HIGH-FREQUENCY VIBRATION ANALYSIS OF A PLATE IN THERMAL ENVIRONMENT BASED ON ENERGY FINE ELEMENT METHOD

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The structure of a hypersonic craft is usually subjected to a service environment of heating and high-frequency exciting. To evaluate the vibration behaviour of the structure in a high-frequency range, the energy density governing equation to analyse high-frequency behaviour of an isotropic plate in uniform thermal environment is derived in this paper. The energy density distribution of a simply supported plate is studied, in which the thermal effect is considered as thermal stress. The energy finite element method (EFEM) formulation based on the governing equation is developed, and a simple numerical example is carried out. The numerical solution is compared with the theory solution (modal decomposition solution) to verify the validity and accuracy of the derived equation. The results show that the derived equation approximates well the smooth time- and locally space-averaged energy density of the structure under high-frequency excitation in a thermal environment, and EFEM needs a smaller number of elements than FEM’s.

1. Introduction

With the great development of hypersonics which are usually subjected to extreme aerodynamic heating and high-frequency exciting during working, there’s a great need for the high-frequency vibration analysis of structures in thermal environments. Thermal environments have series of impacts on material properties, geometry shapes, stress states and so on. Great efforts have been made to analyse the dynamic characters of structures in thermal environments. Jeyaraj et al.\textsuperscript{12} used combined FEM/BEM to analyse the vibration and acoustic radiation characters of an isotropic plate and a composite plate with inherent material damping in a thermal environment, finding that the natural frequencies decrease with the temperature increasing due to interaction between reduced stiffness and enhanced damping. Q. Geng et al.\textsuperscript{4} analysed the dynamic and acoustic radiation characters of a flat plate under thermal environments through both the theory method and combined FEM/BEM, and the two results match well.

The above researchers showed that the FEM could predict well the low-frequency vibrational responses of built-up structures under thermal environments. But with the frequency increase, more and more elements are needed to describe the vibration of the structure and small uncertainties will have more and more impacts on the calculating results. Thus the FEM calculation will not only cost much time, but the error is much bigger. Though SEA(Statistical energy analysis) is used widely to predict the space- and frequency-averaged behaviour of built-up structures at high frequencies...
where the modal density of structures is high\(^4\), it could only approximate a single vibrational energy value for each subsystem of structures.

Due to the disadvantages of the FEM and SEA at high frequency, EFEM (Energy finite element method) which could predict the time- and space- averaged far-field vibrational energy of structures appeared. Nefsk and Sung\(^5\) developed the finite element formulation of the power flow model and applied it to beams. Bouthier and Bernhard extended the work to two-dimension structures and assessed the energy density distribution of a membrane\(^6\) and a Kirchhoff plate\(^7\). D.-H. Park et al. developed the power flow model of orthotropic plate\(^8\) and Mindlin plate\(^9\). Xie Mx et al. applied EFEM to high-frequency structural-acoustic coupling of an aircraft cabin with truncated conical shape\(^10\) and proposed the transient vibrational energy response analysis of a rod under high-frequency excitation\(^11\).

Though EFEM has been developed well since 1980s, few researchers paid attention to the EFEM of structures under thermal environments. In this work, we derived the energy governing equation of an isotropic plate under uniform thermal environments. And the accuracy of the developed EFEM is verified by applying it to a simply supported isotropic plate under thermal environments compared with modal decomposition method. The impact of thermal environments on high frequency vibration energy distribution of plates is also investigated.

2. Energy governing equation for finite plates under uniform thermal environments

In this paper, we consider a simply supported rectangular isotropic thin plate under thermal environments. Assume that the temperature is uniform throughout the plate and the plate is stress-free at the reference temperature \(T_0\). Only the thermal stress is considered, and the change of material properties is not taken into account. So when the temperature changes to \(T\), the transversely vibrating governing equation will be\(^3\)

\[
D_0 \nabla^4 + \rho h \frac{\partial^2 w}{\partial t^2} = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + q 
\]

(1)

Where \(D_0 = E h^3 (1 + \eta) / 12 (1 - \nu^2)\) is the bending stiffness of the plate, \(\eta\) is the hysteretic damping loss factor, \(\rho\) is the density, \(h\) is the thickness, \(q\) is the transverse mechanical load applied on plate, \(j\) is the imaginary unit. \(N_x, N_y\), and \(N_{xy}\) are the thermally induced membrane forces. Considering the uniform thermal environments and the boundary condition, the three thermally induced membrane forces can be expressed as

\[
N_x = -\frac{E \alpha \Delta T h}{1 - \nu} 
\]

(2)

\[
N_y = -\frac{E \alpha \Delta T h}{1 - \nu} 
\]

(3)

\[
N_{xy} = 0 
\]

(4)

Because the complete, general solution of Eq. (1) subjected to various boundary conditions can’t be found, we only utilize the far field solution of Eq. (1) to simplify the energy analysis\(^7\). So the general form of the far field travelling plane wave solution can be expressed as:

\[
w = (A_x e^{-ik_x x} + B_x e^{ik_x x})(A_y e^{-ik_y y} + B_y e^{ik_y y}) e^{iat} 
\]

(5)

where the unknown constants \(A_x, B_x, A_y, B_y\) are the amplitudes of the waves in \(x\) and \(y\) directions. And \(k_x, k_y\) are the complex wave numbers in \(x\) and \(y\) directions, respectively. Assume that \(k_x^2 + k_y^2 = k^2\), so wavenumber can be expressed as
\[
k = \sqrt{\frac{N}{D_0} - \frac{\left(\frac{N}{D_0}\right)^2 - 4\rho \omega^2}{D_0}} = k_i + \frac{1}{2} i \eta k_i \quad \text{and} \quad k_e = \sqrt{\frac{N}{D} - \frac{\left(\frac{N}{D}\right)^2 - 4\rho \omega^2}{D}}
\]  

(6)

As discussed by Bouthier\textsuperscript{7}, the time averaged energy density which includes the kinetic and potential energy densities of the plate across the thickness can be expressed in terms of transverse displacement as:

\[
\bar{e} = \frac{1}{2\pi} e_d \int_0^\pi \left(\sigma_x e_x^* + \sigma_y e_y^* + \sigma_{xy} e_{xy}^*\right) dz
\]  

(7)

For the transversely vibrating governing equation of the plate under thermal environments was derived under the premise that the thermal tresses keep the same while vibrating, the Eq. (7) can be written as:

\[
\bar{e} = \frac{D}{4} \left[\frac{\partial^2 w}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2}\right)^* + \frac{\partial^2 w}{\partial y^2} \left(\frac{\partial^2 w}{\partial y^2}\right)^* + 2\nu \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial^2 w}{\partial y^2}\right)^* + 2\nu \left(1 - \nu\right) \frac{\partial^2 w}{\partial x \partial y} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^* \right]
\]  

(8)

As above, the time averaged energy intensity across the thickness can be expressed in terms of transverse displacement as\textsuperscript{7}:

\[
\bar{T}_x = -M_{xx} \frac{\partial^2 w}{\partial x^2} - M_{xy} \frac{\partial^2 w}{\partial y^2} + Q_x \frac{\partial w}{\partial t}
\]

(9)

\[
\bar{T}_y = -M_{yy} \frac{\partial^2 w}{\partial x^2} - M_{yx} \frac{\partial^2 w}{\partial y^2} + Q_y \frac{\partial w}{\partial t}
\]

(10)

Substituting Eq. (5) into Eq. (8) Eq. (9) and Eq. (10), we can obtain the complete time averaged far field energy density and energy intensity expressions in terms of transverse displacement. For we couldn’t find obvious relations between them, they are averaged spatially over a half wavelength for small damping in the following manner\textsuperscript{7}:

\[
\left\langle \bar{e} \right\rangle = \frac{k_x k_y}{\pi} \int_0^\pi \bar{e}_d dy dx \quad , \quad \left\langle \bar{T}_x \right\rangle = \frac{k_x k_y}{\pi} \int_0^\pi \bar{T}_x dy dx \quad , \quad \left\langle \bar{T}_y \right\rangle = \frac{k_x k_y}{\pi} \int_0^\pi \bar{T}_y dy dx
\]

(11)

Thus the time- and space-averaged density \( \left\langle \bar{e} \right\rangle \) can be written as:

\[
\left\langle \bar{e} \right\rangle = \frac{D}{4} \text{Re} \left[ |A|^2 e^{-(k_{xz} + k_{yz})} + |A|^2 |B|^2 e^{-k_{xz} + k_{yz}} \right]
\]

(12)

The time- and space-averaged intensity \( \left\langle \bar{T}_x \right\rangle \) and \( \left\langle \bar{T}_y \right\rangle \) can be written as:

\[
\left\langle \bar{T}_x \right\rangle = \frac{k_x \omega}{2} \text{Re} \left[ |A|^2 |B|^2 e^{-(k_{xz} + k_{yz})} + |A|^2 |B|^2 e^{-(k_{xz} + k_{yz})} \right]
\]

(13)

and
\[
\langle T_y \rangle = \frac{k_y}{2} \frac{\omega}{2} \text{Re} \left[ k_x^2 \right] + k_x k_y \nu + \nu k_x^2 + \nu (1-\nu)k_x k_y^* \right].
\]
\[
\begin{align*}
|A_x|^2 |A_y|^2 e^{(k_{x2}x-k_{y2}y)} + |A_x|^2 |B_y|^2 e^{-(k_{x2}x-k_{y2}y)} + \\
|B_x|^2 |A_y|^2 e^{(k_{x2}x-k_{y2}y)} + |B_x|^2 |B_y|^2 e^{-(k_{x2}x-k_{y2}y)}
\end{align*}
\] (14)

Obviously the relationship between the above energy density and intensity is:
\[
\langle I \rangle = -\frac{2\omega k^2}{\eta (k^4 + \rho \omega^2) D} \nabla \langle e \rangle
\] (15)

For an elastic medium, the differential steady energy balance equation in terms of time- and space-averaged energy density can be written as:
\[
\vec{\pi}_m - \eta \omega \vec{e} = \nabla \cdot \vec{I}
\] (16)

Combining Eq. (15) and Eq. (16), we can obtain the steady far field energy governing equation of plates under uniform thermal environments:
\[
\left(\frac{\omega k^2}{k^4 + \rho \omega^2} \right) \nabla^2 \langle e \rangle + \eta \omega \langle e \rangle = \langle \vec{\tau}_m \rangle
\] (17)

It is obvious that thermal environments impact the first coefficient of Eq. (17) by impacting wave numbers of the plate, and thus change the energy distribution state of the plate. Compared to the energy governing equation derived by Bouthier\(^7\), Eq. (17) will be the same when the temperature become reference temperature.

3. Verification

In this section, we will verify the accuracy of the new derived energy governing equation of the isotropic plate under thermal environments and investigate how the thermal environments impact the energy distribution of the plate.

As shown in Fig. 1 , a simply supported isotropic plate is studied here. The plate is excited by a harmonic point force at the center. The geometric parameters and material properties are shown in Table 1. According to the information from Table 1, we could carry out the critical buckling temperature of the plate using the following equation:\(^3\):
\[
T_{cr} = \frac{h^2 \pi^2}{12(1+\nu) \alpha} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)
\] (18)

Substituting the parameters in table 1 into Eq. (18), the critical buckling temperature is carried out to be 49\(^\circ\)C.

![Figure 1. Simply supported, point excited isotropic plate](image-url)
The finite element formulation of the new derived Eq. (17) is developed here to calculate the energy distribution of the example. The results are compared to the theory solutions derived by modal decomposition method. As discussed by Qian Geng\(^3\), the modal decomposition solution expression of Eq. (1) with four boundaries simply supported and a harmonic point force in the center is:

\[
 w(x,y,t) = \sum_{m,n} \frac{4q}{\rho L_x L_y \left( \omega_{mn}^2 - \omega^2 \right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{j\omega t}
\]

(19)

where

\[
 \omega_{mn} = \sqrt{\pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \frac{D_0}{\rho h} + \frac{N_a}{\rho h} \left( \frac{m\pi}{a} \right)^2 + \frac{N_y}{\rho h} \left( \frac{n\pi}{b} \right)^2}
\]

(20)

Table 1. Geometric parameters and material properties of the plate

<table>
<thead>
<tr>
<th>E(Gpa)</th>
<th>ν</th>
<th>ρ (kg/m(^3))</th>
<th>(/K)</th>
<th>L(_x)(m)</th>
<th>L(_y)(m)</th>
<th>h(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>72.4</td>
<td>2700</td>
<td>2.3e-5</td>
<td>1</td>
<td>1</td>
<td>0.03</td>
</tr>
</tbody>
</table>

The external input power \( \langle \pi_{in} \rangle \) can be expressed as:

\[
 \langle \pi_{in} \rangle = \frac{1}{2} \text{Re} \left( FV^* \right) \quad \text{or} \quad \langle \pi_{in} \rangle = \frac{1}{2} |F|^2 \text{Re} \left( A_f \right)
\]

(21)

Where \( F \) is the external force, \( V^* \) is the conjugate of the velocity of the exciting point, \( A_f \) is the admittance at the point. The front equation is used here and the velocity could be calculated by modal decomposition method using Eq. (20).

Four different cases are performed here to verify the accuracy of the derived energy governing equation. The temperature, frequency and damping are shown in Table 2. The reference temperature in this work is 0°C.

Table 2. Parameters for each case

<table>
<thead>
<tr>
<th>Case</th>
<th>Frequency(Hz)</th>
<th>Loss factor</th>
<th>T(°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11000</td>
<td>0.05</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>11000</td>
<td>0.05</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>15000</td>
<td>0.05</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>15000</td>
<td>0.05</td>
<td>40</td>
</tr>
</tbody>
</table>

Figure 2 gives the calculated energy density distributions of case1 obtained by both the theory method(modal decomposition method) and EFEM respectively. It is clear that the exact energy density propagates in wave form from the exciting point to the boundary, while EFEM gives an averaged smooth energy distribution. Compared to case 1, the temperature increases by 20°C in case 2. And the energy density distribution of case 2 shown in Fig. 3 shows the same trend with case 1.

The comparison of the energy density along diagonal of case 1 and case 2 between the two methods are shown in Fig. 4. It is obvious that EFEM approximates well the time- and space-averaged far field energy density compared to the modal decomposition solution both for case 1 and case 2. And the results prove that the impact of thermal environments are described well here by Eq. (17).
Figure 2. Energy density distributions of the plate in case 1 calculated both by theory method and EFEM

Figure 3. Energy density distributions of the plate in case 2 calculated both by theory method and EFEM

Figure 4. Energy density distribution comparison along diagonal in case 1 and case 2

Compared with case 1 and case 2, the frequencies are increased to 15000 Hz in case 3 and case 4. And the corresponding energy density distributions along diagonal obtained by both methods are shown in Fig. 5. Also as the first two cases, the EFEM results line locate almost the middle of the modal decomposition results.
From the above 4 cases, the good agreement between EFEM results and modal decomposition results demonstrates that the energy governing equation derived in this equation approximates well the smooth time- and space-averaged energy density distribution of plates under thermal environments below critical buckling temperature.

To investigate the impact of temperature changing on the energy response of the plate, 10 temperature cases in Table 3 are studied for the above plate with the loss factor of 0.05 in both 11000 Hz and 15000 Hz.

Table 3. Temperature cases

<table>
<thead>
<tr>
<th>Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>t/℃</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
<td>45</td>
</tr>
</tbody>
</table>

As shown in Fig. 6, point A is chosen to observe the energy density variation versus temperature. And from both figures the energy densities decrease as the temperatures increase.

4. Conclusion

The energy density governing equation of plates under uniform thermal environments using in high frequency range is derived in this work. The thermal environments are considered that only
impact the thermal stress in the plate. And the far field energy density is utilized as the variable by being averaged in a wavelength and a period. On the base of these, the energy finite element method (EFEM) formulation has been developed. To verify the accuracy of derived equation, series of numerical examples are performed between EFEM and modal decomposition method. The results demonstrate that the energy density calculated by EFEM approximates well the time- and space-averaged energy density. In high frequency range, the energy density of the plate will varies in a small range with temperature changing below critical buckling temperature.

Many structures, like hypersonic, work under thermal environments, subjected to high-frequency exciting. The present study could be a foundation for researchers to do further research on the high-frequency vibration energy response of complex structures under thermal environments.

Acknowledgments

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REFERENCES