# Stability Control of Linear and Nonlinear Dynamic Systems


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(Received 5 November 2014, accepted: 11 March 2016)



The behavior of linear or nonlinear dynamic systems depends on different parameters (identifiable or free) that are involved in their definition. The stability analysis of such dynamical systems is realized by using a domain of selected free parameters. In this paper, we discuss specific theorems that concern the stability of linear dynamical systems, the stability of nonlinear dynamical systems in terms of "first linear approximations", and other stability criteria. We study the stable/unstable separation property in the free parameters domain and present a rigorous mathematical justification of this property with specific examples from various branches of science. Furthermore, we investigate specific conditions when the separation property is passed on to the nonlinear dynamical system from its first order linear approximation. The stable/unstable separation property is also emphasized as an important property of the environment that can contribute to its mathematical modeling.


## 1. INTRODUCTION

In this paper, we analyzed the multiple aspects of the stability control of linear or nonlinear dynamical systems ensured by the property of separation between stable and unstable regions of the free parameters domain.

Numerous authors have studied the problems of dynamic systems stability. We have surveyed some of the relevant literature here.[1–8, 11–13]

Any dynamical system can be considered in terms of its defining parameters without fixing their values as geometrical parameters, physical parameters (in particular mechanical parameters), and possible economical or biological parameters.

Another important idea is that many real-life dynamical systems are considered in the literature (e.g. the Mathieu equation, the Hill equation, the harmonic vibration equation, etc.) and have the property of separation between the stable and unstable region in a selected domain of free parameters. The stable and unstable regions are separated by a boundary in the domain of the free parameters. The property of separation can be described by the fact that the stable and unstable regions, except the points on the boundary, are open sets. This separation aspect, which is considered in this paper, creates the freedom of stability control on a neighborhood of fixed stable point in the open stable region of the dynamical system.

We discovered some mathematical conditions of the stability regions existence for dynamical systems using various results from matrix theory, real analysis, stability theory, and others.

The property of separation of stability regions is an important property of the environment, as one refers to a specific dynamical system locally implemented in the environment.

Our study has not exhausted the subject of stability control. New results in matrix theory, in the linear or nonlinear dynamical system stability theory, and in real analysis will provide further direction.

## 2. ON THE CONTINUITY OF THE REAL MATRIX EIGENVALUES

The real matrix in the discussion was the matrix that defined the linear dynamical system, or the "first approximation", of the nonlinear dynamical system depending on some parameters. The components of the real matrix were assumed to be continuous or piecewise continuous functions of the system parameters (time could also be considered as a parameter).

The dependence of the spectrum of this matrix on the matrix components properties is discussed in the following paragraphs.

### 2.1. QR Algorithm for Hessenberg Form of the Real Matrix

In what follows, we assumed that the $n \times n$ matrix had distinct eigenvalues. The QR algorithm was formulated for the matrices of Hessenberg form, meaning that its entries satisfy $a_{ij} = 0$ for $2 < i \leq n, j < i - 1$.

We then defined the Schur form of the matrix $A$. Let $\lambda$ be a real eigenvalue and $x \in R^{n\times 1}$ the corresponding real eigenvector of the $n \times n$ matrix $A$ so that $Ax = \lambda x$, $x \neq 0$. (We assume $n > 2$.) Let $Q = [x, Y]$, $x \in R^{n\times 1}$, $Y \in R^{n\times(n-1)}$ be an orthogonal base of vectors in $R^n$ that include the eigenvector $x \in R^{n\times 1}$ so that $QQ^T = I_n$. Then $A = Q \begin{bmatrix} \lambda & (x^T AY) \\ 0 & B \end{bmatrix} Q^T$ where, $x^T AY \in R^{1\times(n-1)}$, $0 \in R^{(n-1)\times 1}$ and $B = Y^T AY \in R^{(n-1)\times(n-1)}$. When $B$ (and implicitly $A$) had a pair of complex conjugate eigenvalues $\alpha \pm i\beta$ with associated eigenvectors $u \pm iv$ where $u, v \in R^{(n-1)\times 1}$ were linearly independent real vectors, we can write:

$$X^* = [uv], M = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}; BX^* = X^* M. \qquad (1)$$

Let $Q^* = [X^*, Y^*]$ be an orthonormal basis of $R^{n-1}$ where $X^* = [u, v] \in R^{(n-1)\times 2}$, $Y^* \in R^{(n-1)\times(n-3)}$ and $Q^* Q^{*T} = I_{n-1}$. The matrix $A$ with $\lambda$ and $\alpha \pm i\beta$ eigenvalues had the form:

$$A = Q \begin{bmatrix} \lambda & (x^T AY) \\ 0 & B \end{bmatrix} Q^T,$$
$$B = Q^* \begin{bmatrix} M & (X^{*T} BY^*) \\ 0 & (Y^{*T} BY^*) \end{bmatrix} Q^{*T}. \qquad (2)$$

Thus the matrix $A$ with (possibly complex) distinct eigenvalues, was similar to the associated Schur form (which is a matrix of Hessenberg type).

The QR algorithm, by the Wilkinson's manner, is described below, where the initial real matrix $A$ was denoted $A_1$ in the algorithm, and where the convergence symbol "$\to$" was used.[9]

$$\begin{aligned} A_s = Q_s R_s, A_{s+1} = R_s Q_s, s = 1, 2, \ldots; s \to \infty. \\ A_s = R_{s-1} Q_{s-1}; A_{s-1} = Q_{s-1} R_{s-1}, s = 2, 3, \ldots; \\ R_{s-1} = Q_{s-1}^{-1} A_{s-1}; A_s = Q_{s-1}^{-1} A_{s-1} Q_{s-1}, s = 2, 3, \ldots; \\ A_s = Q_{s-1}^{-1} \ldots Q_1^{-1} A_1 Q_1 \ldots Q_{s-1} = \\ = (Q_1 \ldots Q_{s-1})^{-1} A_1 Q_1 \ldots Q_{s-1}, s = 2, 3, \ldots; \\ Q_1 \ldots Q_{s-1} A_s = A_1 Q_1 \ldots Q_{s-1}, s = 2, 3, \ldots; \\ Q_1 \ldots Q_{s-1} Q_s R_s = A_1 Q_1 \ldots Q_{s-1}, s = 2, 3, \ldots; \\ Q_1 \ldots Q_{s-1} Q_s R_s R_{s-1} \ldots R_1 = \\ = A_1 Q_1 \ldots Q_{s-1} R_{s-1} \ldots R_1, s = 2, 3, \ldots; \\ Q_1 \ldots Q_{s-1} Q_s R_s R_{s-1} \ldots R_1 = A_1^s, s = 1, 2, \ldots \end{aligned} \qquad (3)$$

The matrices $Q_k, k = 1, 2, \ldots$ were orthogonal and the matrices $R_k$ were upper triangular and invertible. The matrices $A_k, A_{k+1}, k = 1, 2, \ldots$ were also of Hessenberg form and similar.

Parlet described the convergence of QR algorithm to the Schur form of the matrix $A$, where the real matrix $A$ was considered in the Hessenberg form.[10]

The above study was performed under the hypothesis that all the eigenvalues of the real matrix were distinct. For the case of real matrix multiple eigenvalues, we used the results from the matrix theory. Hirsch, Smale and Devaney demonstrated, on the normed space of dimension $n$ matrices set $L(R^n)$, the following theorem.[1]

Theorem 1. *The set of matrices with distinct eigenvalues from linear normed space $L(R^n)$ is an open and dense set in the linear space $L(R^n)$.*

The above theorem created the possibilities to justify the transmission of some properties from the real matrices set with distinct eigenvalues to the real matrices set including multiple eigenvalues that could intervene in the stability analysis of linear (can be of "first approximation") dynamical systems.

## 2.2. Transmissibility of the Continuity from the Matrix Elements to the Eigenvalues

The components of the real matrix $A$ that defined the linear dynamical system depending on parameters were assumed to be piecewise continuous in the free parameters. We formulated the following theorem on the transmissibility of the continuity below:

Theorem 2. *If the components of the matrix A are piecewise continuous relative to the free parameters and the sequence of Hessenberg matrices $A_s, s = 1, 2, \ldots$ from the QR algorithm, starting at $A$, is uniform convergent to the Schur form of the matrix $A$, then the eigenvalues of the matrix $A$ are piecewise continuous.*

This theorem was verified using the property from the real analysis that the uniform convergence of continuous functions implied the continuity on the function limit. When the eigenvalue was on the boundary, this eigenvalue had a null real part and the stability character of the point (stable or unstable) was unknown.

The above property was considered in our study using the following property of continuous functions, formulated here for one variable function.

Theorem 3. *Let $f : E \to R$, $E \subset R$ be a continuous function in the inner point $x_0 \in E$ such that $\alpha < f(x_0) < \beta; \alpha, \beta \in R$. Then there exists a neighborhood of the point $x_0 \in E$, where the function values respect the same inequalities.*

Remark: *Theorem 3 ensured that the function $f$ was continuous in the point $x_0 \in E$ and was preserved in the neighborhood of $x_0$, the function sign in $x_0$.*

The mathematical conditions that ensured the separation between stable and unstable regions for the linear dynamical system were described beginning with the following property:

Theorem 4. *Let the linear dynamical system be defined by the differential equation $dydt = Ay(t)$, $y(t) = (y_1(t), \ldots, y_n(t))^T, A = (a_{ij}), i = 1, \ldots, n; j = 1, \ldots, n$, where the symbol $T$ signifies the transposition of the matrix and the values $a_{ij}$ are assumed to be constants. If the real part* *of all eigenvalues of the matrix $A$ is strictly negative, then the solution of the differential equation is asymptotic stable in origin. If the real part of at least one eigenvalue of the matrix $A$ is strictly positive, then the solution of the differential equation is unstable in origin.*

*If the real parts of the matrix $A$, eigenvalues are strictly negative, with the exception of at least one eigenvalue that has null real part, then the stability of the dynamical system in origin is unknown (possible stable or unstable).*

## 3. ON THE SEPARATION OF THE DYNAMICAL SYSTEM STABLE REGIONS

The possible structure of the stable and unstable points of the dynamical system free parameters domain is described by the following cases:

- If the dynamical system was stable in one point of the domain of free parameters and was not on the frontier between stable and unstable regions, then there existed a neighborhood around this point where the dynamical system was also stable in each point of the neighborhood and an analogue property for unstable point.
- The stable or unstable point could be situated on the frontier between stable and unstable regions in the free parameters domain. Any neighborhood of such point was composed of stable and unstable points.

In the case of distinct eigenvalues of the real matrix $A$, to the linear dynamical system was attached to a theorem of separation between stable and unstable regions in the free parameters domain of linear dynamical system, which is formulated below. This theorem was a consequence of the above theorems 2, 3, and 4.

Theorem 5 (Separation theorem). *If the linear dynamic system is defined by the real matrix $A$, which has Hessenberg form and whose entries are piecewise continuous functions of free parameters, and the convergent QR algorithm ensures that the real part of eigenvalue functions of the matrix A are also piecewise continuous, then these conditions determine the separation between stable and unstable regions of the dynamical system in the domain of free parameters.*

Remark: We comment on the possibility of substituting in practice the infinite QR algorithm by a finite one that would simplify the application of the Separation Theorem 5 to the specific examples later on.

### 3.1. The Separation Studied by Nonlinear System "First Approximation"

The stability study for non-null solution of the nonlinear equation $dy/dt = h(t, y)$, $y \neq 0$, could be similar to one corresponding to the null solution. This was the aspect for which only the stability for the null solution of nonlinear dynamical system is analyzed. Another assumption was that the equation of the dynamical system was in the autonomous form $dx/dt = f(x)$. Many particular dynamical systems were in the autonomous form.

The function $f(x)$ was supposed to depend on the variable $x = (x_1, \ldots, x_n)^T$ and its components denoted $f(x) = (f_1(x), \ldots, f_n(x))$. Its components were assumed to have the following Taylor expansion near the origin:

$$\begin{aligned} f_i(x) = f_i(0) + \sum_{j=1}^{n} \left(\partial f_i(x)/\partial x_j\right)|_{x=0} x_j + \\ + \sum_{j=1}^{n}\sum_{k=1}^{n} \left(\partial^2 f_i(x)/\partial x_j \partial x_k\right)|_{x=0} x_j x_k + \\ + \ldots, i = 1, \ldots, n \end{aligned} \quad (4)$$

The above assumptions were permited to consider $f_i(0) = 0, i = 1, \ldots, n$ and used the notations for derivatives of the first order $a_i j = \partial f_i(x)/\partial x_j|_{x=0}$; $i, j = 1, \ldots, n$ we could formulate the equation:

$$dx/dt = [a_{ij}]x + g(x); i, j = 1, \ldots, n. \quad (5)$$

The following theorems were due to Liapunov:

Theorem 6. *The evolution of nonlinear dynamical system in Eq.* (5) *is asymptotic stable in origin if the real parts of all eigenvalues of the matrix $A = [a_{ij}], i, j = 1, \ldots, n$ are strictly negative.*

Theorem 7. *The evolution of the nonlinear dynamical system in Eq.* (5) *is unstable in origin if the real part of at least one eigenvalue of the matrix $A = [a_{ij}], i, j = 1, \ldots, n$ is strictly positive.*

### 3.2. The Separation Studied on Nonlinear System by Indirect Method

The indirect method of stability analysis consists in using the differential equation solution that describes the evolution of the dynamical system.

We again considered the equation $dx/dt = f(x)$, with the solution $x(t) \equiv 0$, $x = (x_1, \ldots, x_n)^T$, and the assumption that the functions $f_i(x), i = 1, \ldots, n$, may be developed into a series around the origin so that the above equation could be expressed in the form in Eq. (5), where it was supposed that the function $x(t)$ was at least $C^2$ class so that the function $g(x) = dx/dt - Ax$ was at least $C^1$ class.

Because matrix $A$ is a Jacobian matrix in origin $x(t) \equiv 0$ of the function $f(x)$, then $g(x)$ had the property, so that for each $\gamma > 0$, there was $\delta(\gamma) > 0$, such that if $|x| < \delta(\gamma)$, then $|g(x)| < \gamma|x|$. This property meant that $g(x)$, which corresponded to "higher order terms" in the series, was developed around the origin and became negligible; at was reported to linear order terms for a sufficiently small $x$.

A theorem that underlines the property of separation in the free parameters domain of nonlinear dynamical systems, using the indirect method is stated below.[2]

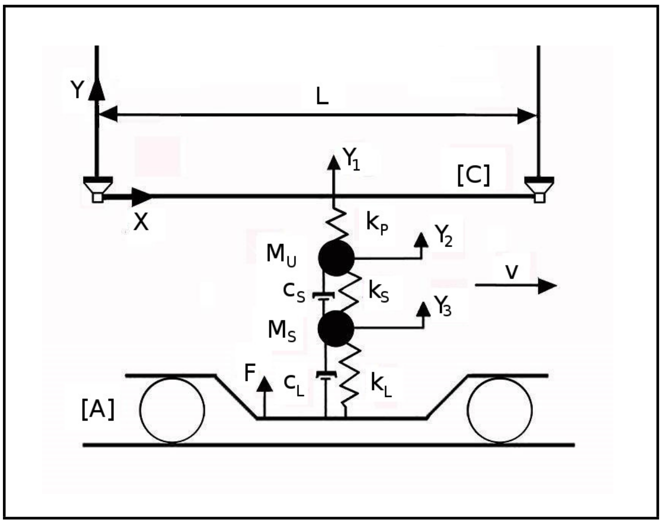


**Figure 1.** Physical model.

Theorem 8. *Consider the dynamical system defined by Eq. (5), where A is a constant $n \times n$ matrix with real entries, the variable $x = (x_1, \ldots, x_n)^T$, the function $x(t) \equiv 0$ is a solution of the equation, and the function $g(x)$ is supposed to be continuous. Furthermore, the property that for each $\gamma > 0$, there is $\delta(\gamma) > 0$, such that if $|x| < \delta(\gamma)$, then $|g(x)| < \gamma|x|$. We also assumed that all the eigenvalues of A have negative real parts, such that Real $\lambda_i \leq -2\alpha < 0$, $i = 1, \ldots, n$.*

Then there exists $\delta_0 > 0, \beta \geq 1$, such that for any $|x_0| < \delta_0$:

$$|x(t; t_0, x_0)| \leq \beta e^{-\alpha(t-t_0)/2}|x_0|, t \geq t_0. \quad (6)$$

In conclusion, if the conditions of theorem 8 are verified, then the origin stability ensures the stability in a whole neighborhood of the origin which in turn implies the separation of the stable regions.

Next, we give a concrete example of a dynamical system with the property of the separation between stable and unstable regions. We also mentioned the stability study of other particular dynamical systems in the references.[11–13]

## 4. STABILITY ANALYSIS OF PANTOGRAPH-CATENARY DYNAMICAL SYSTEM

In this application, we analyzed the property of separation of the stable and unstable regions in the plane of principal parameters for a particular dynamical system depending on parameters, namely the pantograph, which is a catenary dynamical system.[4] An analysis of the stability was performed here on the mathematical model attached to the physical model, as described in Fig. 1, of the electrical locomotive "pantograph - contact wire" dynamical system.

The defined physical model came from a vehicle $[A]$ in a uniform linear motion, which compressed with a force $F$, an oscillating system composed of two sprung concentrated masses $M_u$ and $M_s$ on the wire $[C]$, where $y(x,t)$ was the deflection of the wire, $EI$ was the bending stiffness of the wire for each specified $x$ and $t$ values, $T$ was the horizontal tension in the wire, $\beta$ was the viscous damping of the wire, $m$ was the mass per unit length of the wire, $c_s$ and $c_L$ were the damping coefficients, $k_s$, $k_L$ and $k_p$ were the stiffness elements of the system. The values $y_1$, $y_2$, $y_3$ were respectively, the deflection of the wire compressed by the oscillating system in the contact point, and the deflections from the equilibrium position of the masses $M_u$ and $M_s$. The oscillating system was moving with a constant speed v at the same time with the vehicle $[A]$.

The transformed system of equations were deduced through the transformed parameters using dimensionless parameters of the system:

$$\begin{aligned}
&(1-\mu)\ddot{\tilde{y}}_3 + 2(\varsigma_s\tilde{\omega}_{sn} + \varsigma_L\tilde{\omega}_{nL})\dot{\tilde{y}}_3 - \\
&-2\varsigma_s\tilde{\omega}_{sn}\dot{\tilde{y}}_2 + \left(\tilde{\omega}_{sn}^2 + \tilde{\omega}_{nL}^2\right)\tilde{y}_3 - \tilde{\omega}_{sn}^2\tilde{y}_2 = 0. \\
&\mu\ddot{\tilde{y}}_2 + 2\varsigma_s\tilde{\omega}_{sn}\dot{\tilde{y}}_2 - 2\varsigma_s\tilde{\omega}_{sn}\dot{\tilde{y}}_3 + \tilde{\omega}_{sn}^2\tilde{y}_2 - \tilde{\omega}_{sn}^2\tilde{y}_3 + \\
&+2\varsigma_L\tilde{\omega}_{nL}\dot{\tilde{y}}_3 + \tilde{\Omega}_n^2\left(\tilde{y}_2 - \sum_{j=1}^{k} Tj(\tau)\sin j\tau\right) = 0. \\
&\frac{1}{2\tilde{M}\sin\tau}\left(\frac{d^2T_j}{d\tau^2} + \frac{1.}{\tilde{v}_\beta}\frac{dT_j}{d\tau} + \left(\frac{j^4}{\tilde{v}_{EI}^2} + \frac{j^2}{\tilde{v}_T^2}\right)T_j\right) - \\
&-\tilde{\Omega}_n^2\left(\tilde{y}_2 - \sum_{j=1}^{k} Tj(\tau)\sin j\tau\right) = 0, j = 1, \ldots, k. \quad (7)
\end{aligned}$$

For more detailed data one, the references.[4]

The matrix of the system in Eq. (7), denoted $A$, in the case $k = 1$, had the unknown vector components:

$$\begin{aligned}
&X_1(\tau) = \mu\tilde{y}_2, X_2(\tau) = \mu\dot{\tilde{y}}_2, X_3(\tau) = (1-\mu)\tilde{y}_3, \\
&X_4(\tau) = (1-\mu)\dot{\tilde{y}}_3, X_5(\tau) = \frac{1}{2\tilde{M}\sin\tau}T_1(\tau), \\
&X_6(\tau) = \frac{1}{2\tilde{M}\sin\tau}\dot{T}_1(\tau) \quad (8)
\end{aligned}$$

We did not specify the components of the matrix $A$ from the equation $dX/d\tau = AX$ of our dynamical system model, where the vector $X$ was defined by its components from Eq. (8).

The stability of the dynamical system was studied in the following case of fixed parameters:

$$\begin{aligned}
&\tilde{\Omega}_n = 3.185, \tilde{v}_\beta = 19.8, \tilde{\omega}_{nL} = 0.48, \\
&\tilde{v}_{EI} = 90.96, \varsigma_s = 0.3, \varsigma_L = 0.3. \quad (9)
\end{aligned}$$

The chosen free parameters were the transformed variables $\tilde{\omega}_{ns}$ and $\tilde{v}_T$ corresponded to the free dimensional parameters of the dynamical system, respectively, the stiffness parameter $k_s$ of the system and the horizontal tension $T$ in the wire.

We analyzed the stability of motion for the displacement $\tilde{y}_2$ of the concentrated mass $M_u$ in the specified free parameters domain of interest.

The frontier curve of stable and unstable separation regions of the displacement $\tilde{y}_2$ was plotted with a continuous line, as seen in Fig. 2. It was done for two chosen parameters as defined by the variables $\tilde{\omega}_{ns}$ and $\tilde{v}_T$ for a selected domain by using an algorithm elaborated by the authors, which is explained below and in some previous papers.

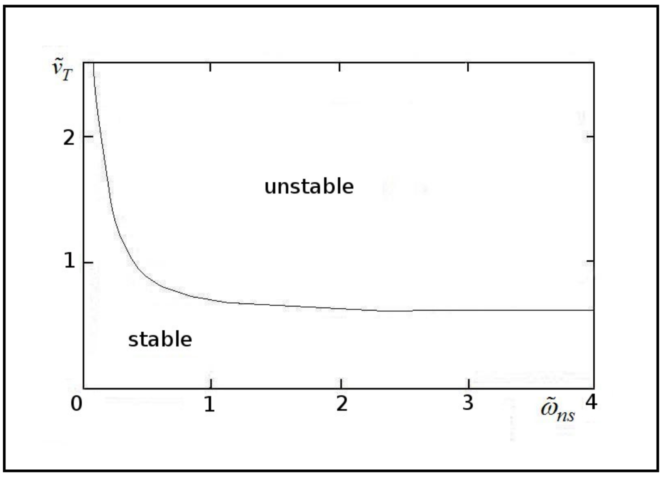


**Figure 2.** Separation of stable zones.

Basically, the procedure to identify the frontier between the stable and unstable regions is ensuring that the domain was covered with a sufficient fine mesh and analyzed the stability for each specified pair $\tilde{\omega}_{ns}$ and $\tilde{v}_T$. A refined mesh in the identification algorithm could be deduced by the bisection of the interval between two neighboring points of the mesh for each free parameter and the system solution successive values were compared in order to decide the accepted value. The property of separation could be justified using a finite QR algorithm in our separation theorem 5 for the system $dX/d\tau = AX$ attached to physical model from Fig. 1.

Again, we specified that the property of separation described above permitted a selection of the free parameters, in a fixed neighborhood of the stable region inner point in order to respect one compatible optimized criterion.

## 5. CONCLUSIONS

A mathematical analysis of the possible structure of the stable and unstable points of the free parameters domain is performed in this paper. Some mathematical conditions that ensure the separation between stable and unstable regions of the free parameters of the linear or nonlinear dynamical system are set off. The property of separation described here is also encountered in many defined dynamical systems from the literature, without mathematical justification, which is important because it ensures the possibility that the evolution, stability, and control of a dynamical system may be optimized using the compatible criterion in the stability regions. A defined dynamical system that has the property of separation between stable and unstable regions is described in this paper. We acknowledge that our study has not exhausted the problem of dynamical systems stability and control. However, an interesting domain of scientific research has been opened.

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