0, \quad \lambda > 0
\]

Defining \( p = \cos(\lambda) \) and \( q = e^\lambda \), Eq. (4) yields the two separate equations.
\[ p = \frac{2q}{1+q^2} \]  
\[ p = \cos\left(\log(q)\right) \]  

where \( q > 1 \) because \( \lambda > 0 \). It is worthwhile to notice that \( p > 0 \) in Eq. (5) because \( q > 1 \), and \( p < 1 \) due to \((1+q^2)-2q=(q-1)^2 > 0\) for \( q > 1 \). In an effort to express \( q \) in Eq. (5) in terms of \( p \), Eq. (5) is transformed into the quadratic equation.

\[ pq^2-2q+p=0, \quad 0 < p < 1, \quad q > 1 \]  

where \( p \) is considered a parametric coefficient. The discriminant of Eq. (7) is \( 1-p^2 \) and its sign is examined to understand the characteristics of the roots. As \( 1-p^2 > 0 \) due to \( 0 < p < 1 \), Eq. (7) has two distinct real roots denoted by \( q_1 \) and \( q_2 \).

\[ q_1 = \frac{1+\sqrt{1-p^2}}{p}, \quad 0 < p < 1 \]  
\[ q_2 = \frac{1-\sqrt{1-p^2}}{p}, \quad 0 < p < 1 \]

The Minkowski’s inequality verifies \( q_2 \leq 1 \) as follows.

\[ q_2 = \frac{1}{p} - \sqrt{\frac{1}{p^2} - 1} \leq \frac{1}{p} - \sqrt{\frac{1}{p^2} + 1} = 1 \]

Since \( q_2 \) does not meet \( q > 1 \), \( q_2 \) is no longer considered. On the other hand, \( q_1 > 1 \) because \( 1 < 1+\sqrt{1-p^2} < 2 \) for \( 0 < p < 1 \). Therefore, only \( q_1 \) is considered hereinafter. In order to solve Eq. (8) and Eq. (6) simultaneously, Eq. (6) is substituted into Eq. (8), which yields

\[ q \cos\left(\log(q)\right) = 1+\sqrt{1-\cos^2\left(\log(q)\right)} \]
\[ = 1+\left|\sin\left(\log(q)\right)\right| \]  

Equation (11) is divided into two separate cases depending on the sign of \( \sin\left(\log(q)\right) \).

\[ e^\lambda \cos(\lambda) = \begin{cases} 1-\sin(\lambda) & \text{if} \quad \sin(\lambda) < 0 \quad \text{(Case A)} \\ 1+\sin(\lambda) & \text{if} \quad \sin(\lambda) > 0 \quad \text{(Case B)} \end{cases} \]

where \( \lambda = \log(q) > 0 \). Note that Eq. (12) does not hold for \( \sin(\lambda) = 0 \) so this case is neglected. Equation (12) solved for \( \sin(\lambda) < 0 \) is referred to as Case A whereas Case B refers to Eq. (12) solved for \( \sin(\lambda) > 0 \). Solving Eq. (12) for Case A, the roots of Eq. (12) should be found in the region \( D_A \) where \( \lambda > 0 \), \( \sin(\lambda) < 0 \), and \( 0 < \cos(\lambda) < 1 \).
Equation (12) for Case A is transformed such that it includes only trigonometric functions by using the relationship between the exponential function and the tangent function.

\[
e^{i} = \tan \left( \frac{\pi}{4} + \frac{1}{2} \text{gd}(\lambda) \right)
\]

where \( \text{gd}(\lambda) \) represents the Gudermannian function. In addition, \( \sin(\lambda) \) and \( \cos(\lambda) \) in Eq. (12) are converted to the tangent function by means of the trigonometric identities.

\[
\sin(\lambda) = \frac{2 \tan \left( \frac{\lambda}{2} \right)}{1 + \tan^2 \left( \frac{\lambda}{2} \right)} , \quad \cos(\lambda) = \frac{1 - \tan^2 \left( \frac{\lambda}{2} \right)}{1 + \tan^2 \left( \frac{\lambda}{2} \right)}
\]

In view of Eqs. (14) and (15), Eq. (12) for Case A becomes

\[
\tan \left( \frac{\pi}{4} - \frac{\lambda}{2} + n\pi \right) = \tan \left( \frac{\pi}{4} + \frac{1}{2} \text{gd}(\lambda) \right), \quad n = 0, \pm 1, \pm 2, \ldots
\]

where \( \tan(\lambda) = \tan(\lambda + n\pi), \ n = 0, \pm 1, \pm 2, \ldots \) is used in deriving Eq. (16). Notice that the roots of Eq. (16) can be acquired from the simpler form of the frequency equation.

\[
-\lambda_n + 2n\pi = \text{gd}(\lambda_n), \quad n = 0, \pm 1, \pm 2, \ldots
\]

where \( \lambda_n \) denotes the \( n^{th} \) root of Eq. (17). The properties of \( \text{gd}(\lambda) \) below signifies that \( \text{gd}(\lambda) \) increases monotonically for \( \lambda > 0 \) and approaches \( \pi/2 \) asymptotically.

\[
gd(0) = 0, \quad \gd'(0) > 0, \quad \lim_{\lambda \to \infty} \gd'(\lambda) = \frac{\pi}{2}, \quad \gd'(\lambda) > 0 \text{ for } \lambda > 0
\]

Equation (18) means that \( 0 < \text{gd}(\lambda) < \pi/2 \) for \( \lambda > 0 \). Therefore, Eq. (17) must satisfy

\[
0 < -\lambda_n + 2n\pi < \frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

Equation (19) implies that \( \lambda_n \) determined for \( n = 1, 2, \ldots \) are the roots of Eq. (17) because they are positioned in the region \( D_4 \) while those collected for \( n = 0, -1, -2, \ldots \) are abandoned because they do not meet Eq. (13). Since \( \text{gd}(\lambda) \) tends to be \( \pi/2 \) as \( \lambda \) increases, \( \lambda_n \) is approximated as follows.

\[
-\lambda_n^A + 2n\pi \approx \frac{\pi}{2} , \quad n = 1, 2, 3, \ldots
\]
where \( \lambda_n^d \) denotes the root of Eq. (20) decided for mode number \( n \) and also means Frequency Root for Case A.

**Figure 1.** (a) ● : Roots of \( \cos(\lambda) \cosh(\lambda) = 1 \), ○ : Roots of \(-\lambda_n + 2n\pi = \text{gd}(\lambda_n)\), (b) * : Roots of \(-\lambda_n^d + 2n\pi = \pi / 2\), \( n = 1, 2, 3, \ldots \).

The transformed frequency equation in Eq. (17) is validated by comparing its roots with those of the original frequency equation in Eq. (3) collected for \( \sin(\lambda) < 0 \). The roots of Eq. (17) are determined by finding the circles ○ in Fig. 1(a) where \( \eta = \text{gd}(\lambda) \) meets \( \eta = -\lambda_n + 2n\pi \). Figure 1(a) shows that \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are the same as the roots of Eq. (3) decided for the first, third, and fifth modes. As a result, Eq. (17) enables collecting the roots of Eq. (3) for \( \sin(\lambda) < 0 \). The roots of Eq. (20) are compared with those of Eq. (17) to examine the accuracy of \( \lambda_n^d \) in approximating \( \lambda_n \). Figure 1(b) demonstrates that \( \lambda_n^d \) agrees fairly well with \( \lambda_n \), and \( \lambda_n^d \) tends to be accurate as \( \lambda \) increases because \( \eta = \text{gd}(\lambda) \) becomes almost the same as \( \eta = \pi / 2 \) for large values of \( \lambda \).

The roots of Eq. (12) for Case B must exist in the region \( D_B \) where \( \lambda > 0 \), \( \sin(\lambda) > 0 \), and \( 0 < \cos(\lambda) < 1 \).

\[
D_B = \left\{ \lambda \mid 2m\pi < \lambda < 2m\pi + \frac{\pi}{2}, \ m = 0, 1, 2, \ldots \right\} \tag{21}
\]

Applying Eqs. (14) and (15) to Eq. (12) for Case B, Eq. (12) becomes

\[
\tan \left( \frac{\pi}{4} + \frac{\lambda}{2} \right) = \tan \left( \frac{\pi}{4} + \frac{1}{2} \text{gd}(\lambda) + n\pi \right), \ n = 0, \pm 1, \pm 2, \ldots \tag{22}
\]

Then, Eq. (22) yields the final frequency equation that yields the roots of Eq. (3) for \( \sin(\lambda) > 0 \).

\[
\lambda_n - 2n\pi = \text{gd}(\lambda_n), \ n = 0, \pm 1, \pm 2, \ldots \tag{23}
\]

where \( \lambda_n \) denotes the root of Eq. (23) for mode number \( n \). Since \( 0 < \text{gd}(\lambda) < \pi / 2 \) for \( \lambda > 0 \), \( \lambda_n \) in Eq. (23) must satisfy the following relationship.

\[
0 < \lambda_n - 2n\pi < \frac{\pi}{2}, \ n = 0, \pm 1, \pm 2, \ldots \tag{24}
\]
Equation (24) signifies that \( \lambda_n \) determined for \( n=1,2,3,\ldots \) are located in the region \( D_B \) so they are the roots of Eq. (23) for \( \sin(\lambda) > 0 \). In contrast, \( \lambda_n \) collected for \( n=0,-1,-2,\ldots \) do not meet Eq. (24) so they are ruled out. In view of the asymptotic properties of \( \text{gd}(\lambda) \) in Eq. (18), the approximate roots of Eq. (23) are obtained from the following equation.

\[
\lambda_n^B - 2n\pi \approx \frac{\pi}{2}, \ n=1,2,3,\ldots
\]

where \( \lambda_n^B \) denotes the root of Eq. (25) decided for the \( n^{th} \) mode and also represents Frequency Root collected for Case B.

![Image](image.png)

**Figure 2.** (a) ● : Roots of \( \cos(\lambda)\cosh(\lambda) = 1 \), ○ : Roots of \( \lambda_n - 2n\pi = \text{gd}(\lambda_n) \), (b) * : Roots of \( \lambda_n - 2n\pi = \text{gd}(\lambda_n) \), ○ : Roots of \( \lambda_n^B - 2n\pi = \pi/2, \ n=1,2,3,\ldots \).

The roots of Eq. (23), \( \lambda_n \), are compared with those of Eq. (3) determined for \( \sin(\lambda) > 0 \) in Fig. 2(a) to validate Eq. (23). It is exhibited in Fig. 2(a) that \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are the same as the roots of Eq. (3) collected for the second, fourth, and sixth modes, which means that Eq. (23) decides the roots of Eq. (3) satisfying \( \sin(\lambda) > 0 \). In Fig. 2(b), \( \lambda_n^B \) is compared with \( \lambda_n \) to comprehend how much accurate \( \lambda_n^B \) is in estimating \( \lambda_n \). It is verified that the difference between \( \lambda_n^B \) and \( \lambda_n \) tends to diminish as \( \lambda \) increases in Fig. 2(b). Therefore, \( \lambda_n^B \) is a good approximation to \( \lambda_n \) for large mode numbers. Equation (25) becomes identical to Eq. (23) for large values of \( \lambda \) and enables collecting the precise roots of Eq. (3) for the higher modes.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<tbody>
<tr>
<td>( \lambda_n^{Timo} )</td>
<td>( \lambda_1^{Timo} = \frac{3}{2} \pi )</td>
<td>( \lambda_2^{Timo} = \frac{5}{2} \pi )</td>
<td>( \lambda_3^{Timo} = \frac{7}{2} \pi )</td>
<td>( \lambda_4^{Timo} = \frac{9}{2} \pi )</td>
<td>( \lambda_5^{Timo} = \frac{11}{2} \pi )</td>
<td>( \lambda_6^{Timo} = \frac{13}{2} \pi )</td>
</tr>
<tr>
<td>( \lambda_n^{A}, \lambda_n^{B} )</td>
<td>( \lambda_1^{A} = \frac{3}{2} \pi )</td>
<td>( \lambda_2^{B} = \frac{5}{2} \pi )</td>
<td>( \lambda_3^{A} = \frac{7}{2} \pi )</td>
<td>( \lambda_4^{B} = \frac{9}{2} \pi )</td>
<td>( \lambda_5^{A} = \frac{11}{2} \pi )</td>
<td>( \lambda_6^{B} = \frac{13}{2} \pi )</td>
</tr>
</tbody>
</table>

Timoshenko et al. proposed \( \lambda_i^{Timo} = (i + 0.5) \pi \) to predict the roots of Eq. (3). \( ^1 \lambda_i^{Timo} \) denotes the Timoshenko Root and is compared with \( \lambda_i^{A} \) and \( \lambda_i^{B} \) in Table 1 to figure out the accuracy of \( \lambda_i^{Timo} \) in deciding the roots of Eq. (3). It is verified that \( \lambda_i^{Timo} \) collected for the first six modes are the
same as $\lambda_n^A$ and $\lambda_n^B$ that are listed such that their magnitude increases in Table 1. In consequence, the Timoshenko Root ($\lambda_n^{Timo}$) has the same properties as the Frequency Root ($\lambda_n^A$, $\lambda_n^B$), which means $\lambda_n^{Timo}$ also determines the roots in a precise manner for large mode numbers. The same conclusion can be drawn for a fixed-free beam through the analysis procedure presented in this section.

3. Fixed-Hinged Beam

The asymptotic properties of the roots for a fixed-hinged beam are investigated in this section. The frequency equation for this case is

$$\cos(\lambda)\sinh(\lambda) - \sin(\lambda)\cosh(\lambda) = 0, \quad \lambda > 0 .$$  \hspace{1cm} (26)

Expressing $\sinh(\lambda)$ and $\cosh(\lambda)$ in Eq. (26) in terms of $e^\lambda$ using Euler’s formula, Eq. (26) leads to

$$\cos(\lambda)(e^{2\lambda} - 1) - \sin(\lambda)(e^{2\lambda} + 1) = 0, \quad \lambda > 0 .$$  \hspace{1cm} (27)

Equation (27) can be written in terms of the tangent function by means of trigonometric identities.

$$e^{2\lambda} = \frac{1 + \tan(\lambda)}{1 - \tan(\lambda)} = \tan\left(\frac{\pi}{4} + \lambda\right)$$  \hspace{1cm} (28)

Equation (28) is transformed into the following equation according to Eq. (14).

$$\tan\left(\frac{\pi}{4} + \frac{1}{2}\text{gd}(2\lambda) + n\pi\right) = \tan\left(\frac{\pi}{4} + \lambda\right), \quad n = 0, \pm 1, \pm 2, \cdots$$  \hspace{1cm} (29)

Then, Eq. (29) reduces to the final frequency equation.

$$2\lambda_n - 2n\pi = \text{gd}(2\lambda_n), \quad n = 0, \pm 1, \pm 2, \cdots$$  \hspace{1cm} (30)

where $\lambda_n$ denotes the $n^{th}$ root of Eq. (30) and must satisfy the relationship below considering $0 < \text{gd}(2\lambda) < \pi / 2$ for $\lambda > 0$.

$$0 < 2\lambda_n - 2n\pi < \frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \cdots$$  \hspace{1cm} (31)

Equation (31) implies that $\lambda_n > 0$ is satisfied for $n = 1, 2, 3, \cdots$ whereas $\lambda_n > 0$ does not hold for $n = 0, -1, -2, \cdots$. Therefore, the roots of Eq. (30) decided for $n = 1, 2, 3, \cdots$ are the roots of Eq. (26) and those collected for $n = 0, -1, -2, \cdots$ are rejected. The asymptotic behaviour of $\text{gd}(2\lambda)$ in Eq. (18) enables approximating $\lambda_n$ as follows.

$$2\lambda_n^{FH} - 2n\pi = \frac{\pi}{2}, \quad n = 1, 2, 3, \cdots$$  \hspace{1cm} (32)

where $\lambda_n^{FH}$ represents the $n^{th}$ root of Eq. (32) decided for the fixed-hinged beam case.
The transformed frequency equation in Eq. (30) is validated in comparison with Eq. (26). Figure 3(a) shows that $\lambda_n$ collected for $n = 1, 2, 3, \cdots$ are the same as the roots of Eq. (26). Therefore, Eq. (30) also yields the roots of Eq. (26). $\lambda_{n}^{FH}$ is compared with $\lambda_n$ in Fig. 3(b) to comprehend the accuracy of $\lambda_{n}^{FH}$. Figure 3(b) displays that $\lambda_{n}^{FH}$ shows a good agreement with $\lambda_n$ and its accuracy tends to improve as $\lambda$ increases because $gd(2\lambda)$ tends to be $\pi/2$ as $\lambda$ increases.

Timoshenko et al. suggested utilizing $\lambda_{n}^{Timo} = (i + 0.25)\pi$ in collecting the roots of Eq. (26). It is worth noticing that $\lambda_{n}^{Timo}$ supplies the same numerical values as $\lambda_{n}^{FH}$. Therefore, $\lambda_{n}^{Timo}$ is equivalent to $\lambda_{n}^{FH}$ and possesses the same asymptotic properties as $\lambda_{n}^{FH}$. Consequently, $\lambda_{n}^{Timo}$ also determines the roots of the frequency equation in Eq. (26) in a highly accurate manner and tends to be precise for large values of $\lambda$. Frequency Root for a fixed-sliding beam can also be obtained through an analogous analysis discussed in this section and exhibits the same properties as $\lambda_{n}^{FH}$.

4. Conclusion

The frequency equation was converted to another form of the equation that unveils the asymptotic properties of the roots. Frequency Root for fixed-fixed and fixed-hinged end conditions was derived to approximate the roots on the basis of their asymptotic characteristics. It has been verified that the Frequency Root is a decent approximation to the roots and becomes precise for large mode numbers. Timoshenko Root for these cases was corroborated by demonstrating that the roots determined using the Timoshenko Root are identical to those decided by the Frequency Root. One is encouraged to exploit either the Frequency Root or the Timoshenko Root for the analysis of a structure driven by high-frequency excitation forces or for the operation of a multifrequency AFM probe.

REFERENCES


