Vibrations of Completely Free Rounded Regular Polygonal Plates

C. Y. Wang

Departments of Mathematics and Mechanical Engineering Michigan State University, East Lansing, MI 48824

(Received 28 February 2013; accepted 5 March 2014)

The vibrations of completely free polygonal and rounded polygonal plates are important for large floating or space platforms. The problem is solved by an improved Ritz method on a class of homotopy shapes. The first five frequencies are determined, and interesting evolutions of mode shapes are shown.

1. INTRODUCTION

Vibration of elastic plates is essential in structural mechanics. Basic data (frequencies, mode shapes) of plate vibrations can be found in the works of Leissa[1] and Blevins.[2] Exact solutions exist for the vibration of circular and annular plates, some simply supported triangular plates, and rectangular plates with two simply supported opposite edges.[3] For all other shapes or boundary conditions numerical means are necessary.

The vibrations of regular polygonal plates have also been studied, notably Conway[4] using point match, Shahady et al.[5] using conformal mapping, Irie et al.[6] using membrane analogy, Liew and Lam[7] using a Ritz method, and Ghazi et al.[8] using finite elements. These sources however, only consider simply supported or clamped edges.

We are interested in the vibration of completely free plates, i.e. all edges are free. The study is important in the design of very large ocean floating structures (e.g.[9]) and also large structures in space, such as platforms and solar panels (e.g.[10]). These structures have nominal dimensions in kilometres, much larger than their thicknesses, thus can be modelled as thin plates.

Aside from the data presented by Leissa,[1] the vibrations of completely free rectangular plates includes Leissa[11] using the Ritz method, Gorman[12] using a superposition method, Behnke and Mertins[13] using the Ritz method, and Mochida and Ilanko[14] using superposition and finite differences. The free triangular plate was considered by Leissa and Jaber[15] and the free trapezoidal plate by Qatu et al.,[16] both sources using the Ritz method. However, the natural vibration of the free regular polygonal plate has not been fully investigated.

The purpose of the present note is to study not only the vibrations of completely free regular polygonal plates, but also the more general class of rounded regular polygonal plates, which includes the polygonal plates as special cases. Rounding the corners of a polygonal plate has definite advantages in terms of savings in material, weight, and boundary length, while it changes little in the structural strength or vibration properties.

2. THE HOMOTOPY SHAPES

The homotopy shapes are first introduced by Wang[17] and applied to the vibration of membranes. The homotopy transform

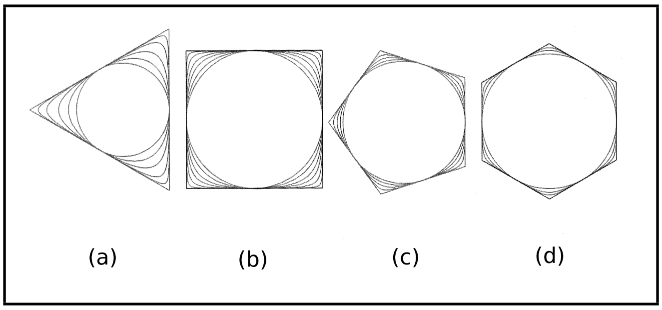

Figure 1. (a) Rounded triangular plate. From outside, $\alpha = 0$, 0.01, 0.05, 0.2, 0.5, 1. (b) Rounded square plate. From outside, $\alpha = 0$, 0.01, 0.05, 0.15, 0.3, 0.6, 1. (c) Rounded pentagonal plate. From outside, $\alpha = 0$, 0.05, 0.2, 0.5, 1. (d) Rounded hexagonal plate. From outside, $\alpha = 0$, 0.05, 0.2, 1.

is briefly described as follows: Let all lengths be normalized by the radius of the inscribing circle of the polygon; for a rounded equilateral triangular plate we set

$$H = \alpha(1 - x^2 - y^2) + (1 - \alpha)(1 - x)\left[\left(1 + \frac{x}{2}\right)^2 - \frac{3}{4}y^2\right] = 0; \qquad (1)$$

where $\alpha = 0$ is an equilateral triangle of edge length $2\sqrt{3}$, and $\alpha = 1$ is a circle of radius one. The homotopy, as α is increased from 0 to 1, gives a family of rounded triangles shown in Fig. 1(a). The maximum distance from the centroid is found to be

$$l = \frac{2}{1 + \sqrt{\alpha}}. \qquad (2)$$

The degree of rounding is represented by the distance d to the original corner

$$d = 2 - l. \qquad (3)$$

For a rounded square plate (Fig. 1(b)) we set

$$H = \alpha(1 - x^2 - y^2) + (1 - \alpha)(1 - x^2)(1 - y^2). \qquad (4)$$

The maximum distance and degree of rounding are

$$l = \sqrt{\frac{2}{1 + \sqrt{\alpha}}}, d = \sqrt{2} - l. \qquad (5)$$

The rounded regular pentagonal plate is shown in Fig. 1(c).

The boundary is given by

$$H = \alpha(1 - x^2 - y^2) + (1 - \alpha)(1 - x) \left[\left(1 + \frac{y_2 - y_1}{y_1 x_2 - y_2} x\right)^2 - \left(\frac{x_2 - 1}{y_1 x_2 - y_2} y\right)^2\right] \times \left[\left(1 - \frac{x}{x_3}\right)^2 - \left(\frac{x_2 - x_3}{y_2 x_3} y\right)^2\right]. \quad (6)$$

Here

$$y_1 = \tan(\pi/5), x_2 = -\sec(\pi/5)\cos(2\pi/5), \quad y_2 = \sec(\pi/5)\sin(2\pi/5), x_3 = -\sec(\pi/5). \quad (7)$$

There is no closed form solution for the maximum distance l. For each α we set $y = 0$ in $H = 0$. The smallest root is $x = -l$. Then

$$d = \sec(\pi/5) - l. \quad (8)$$

Fig. 1(d) shows the rounded regular hexagonal plate given by

$$H = \alpha(1 - x^2 - y^2) + (1 - \alpha)(1 - x^2)\cdot \left[\left(1 + \frac{x}{2}\right)^2 - \frac{3}{4}y^2\right]\left[\left(1 - \frac{x}{2}\right)^2 - \frac{3}{4}y^2\right]. \quad (9)$$

We find

$$l = \frac{1}{3}\sqrt{\frac{12 - 4\alpha - 4\sqrt{\alpha(3+\alpha)}}{1 - \alpha}}, \quad d = \frac{2}{\sqrt{3}} - l. \quad (10)$$

Rounded regular polygons of more than six sides can be described similarly, but there are not considered in this paper.

3. THE RITZ METHOD

The total energy functional for a thin, isotropic (Kirchhoff) vibrating plate is as in (11)[17,18]

Here ρ is the density, h is the thickness, ω is the frequency, w is the deflection, D is the flexural rigidity, v is the Poisson ratio ($v = 0.3$ in all our computations), and Ω is the domain with boundary S of the plate. The last integral is the work done on the boundary, where $\hat{M}_n$ is the applied moment, $\hat{V}_n$ is the applied edge force per length, n is the unit normal to the boundary and s is the unit tangent. The minimization of E is equivalent of setting its variation to zero. From Eq. (11), after some work, see (12).

In (13) and (14), where $k = \omega L^2\sqrt{ph/D}$ is the normalized frequency, and n_x, n_y are the direction normals. Since δw is arbitrary in the interior, we recover the governing plate equation

$$\nabla^4 w - k^2 w = 0. \quad (15)$$

For completely free boundaries, $M_n = \hat{M}_n = 0$ and $V_n = \hat{V}_n = 0$. Thus the displacement w and its normal derivative should be arbitrary on the edges. Solving Eq. (15) is equivalent to minimizing Eq. (11) with the line integral in Eq. (11) set to zero.

For the Ritz method, we express the deflection in terms of a linear sum of Ritz functions taken as integer powers of x and y. They are independent but need not be orthogonal. For the completely free plate, there are no constraints for Ritz functions. Since the polynomials are complete, the variational solution converges to the true solution as the number of terms N is increased. Let

$$w = \sum_{i=1}^{N} c_i \varphi_i(x, y); \quad (16)$$

where $\varphi_i(x, y)$ are the Ritz functions and c_i are the weights to be determined.

Now Eq. (16) is substituted into Eq. (11) with the last boundary integral set to zero. In order to minimize E, a necessary condition is

$$\frac{\partial E}{\partial c_j} = 0, j = 1 to N. \quad (17)$$

After some work, Eq. (17) reduces to

$$\sum_{i=1}^{N}\left(A_{ji} - k^2 B_{ji}\right) c_i = 0, j = 1 to N; \quad (18)$$

Here

$$A_{ji} = A_{ji} = \iint_\Omega [\varphi_{ixx}\varphi_{jxx} + \varphi_{iyy}\varphi_{jyy} + 2(1-\nu)\varphi_{ixy}\varphi_{jxy} + \nu(\varphi_{ixx}\varphi_{jyy} + \varphi_{jxx}\varphi_{iyy})]\, dxdy; \quad (19)$$

$$B_{ji} = B_{ij} = \iint_\Omega \varphi_i\varphi_j dxdy. \quad (20)$$

Equation (18) is a set of linear homogeneous algebraic equations. For non-trivial c_i, the determinant of the coefficient matrix is set to zero

$$\left|A_{ji} - k^2 B_{ji}\right| = 0. \quad (21)$$

This is a characteristic equation for the eigenvalue k^2. A simple bisection algorithm on Eq. (21) yields the normalized frequencies k; the first root would be the first frequency, the second root the second frequency, and so forth. Notice that the area integrals can be performed once and for all, i.e. increasing N would not affect the previously calculated integrals.

Also, depending on the geometry, we need not take all possible polynomial powers. For plates with an odd number of sides, the vibration modes can be either even in y or odd in y. Set

$$\{\varphi_i\} = q\{1, x, x^2, y^2, x^3, xy^2, x^4, x^2y^2, y^4, x^5, x^3y^2, xy^4, x^6, x^4y^2, x^2y^4, y^4, \cdots\}. \quad (22)$$

For modes even in y, let $q = 1$; for modes odd in y, let $q = y$. For plates with an even number of sides (or the circle), the modes can be symmetric (even) in both x and y (SS mode), anti-symmetric in both x and y (AA mode), symmetric in x and anti-symmetric in y (SA mode), and anti-symmetric in x and symmetric in y (AS mode). Let

$$\{\varphi_i\} = q\{1, x^2, y^2, x^4, x^2y^2, y^4, x^6, x^4y^2, x^2y^4, y^4, x^8, x^6y^2, x^4y^4, x^2y^6, y^8, \cdots\}. \quad (23)$$

$$E = \frac{1}{2}\iint_{\Omega} D\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} dxdy - \frac{\rho h}{2}\omega^2 \iint_{\Omega} w^2 dxdy + \oint_S \left[\hat{M}_n \frac{\partial w}{\partial n} - \hat{V}_n w\right] ds; \qquad (11)$$

$$\delta E = D \iint_{\Omega} \left(\nabla^4 w - k^4 w\right) \delta w dxdy + \oint_S \left(M_n - \hat{M}_n\right)\delta\left(\frac{\partial w}{\partial n}\right) ds - \oint_S \left(V_n - \hat{V}_n\right)\delta w ds = 0; \qquad (12)$$

$$M_n = -D\left\{(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}n_x^2 + 2\frac{\partial^2 w}{\partial x \partial y}n_x n_y + \frac{\partial^2 w}{\partial y^2}n_y^2\right] + \nu \nabla^2 w\right\}; \qquad (13)$$

$$V_n = D\left\{\begin{array}{c}(1-\nu)\frac{\partial}{\partial s}\left[\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right)n_x n_y - \frac{\partial^2 w}{\partial x \partial y}\left(n_x^2 - n_y^2\right)\right] \\ -\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right)n_x - \left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y \partial x^2}\right)n_y\end{array}\right\}; \qquad (14)$$

Here $q = 1, xy, y, x$ for the SS, AA, SA, and AS modes respectively. The number of terms N taken includes the highest homogeneous powers.

Table 1 shows some typical convergence rates. It is seen that for the even number of sides, including the circle, $N = 28$ is adequate, while for the odd number of sides, $N = 35$ is adequate for a five-figure accuracy. Table 2 shows a comparison with known results. The frequency for the free circular plate has an exact formula:[3]

$$\left\{k^{3/2}I_n'(\sqrt{k}) - (1-\nu)n^2\left[\sqrt{k}I_n'(\sqrt{k}) - I_n(\sqrt{k})\right]\right\} \times \left\{kJ_n(\sqrt{k}) + (1-\nu)\left[\sqrt{k}J_n'(\sqrt{k}) - n^2 J_n(\sqrt{k})\right]\right\} - \left\{k^{3/2}J_n'(\sqrt{k}) + (1-\nu)n^2\left[\sqrt{k}J_n'(\sqrt{k}) - J_n(\sqrt{k})\right]\right\} \times \left\{kI_n(\sqrt{k}) - (1-\nu)\left[\sqrt{k}I_n'(\sqrt{k}) - n^2 I_n(\sqrt{k})\right]\right\} = 0; \qquad (24)$$

where J and I are Bessel functions and modified Bessel functions, respectively. Our Ritz results for the circle are identical to the exact solution. The results for the square are very close to those of Mochida and Ilanko[14] (using finite differences), and the equilateral triangle results are very close to Leissa and Jaber.[15]

After the frequencies are determined, the mode shapes can be obtained from Eq. (18), using an arbitrary amplitude, say $c_1 = 1$, and solving for the other weights.

4. RESULTS

Table 3 shows the first five frequencies for the rounded triangular plate. $\alpha = 0$ is the equilateral triangle, and $\alpha = 1$ is the circle. Tables 4-6 show the first five frequencies for the rounded square plate, rounded pentagonal plate, and rounded hexagonal plate. We note that although some modes are different, the frequencies are same (up to some numerical error in the fifth digit). The means that the modes could be linearly superposed; for example, the first and second modes of the pentagonal and hexagonal plates, and also the fourth and fifth modes of the pentagonal plate could be linearly superposed. Due to symmetry in both x and y directions, the fourth frequency or the fifth frequency of the square plate represent both AS or SA modes.

Most interesting is the evolution of the mode shapes as rounding is increased. Figure 2 shows the mode shapes (nodal lines) of the rounded triangular plate. The top row shows the modes corresponding to the first (fundamental) frequency, and the second row the second frequency, and so forth. It is ob-

Table 1. Typical convergence rates of the frequency. Parentheses indicate the number of terms used.

Circle 3rd mode	Square 2nd mode	Triangle 2nd mode	Pentagon 1st mode
9.1265 (6)	4.9436 (6)	3.0499 (6)	4.5318 (6)
9.0035 (10)	4.8993 (10)	3.0276 (9)	4.5275 (9)
9.0031 (15)	4.8992 (15)	3.0072 (12)	4.5158 (12)
9.0031 (21)	4.8990 (21)	3.0060 (16)	4.4829 (16)
	4.8990 (28)	3.0052 (20)	4.4798 (20)
		3.0052 (25)	4.4798 (25)
			4.4797 (30)
			4.4797 (35)

Table 2. Comparison of the first four frequencies. Asterisks denote the exact solution from Eq. (17), square brackets from Mochida and Ilanko,[14] flower brackets from Leissa,[11] and parentheses from Leissa and Jaber.[15]

Circle	Square	Equilateral triangle
5.3583 5.3583*	3.3671 [3.368] {3.372}	2.8565 (2.8566)
9.0031 9.0031*	4.8990 [4.900] {4.947}	3.0051 (3.0052)
12.439 12.439*	6.0676 [6.068] {6.108}	7.0569 (7.0569)
20.475 20.475*	8.7004 [8.700] {8.756}	13.521

Table 4. The first five frequencies of the free rounded square plate. The mode shape characteristics are indicated by subscripts. ASA means either AS or SA. The degree of rounding d is in parentheses.

$\alpha = 0$ (0)	$\alpha = 0.01$ (0.0658)	$\alpha = 0.05$ (0.1357)	$\alpha = 0.15$ (0.2134)	$\alpha = 0.3$ (0.2775)	$\alpha = 0.6$ (0.3526)	$\alpha = 1$ (0.4142)
3.3671_{AA}	3.4699_{AA}	3.6913_{AA}	4.0302_{AA}	4.3775_{AA}	4.8721_{AA}	5.3583_{AA}
4.8990_{SS}	4.9097_{SS}	4.9466_{SS}	5.0202_{SS}	5.1058_{SS}	5.2337_{SS}	5.3583_{SS}
6.0676_{SS}	6.2637_{SS}	6.6632_{SS}	7.2333_{SS}	7.7686_{SS}	8.4426_{SS}	9.0031_{SS}
8.7003_{ASA}	8.9515_{ASA}	9.4507_{ASA}	10.156_{ASA}	10.823_{ASA}	11.687_{ASA}	12.439_{ASA}
15.274_{ASA}	15.562_{ASA}	16.167_{ASA}	17.085_{ASA}	18.009_{ASA}	19.282_{ASA}	20.475_{ASA}

Table 3. The first five frequencies of the free rounded triangular plate. Subscript A indicates mode shape anti-symmetric with respect to y, and subscript S indicates symmetric with respect to y. The degree of rounding d is in parentheses.

$\alpha = 0$ (0)	$\alpha = 0.01$ (0.1818)	$\alpha = 0.05$ (0.3655)	$\alpha = 0.2$ (0.6180)	$\alpha = 0.5$ (0.8284)	$\alpha = 1$ (1)
2.8565_S	3.1356_S	3.3913_S	3.9212_S	4.5838_S	5.3583_S
3.0051_S	3.1360_A	3.3913_A	3.9212_A	4.5838_A	5.3583_A
3.0051_A	3.1601_S	3.7869_S	5.2141_S	7.1317_S	9.0031_S
7.0569_A	7.7653_A	8.1649_A	9.2045_A	10.663_A	12.439_A
7.0573_S	7.7695_S	9.1236_S	11.000_S	11.378_S	12.439_S

Table 5. First five frequencies of the free rounded pentagonal plate. Subscript A indicates mode shape anti-symmetric with respect to y, and subscript S indicates symmetric with respect to y. The degree of rounding d is in parentheses.

$\alpha = 0$ (0)	$\alpha = 0.05$ (0.0820)	$\alpha = 0.2$ (0.1418)	$\alpha = 0.5$ (0.1933)	$\alpha = 1$ (0.2361)
4.4797_A	4.6616_A	4.8960_A	5.1372_A	5.3586_A
4.4798_S	4.6616_S	4.8960_S	5.1372_S	5.3583_S
7.4238_S	7.7695_S	8.1984_S	8.6252_S	9.0031_S
10.597_A	10.943_A	11.418_A	11.937_A	12.439_A
10.598_S	10.943_S	11.418_S	11.947_S	12.439_S

vious that there are mode switches between the first column (equilateral triangle,$\alpha = 0$) and the second column (rounded, $\alpha = 0.01$). The first mode of $\alpha = 0$ morphs to the third mode of $\alpha = 0.01$, The second mode of $\alpha = 0$ turned into the first mode of $\alpha = 0.01$, and the third mode of $\alpha = 0$ becomes the second mode of $\alpha = 0.01$. As the rounding is further increased, the modes gradually change into the circular modes. The higher modes also exhibit switching. The fourth modes of $\alpha = 0$ and $\alpha = 0.01$ disappear into the sixth mode (not shown) of $\alpha = 0.05$, and the sixth mode of $\alpha = 0.01$ morphs into the fourth mode of $\alpha = 0.05$. Similarly there are also a mode switches between the fifth modes of $\alpha = 0.05$ and $\alpha = 0.2$.

Figure 3 shows the mode shapes as the square is rounded to a circle. There are no mode switches for the first five frequencies. Due to the orthotropic symmetry of the rounded square, the fourth and fifth modes all can self- rotate 90 degrees but are classified as the same mode.

Table 6. The first five frequencies of the free rounded hexagonal plate. The mode shape characteristics are indicated by subscripts. The degree of rounding d is in parentheses.

$\alpha = 0$ (0)	$\alpha = 0.05$ (0.0604)	$\alpha = 0.2$ (0.1006)	$\alpha = 0.5$ (0.1318)	$\alpha = 1$ (0.1547)
4.7887_{AA}	4.9262_{AA}	5.0892_{AA}	5.2388_{AA}	5.3583_{AA}
4.7887_{SS}	4.9262_{SS}	5.0892_{SS}	5.2388_{SS}	5.3583_{SS}
8.0045_{SS}	8.2554_{SS}	8.5429_{SS}	8.8008_{SS}	9.0031_{SS}
10.228_{SA}	10.780_{SA}	11.409_{SA}	11.981_{SA}	12.439_{SA}
12.069_{AS}	12.127_{AS}	12.232_{AS}	12.345_{AS}	12.349_{AS}

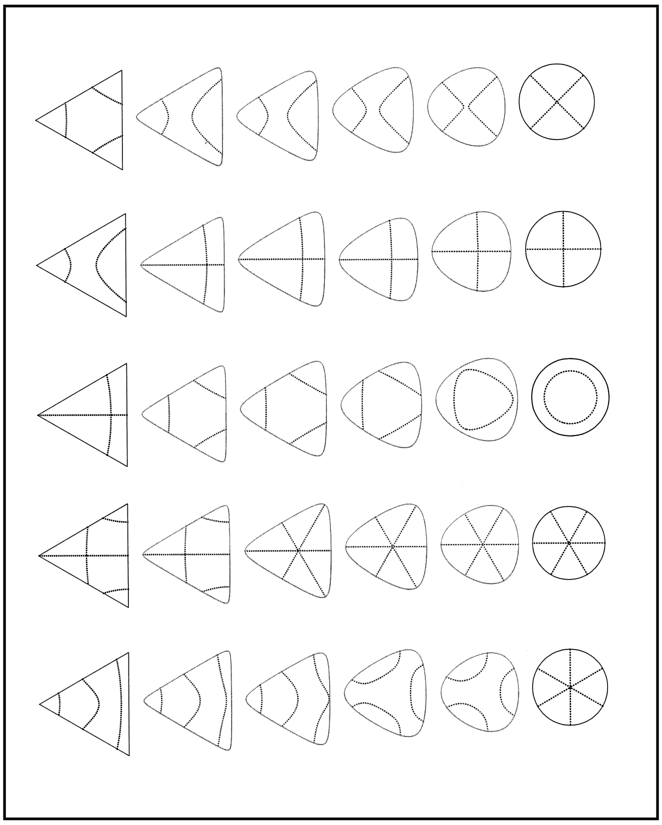

Figure 2. Mode shapes of the free rounded triangular plate. Fundamental modes are on top row, successive higher modes on lower rows. Columns from left: $\alpha = 0$, 0.01, 0.05, 0.2, 0.5, 1.

Figures 4 and 5 show the mode shapes for the rounded pentagon and the rounded hexagon. There are no mode switches for the first five frequencies.

Mode switches can be explained in terms of the total energy E. For example, for a given geometry the fundamental vibration mode corresponds to the absolute minimum of E. Consider free rectangular plates where the aspect ratio changes from much less than unity to much larger than unity. The fundamental mode (with two nodal lines) will always be perpendicular to the long sides for minimum E, i.e. switches to another direction. We find no mode switches for polygons of four or more sides morphing into a circle. Perhaps the geometry is similar, but the triangle is very sensitive to the rounding of sharp corners. Even a small rounding of $\alpha = 0.01$ (Fig. 1(a)) switches the fundamental mode.

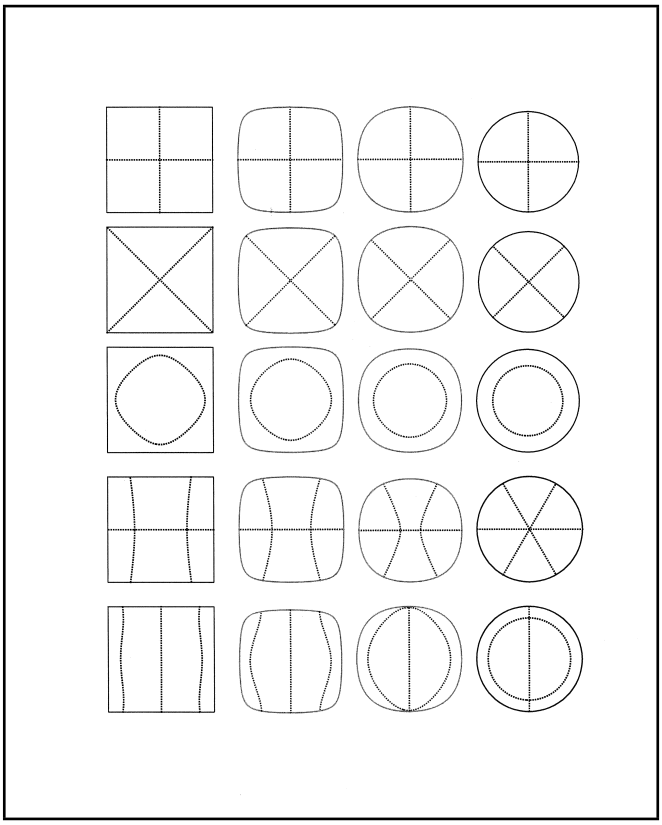

Figure 3. Mode shapes of the free rounded square plate. Fundamental modes are on top row, successive higher modes on lower rows. Columns from left: $\alpha = 0, 0.15, 0.6, 1$.

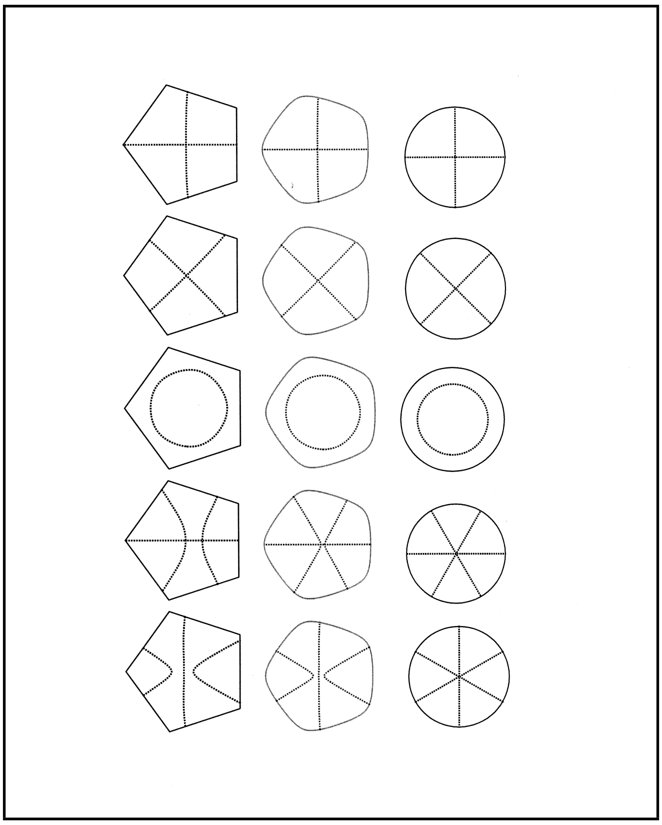

Figure 4. Mode shapes of the free rounded pentagonal plate. Fundamental modes are on top row, successive higher modes on lower rows. Columns from left: $\alpha = 0, 0.2, 1$.

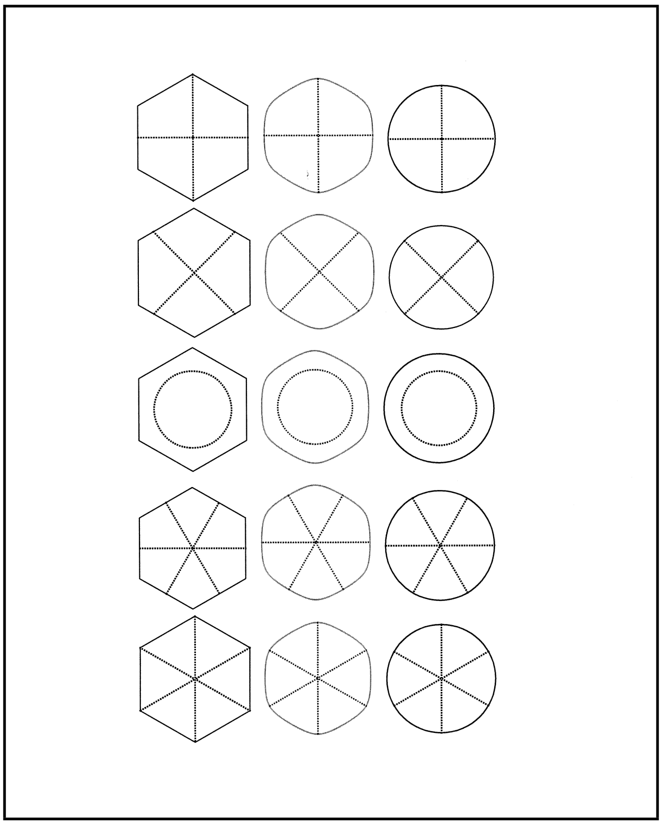

Figure 5. Mode shapes of the free rounded hexagonal plate. Fundamental modes are on top row, successive higher modes on lower rows. Columns from left: $\alpha = 0, 0.2, 1$.

5. DISCUSSION AND CONCLUSIONS

The present paper determines, for the first time, the natural frequencies and mode shapes for the vibration of completely free rounded polygonal plates. The computed results for the regular pentagonal and hexagonal plate are also new.

Using the recently introduced homotopy transformation,[17] rounded polygonal plates can be described analytically. The present plate problem however, is higher order and much more difficult than the previously studied membrane problem.[17]

The Ritz method (not Rayleigh-Ritz[20]) is accurate and efficient. For completely free plates, the tedious conditions of zero moment and zero edge forces are automatically satisfied. In comparison, all other methods including finite differences, finite elements, conformal mapping, superposition, etc. must deal with the boundary conditions and also the scaling problems at the small rounded corners. A further simplification is that, by classifying whether the mode shapes are symmetric or anti-symmetric with respect to the axes, the Ritz sequence is simplified, i.e. not all polynomial powers are needed.

Rounding of the polygonal corners increases the natural frequencies based on the radius of the inscribing circle. Our Tables 3-6 would be useful in the design of completely free plates, which model large space structures.

As rounding parameter α increases, the plate gradually changes from a regular polygon to a circle. The vibration modes also morph to that of a circular plate. We find several mode switches for the rounded triangular plate, while the rounded square, pentagonal, hexagonal plates have no such mode switches, at least for the first five modes.

Different modes with the same frequency (and the same boundary) can be linearly superposed, creating a profusion of mode shapes, as evidenced by the vibration of a free square plate.[1] Again, by using symmetric or anti-symmetric mode properties, systematic classifications of the fundamental mode shapes are possible, such as those depicted in Figs. 3-5. Waller[21] experimentally found many of the mode shapes of regular polygons, but her classification scheme is more complicated than ours.

REFERENCES

1. Leissa, A. W. *Vibration of Plates*, NASA , Washington D.C., (1969).
2. Blevins, R.D. *Formulas for Natural Frequency and Mode Shape*, Krieger, Florida, (1993).
3. Wang, C. Y. and Wang, C. M. *Structural Vibration- Exact Solutions for Strings, Membranes, Beams and Plates*. CRC Press, Boca Raton, Florida, (2013).
4. Conway, H. D. The bending, buckling and flexural vibration of simply supported polygonal plates by point-matching, *Journal of Applied Mechanics*, **28**, 288–291 (1961).
5. Shahady, P. A., Pasarelli, R., and Laura, P. A. A. Application of complex variable theory to the determination of the fundamental frequency of vibrating plates, *Journal of the Acoustical Society of America*, **42**, 806–809 (1967).
6. Irie, T., Yamada, G., and Umesato, K. Free vibration of regular polygonal plates with simply supported edges, *Journal of the Acoustical Society of America*, **69**(5), 1330–1336 (1981).
7. Liew, K. M., and Lam, K. Y. A set of orthogonal plate functions for flexural vibration of regular polygonal plates, *Journal of Vibration and Acoustics*, **113**, 182–186 (1991).
8. Ghazi, S. S. A., Barki, F. A. and Safwat, H. M. Free vibration of penta, hepta-gonal shaped plates, *Computers and Structures*, **62**(2) 395–407 (1997).
9. Duan, W. H., Wang, C. M., and Wang, C. Y. Modification of fundamental vibration modes of circular plates with free edges, *Journal of Sound and Vibration*, **317**, 709–715 (2008).
10. Wie, B. and Roithmayr, C. M., Attitude and orbit of a very large geostationary solar power satellite, *Journal of Guidance Control and Dynamics*, **28**, 439–451 (2005).
11. Leissa, A. W., The free vibration of rectangular plates, *Journal of Sound and Vibration*, **31**(3), 257–293 (1973).
12. Gorman, D. J., Free vibration analysis of the completely free rectangular plate by the method of superposition, *Journal of Sound and Vibration*, **57**, 437–447 (1978).
13. Behnke, H. and Mertins, U., Eigenwertschranken fur das problem der frei schwingenden rechteckigen platte und untersuchungen zum ausweichphanomen, *Zeitschrift fur angewandte Mathematik und Mechanik*, **75**, 343–363 (1995).
14. Mochida, Y. and Ilanko, S., Bounded natural frequencies of completely free rectangular plates, *Journal of Sound and Vibration*, **311**, 1–8 (2008).
15. Leissa, A. W. and Jaber, N. A. Vibrations of completely free triangular plates, *International Journal of Mechanical Sciences*, **34**(8) 605–616 (1992).
16. Qatu, M. S., Jaber, N. A. and Leissa, A. W. Natural frequencies for completely free trapezoidal plates, *Journal of Sound and Vibration*, **167**(1), 183–191 (1993).
17. Wang, C. Y., Fundamental frequencies of rounded polygonal membranes- A class of homotopy shapes, *Journal of Vibration and Acoustics*, **135**, 014501 (2013).
18. Timoshenko, S. and Woinowsky-Krieger, S. *Theory of Plates and Shells*, McGraw-Hill, New York, (1959).
19. Washizu, K. *Variational Methods in Elasticity and Plasticity*, Pergamon, Oxford, (1982).
20. Leissa, A. W. The historical basis of the Rayleigh and Ritz methods, *Journal of Sound and Vibration*, **287**, 961–978 (2005).
21. Waller, M. D. Vibrations of free plates: line symmetry; corresponding modes, *Proceedings of the Royal Society of London*, Ser. A, **211**, 265–276 (1952).