MATHEMATICAL ASPECTS ON THE STABILITY OF NONLINEAR DYNAMICAL SYSTEMS

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The nonlinear dynamical system is considered that depends of parameters because the possible geometrical, physical, chemical, biological, economical, parameters that can intervene in definition of the system, are considered no précised. The analysis of such nonlinear dynamical system is realized using the plane of two selected principal parameters, preferred in the literature, but also possible using of one or more free parameters and supplementary the linear dynamical system of “first approximation” attached to nonlinear dynamical system. Some theorems on the dependence of the nonlinear dynamical system stability by the “first approximation” linear dynamical system stability, or other conditions, are analyzed. The mathematical aspects discovered by us, on the property of separation, in the plane of principal parameters, between stable and unstable zones of the dynamical systems, the general case meted in the specialized literature on the concrete dynamical systems, without justifying mathematical motivations, are reminded. The conditions of the separation property transmission from the “first approximation” linear dynamical system to corresponding nonlinear system that facilitates the stability control of the nonlinear dynamical system are studied. A concrete nonlinear dynamical system is described and analyzed in this theory.

1. Introduction

The identifying of the mathematical properties of the nonlinear dynamical system mathematical model that assure the stability control possibilities of the nonlinear dynamical systems treated as nonlinear dynamical systems expressed in function of no précised geometrical, physical, …etc., parameters, is described below by identifying of the separation mathematical conditions between stable and unstable zones, performed using some selected free parameters (one, two or more).

The property of separation meted for all concrete linear or nonlinear dynamical system, modeled from the reality, means that in one stable point, which is situated in the inner of the stable zone of the nonlinear dynamical system, there is a neighborhood of the stable point where the dynamical system is also stable and analog property for unstable point of dynamical system situated in the inner of the unstable zone.

Our identified mathematical properties on the separation have been possible, in the first, using the discoveries from the specialized literature referred to assurance of any dynamical system stability. From this point of view, possible new results are welcome. Another domain of research, used in
our study, is from matrix theory. One matrix of parameters functions, can define the first approximation linear system attached to non linear dynamical system or any linear dynamical system mathematical model. The mathematical properties of the dynamical system matrix eigenvalues, dependent of the system parameters, can justify the property of separation for the dynamical system stable and unstable zones. The mathematical studies of Bauer (1958), Hausholder (1958), Rutishauser (1958), Francis (1961, 1962), Wilkinson (1958-1965), Parlet (1965) and of others researchers, in the matrix theory, referred to convergence of the concrete algorithms, initialized by a real matrix to a matrix that express the eigenvalues of the starting matrix, are used. The result from the mathematical analysis field referred to conditions of matrix convergent sequence terms continuity transmission to their limit is also used in our mathematical study.

2. Nonlinear dynamical systems stability comments

The dependence of the nonlinear dynamical system Liapunov stability by the first approximation attached linear system or other possibilities are discussed \(^{1-12}\). Some results are described for autonomous nonlinear system defined using the large or short equation expressions as below:

\[
\frac{dx_i}{dt} = f_i(x(t),...,x_n(t)), \quad i = 1,2,...,n
\]

\[
x(t) = (x_1(t), x_2(t),..., x_n(t))^T, \quad f(x) = (..., f_i(x(t),...,x_n(t)))^T,
\]

\[
\frac{dx}{dt} = f(x)
\]

The system (1) is supposed to respect the Cauchy conditions of solution existence and uniqueness and have null solution that can eventually assured by a transformation of the system (1). The expression of the system (1), using the Taylor developing of the functions \(f_i(x_1,...,x_n), \quad i = 1,...,n\), around origin, is described as below:

\[
\frac{dx_i}{dt} = a_{ij}x_j + ... + a_m x_n + F_i(x(t),...,x_n(t)), \quad a_{ij} = \frac{\partial f_i}{\partial x_j}(0,...,0),
\]

\[
A = (a_{ij}), \quad x = (x_1,...,x_n)^T, \quad F(x) = (...F_i(x(t),...,x_n(t)))^T, \quad i, j = 1,...,n
\]

\[
\frac{dx}{dt} = Ax + F(x)
\]

The system (2) without the polynomials \(F_i(x(t),...,x_n(t)), \quad i = 1,...,n\), of degree greater or equal with two in rapport with \(x\), represent the mathematical model of first approximation dynamical system attached to nonlinear dynamical system. The linear first approximation dynamical system, derived from (2) can be expressed by the compact notation:

\[
\frac{dx}{dt} = Ax
\]

The following two Liapunov theorems are formulated \(^4\):

**Theorem 1.**

The evolution of the non linear dynamical system (2) is asymptotic stable in origin if the real part of all roots for the characteristic polynomial equation \(|A - \lambda I| = 0\) is strict negative.

**Theorem 2.**

The evolution of the non linear dynamical system (2) is unstable in origin if the real part at least one roots of the characteristic polynomial equation \(|A - \lambda I| = 0\) is strict positive.

The below theorem is also true \(^2\):

**Theorem 3.**

Let a nonlinear dynamical system be of the form \(\frac{dx}{dt} = Ax + F(x)\), where the quadrature matrix \(A\) has the dimension \(n \times n\) and the continuous function \(F\) respect the property that for \(\gamma > 0\)
there is $\delta(\gamma) > 0$ such that if $|x| < \delta(\gamma)$ then $|F(x)| < \gamma|x|$. 

If all real parts $\alpha_i, i = 1,\ldots,n$ of the matrix $A$ eigenvalues have the property $\alpha_i \leq -2\alpha < 0$, $i = 1,\ldots,n$, then there is $\delta_0 > 0$, $\beta \geq 1$ such that for $|x_0| < \delta_0$ it follows the inequality:

$$\left|x(t; t_0, x_0)\right| \leq \beta e^{-\alpha(t-t_0)/2} |x_0|$$

(4)

**Observations:**

The property supposed to be respected by the function $F$ in the equation $dx/ dt = Ax + F(x)$ is respected in the case of nonlinear system $dx/ dt = f(x)$ in particular form expressed in the text of theorem 3, where the matrix $A$ is, in this case, the Jacobian matrix of the function $f(x)$ and define the linear dynamical system of first approximation attached to nonlinear system.

The theorem 3 emphasizes the asymptotic stability of the solution $x(t; t_0, x_0)$ in the neighborhood of origin, also stable solution of the nonlinear dynamical system, such that can ascertain the property of separation, around origin, between stable and unstable zones of the nonlinear dynamical system.

Two necessary and sufficient conditions on the real matrix $A$ terms to assure the negative real part of all eigenvalues of the matrix, property that intervene in stability analysis of linear dynamical systems or in stability analysis of linear first approximation dynamical system attached to autonomous nonlinear dynamical system are described in the following.

**Theorem 4** (Routh and Hurwitz).

Let the real matrix $A$ and the characteristic polynomial $|A-\lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n$ with real coefficients, $a_0 > 0$, and Hurwitz determinants:

$$
D_1 = a_1, D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ldots, D_n = \begin{vmatrix} a_1 & a_0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \ldots & a_n \end{vmatrix}
$$

(5)

A necessary and sufficient condition on the real matrix $A$ terms to assure the negative real part of all eigenvalues of the matrix is that $D_k > 0, k = 1,2,\ldots,n$.

**Theorem 5** (Lienard and Chipart).

Let the real matrix $A$ and the characteristic polynomial $|A-\lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n$ with real coefficients, $a_0 > 0$, and Hurwitz determinants from (5).

A necessary and sufficient condition on the real matrix $A$ terms to assure the negative real part of all eigenvalues of the matrix is expressed below:

$$a_i > 0, \quad i = 1,2,\ldots,n; \quad D_{n-1} > 0, \quad D_{n-3} > 0, \ldots$$

(6)

**Observation:**

The criterion 5 is faster than criterion 4 because the calculated determinants number is smaller than for criterion 4.

The comments from this paragraph permit to establish in some cases the stability or instability of nonlinear dynamical system using stability nature of linear first approximation dynamical system attached to nonlinear dynamical system. The separation conditions may be also identified.
3. QR algorithm for real Hessenberg form matrix

In this section we analyse for a real matrix of functions that depend of parameters, intervening in definition of linear dynamical system, the mathematical aspects that create the possibilities to transmission the property of linear dynamical system functions continuity to eigenvalues of the matrix.

Let matrix $A \in \mathbb{R}^{m \times n}$ be with elements $a_{ij}$, $i = 1, \ldots, n$; $j = 1, \ldots, n$. The matrix $A$ is named of Hessenberg form if the elements $a_{ij} = 0$ in the cases $2 < i \leq n$, $j < i - 1$. Any real matrix can be substituted by a similar matrix of Hessenberg form. For each column $r = 2, \ldots, n - 1$, of matrix $A$, to perform the Hessenberg form of the matrix, we can substitute the matrix $A$ with the similar matrix $M_r A M_r^{-1}$, using the below elementary matrices:

$$
M_r = \begin{bmatrix}
I_{(r-1)\times(r-1)} & 0_{(r-1)\times(n-r+1)} \\
0 & \ddots & 1 & 0 & \ddots & 0 \\
0 & \ddots & m_{r+1,r} & 1 & \ddots & 0 \\
0 & \ddots & m_{r+2,r} & 0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & m_{n,r} & 0 & \ddots & 1
\end{bmatrix},
M_r^{-1} = \begin{bmatrix}
I_{(r-1)\times(r-1)} & 0_{(r-1)\times(n-r+1)} \\
0 & \ddots & 1 & 0 & \ddots & 0 \\
0 & \ddots & -m_{r+1,r} & 1 & \ddots & 0 \\
0 & \ddots & -m_{r+2,r} & 0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & -m_{n,r} & 0 & \ddots & 1
\end{bmatrix}, \quad (7)
$$

The notations from (7) signify the expressions: $m_{k,r} = a_{k,r-1}/a_{r,r-1}$; $a_{r,r-1} \neq 0$, $r = 2, \ldots, n - 1$; $k = r + 1, \ldots, n$, the matrix $I_{(r-1)\times(r-1)}$ signify the unity matrix of order précised, the matrix $0_{(r-1)\times(n-r+1)}$ signify zero matrix also of order précised and matrix $M_r^{-1}$ represent inverse matrix of matrix $M_r$.

The QR algorithm is formulated in hypothesis that matrix $A$ has Hessenberg form to permit that the complex eigenvalues $\alpha \pm i \beta$, if there exists, to be represented in real final Schur form of the matrix $A$, deduced by QR algorithm convergence, using the real matrix of two order

$$
\begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix}
$$
on the diagonal. The algorithm QR is described below:

$$
A_0 = A, A_k - \mu_k I = Q_k R_k, \; k \geq 0, A_k x \neq \mu_k x, \; \forall x \in \mathbb{R}^{n \times 1}, \; x \neq 0
\quad (8)
$$

The matrices $Q_k$ are orthogonal and matrices $R_k$ upper triangular, invertible by a corresponding selection of parameter $\mu_k \neq 0$. The matrices $A_k, A_{k+1}, k \geq 0$, are also of Hessenberg form and similar, the last property deduced from the relations below:

$$
A_0 = A, A_k - \mu_k I = Q_k R_k, \; k \geq 0 \Rightarrow R_k = Q_k^T (A_k - \mu_k I) \\
A_{k+1} = R_k Q_k + \mu_k I = Q_k^T (A_k - \mu_k I) Q_k + \mu_k I = Q_k^T A_k Q_k, \; k \rightarrow \infty. \quad (9)
$$

The convergence study of QR algorithm or other algorithm, with real matrix $A$ in Hessenberg form or in arbitrary other form, can be seen analysed in the literature13-17.

4. Mathematical conditions to stability zones separation

The mathematical model described by Eq. (3), for the linear dynamical systems that approximate the real phenomenon, supposes that the system parameters dependent functions, which intervene in the definition of the matrix $A$ components, are continue or continue on piecewise.
The construction of matrices $A_i$ sequence, by the QR algorithm, assures the continuity on piecewise of their components. The sequence $A_i$ is convergent to the limit matrix, possible also continue, and implies, in this last case, the continuity of the matrix $A$ eigenvalues.

The following theorem on the convergence of the continue functions sequence is mentioned.

**Theorem 6.**

Let $\{f_n\}_{n \in \mathbb{N}}$ be a functions sequence, convergent uniform on the set $E \subset \mathbb{R}$, with the limit function $f$. If functions $f_n$ are continue in the point $a \in E$ then limit function $f$ is also continue in same point $a \in E$.

This classical property, formulated in one dimension can be formulated also in more dimensions.

Let $f: E \to \mathbb{R}$ be a function defined for the set $E \subset \mathbb{R}$. The following theorem, formulated here for one dimension, will be used in our considerations:

**Theorem 7.**

Let the function $f: E \to \mathbb{R}$ be continuous in the point $x_0 \in E$ and $f(x_0)$ respecting the inequalities $\alpha < f(x_0) < \beta$; $\alpha, \beta \in \mathbb{R}$, then there is a neighborhood $V(x_0)$ such that $\alpha < f(x) < \beta$ for all $x$ in the set $V(x_0) \cap E$.

**Observation:**

Theorem 4 assures that the continuous function $f$ in the point $x_0 \in E$, $x_0 \neq 0$ conserve in the point $x_0$ neighborhood the sign from $x_0$.

The properties described in this paper, up to here, permit to write the following theorem:

**Theorem 8.**

Let the dynamical system, defined by the Eq. (3), be with the components of the matrix $A$, continue functions or continue on piecewise, referred to system parameters, and QR algorithm assuring the matrix $A$ eigenvalues functions continuity or continuity on piecewise, then these conditions determine the separation between stable and unstable zones of the dynamical system, in rapport with free parameters selected.

**Observation:**

By separation between the stable and unstable zones we understand that the dynamical system preserves the quality of stability or of instability in a neighborhood of free parameters fixed values situated in the inner of stable or unstable zone.

The conditions of separations formulated above for linear dynamical system may correspond to one linear systems of first approximation and transmissibility to corresponding nonlinear system may be identified using the adequate theorem.

### 5. Analysis of concrete nonlinear dynamical system

The concrete dynamical system consists from the aircraft with your mass concentrated in the centroid of the aircraft and with supposed evolution in one vertical plane. The surface of the earth is supposed a horizontal plane. The mathematical model is described below (Fig. 1):

$$ m(d\vec{v}/dt)=\vec{T}+\vec{G}+\vec{R}_a $$  \hspace{1cm} (10)

The notations from (10) signify as follows: $m$, the mass of aircraft; $\vec{v}$, speed of the aircraft; $\vec{T}$, traction force (propulsion); $\vec{G}$, the weight of aircraft where the gravity force ($g$) is supposed constant with altitude and $\vec{R}_a$, resultant of aerodynamically forces.
The resultant force $\vec{R}_a$ is compounded from two forces, a force $\vec{R}$ (collinear and opposite to speed $\vec{v}$) named aerodynamic drag and a lifting force $\vec{P}$.

![Figure 1. Physical model of aircraft](image)

The unit vectors of $\vec{v}$ and $\vec{P}$ are denoted respectively $\vec{v}_v$ and $\vec{v}_p$. We can write:

$$\frac{dv}{dt} = \frac{dv}{dt} \vec{v}_v + v \frac{d\theta}{dt} \vec{v}_p; \quad \vec{T} = T(\cos \varepsilon \vec{v}_v + \sin \varepsilon \vec{v}_p)$$

$$\vec{G} = -mg(\sin \theta \vec{v}_v + \cos \theta \vec{v}_p); \quad \vec{R} = -R \vec{v}_v; \quad \vec{P} = P \vec{v}_p$$

The equation (10), in hypothesis $\varepsilon = 0$, projected on the directions $\vec{v}_v$ and $\vec{v}_p$, gives:

$$\frac{dv}{dt} = T/m - R/m - g \sin \theta$$

$$\frac{d\theta}{dt} = -(g \cos \theta)/v + P/m/v$$

The relations $\frac{dx_1}{dt} = v \cos \theta$, $\frac{dx_2}{dt} = v \sin \theta$ attached to relations (12) define the state vector of the system. The analysis of the equation (12) will be simplified denoting $T/m$ by $T$, $R/m$ by $R$ and $P/m$ by $P$.

For the lifting force and aerodynamic drag the expressions are considered:

$$P = k_0 v^2; \quad R = (k_1 v^2 + k_2 / v^2)/2, \quad k_0, k_1, k_2 > 0. \quad (13)$$

The system (12) for $T = R$ admits the particular solution $v = v_0$, $\theta = 0$ if $k_0 v_0^2 = g$. We select a perturbation for the system (12) of the form $v = v_0 + x_1$, $\theta = x_2$ and substitute in (12), using also the relations $T = (k_1 v^2 + k_2 / v^2)/2$, $g = k_0 v_0^2$ and developing in series. The following system of equations is obtained:

$$\frac{dx_1}{dt} = (-k_1 v_0 + k_2 / v_0^2)x_1 - g x_2 + ...$$

$$\frac{dx_2}{dt} = (k_0 + g / v_0^2)x_1 + ... \quad (14)$$

The characteristic equation of first approximation attached to system (14) is of the form:

$$\lambda^2 + (k_1 v_0 - k_2 / v_0^2) \lambda + g(k_0 + g / v_0^2) = 0. \quad (15)$$

The Hurwitz criterion permits to affirm, in the case $v_0^4 > k_2 / k_1$, that the linear system of first approximation is asymptotic stable and by Liapunov criterion that the afferent nonlinear system is also asymptotic stable in origin with analogue observations referred to separation of stable zones.
6. Conclusions

The necessary or sufficient mathematical conditions to constat the punctual stability or separation of stable zones for linear dynamical systems, in particular linear systems of “first approximation”, attached to nonlinear systems, are reminded. The conditions are analysed for the matrix that defines the linear mathematical model, with components supposed function continue or continue on piecewise. The QR-algorithm for matrix in Hessenberg form that starts with a real matrix mentioned and convergent to the matrix that describes the eigenvalues of the initial matrix is studied. Theorems on punctual stability or on separation conditions of the stable zones are formulated for some linear or nonlinear cases. A stability of concrete nonlinear dynamical system, from the flight theory, using the dynamical system of first approximation, is analysed in the paper. An interesting large investigation direction in all implied mathematical domains to assure the separation of stable zones of dynamical system that permits to search optimised values of parameters in a stable zone, is opened.

REFERENCES